TWMS J. App. Eng. Math. V.9, N.4, 2019, pp. 949-956

SOME BOUNDS ON THE SEIDEL ENERGY OF GRAPHS

GÜLİSTAN KAYA GÖK[1], §

Abstract. This paper includes new bounds concepting the Seidel incidence energy. In the sequel, improved bounds about the Seidel Laplacian energy concerned with the edges and the vertices are established.
Keywords: Seidel Laplacian energy, Seidel incidence energy.

AMS Subject Classification: 05C22, 05C50

1. Introduction

Let G be a simple, finite, connected graphs with the vertex set $V(G)$ and the edge set $E(G)$. The maximum degree is denoted by Δ and the minimum degree is denoted by δ.

The Seidel matrix is defined as nxn real symmetric matrix $S(G) = s_{ij}$ where $s_{ij} = -1$ if the vertices v_i is adjacent to v_j, $(v_i \sim v_j)$, $s_{ij} = 1$ if the vertices $v_i \nsim v_j$ and $s_{ij} = 0$ if $i = j$. The eigenvalues of the Seidel matrix labeled as $\mu_1 \geq \mu_2 \geq ... \geq \mu_n$.

The Seidel Laplacian matrix $S_L(G)$ is $D_S(G) - S(G)$. The Seidel Laplacian energy of $S(G)$ is $SLE = SLE(G) = \sum_{i=1}^{n} |\mu_i^L - (n-1) + \dfrac{4m}{n}|$. Let μ_1^L,μ_2^L,...,μ_n^L be eigenvalues of the Seidel Laplacian matrix of G. Also, let $\mu_i^L - (n-1) + \dfrac{4m}{n}$ be denoted by T_i. So, $SLE(G) = \sum_{i=1}^{n} |T_i|$. [4, 11, 12] contain these equations.

The Seidel signless Laplacian matrix is represented by $S_{L^+}(G)$ such that $\mu_1^{L^+}$, $\mu_2^{L^+}$,...,$\mu_n^{L^+}$ are the eigenvalues of the Seidel signless Laplacian matrix. In this matrix, $\mu_1^{L^+} \leq \max_{i \in V} 2d_i$. See details in [3].

The structure of this paper is as the following: In Section 2, some proved lemmas are focused. In third section, some results on upper and lower bound for Seidel incidence energy of graphs with some fixed parameters are obtained. Also, Seidel signless Laplacian energy of graphs and Seidel Laplacian energy are pointed out. In the sequel, some lemmas are outlined and some results using the vertices, the edges and the degrees are improved.

[1] Hakkari University, Faculty of Education, Department of Mathematics Education, Hakkari, Turkey, e-mail: gulistankayagok@hakkari.edu.tr; ORCID: https://orcid.org/0000-0001-9059-1606.
§ Manuscript received: November 25, 2017; accepted: April 4, 2018.

2. Preliminaries

In this section, some back-ground material that is needed for later sections will be given.

Lemma 2.1 (10). *Let $k_i, t_i \in R$, $i = 1, 2, ..., n$. Let k, t, K and T be real constants such that, $0 < k \leq k_i \leq K$ and $0 < t \leq t_i \leq T$. Then,*

$$|n \sum_{i=1}^{n} k_i t_i - \sum_{i=1}^{n} k_i \sum_{i=1}^{n} t_i| \leq \gamma(n)(K-k)(T-t)$$

where $\gamma(n) = n[\frac{n}{2}]([1 - \frac{1}{n}][\frac{n}{2}])$.

Lemma 2.2 (5). *Let $a_i, b_i \in R^+$, $1 \leq i \leq n$. Then,*

$$\sum_{i=1}^{n} b_i^2 + rR \sum_{i=1}^{n} a_i^2 \leq (r+R) \sum_{i=1}^{n} a_i b_i$$

where $r, R \in R$ such that $ra_i \leq b_i \leq Ra_i$. ($R = maxs_i, r = mins_i$).

Let $x_i \in R^+$ for $1 \leq i \leq s$. M_t is defined as

$$\begin{aligned}
M_1 &= \frac{x_1 + x_2 + ... + x_s}{s}, \\
M_2 &= \frac{x_1x_2 + x_1x_3 + ... + x_1x_s + x_2x_3 + ... + x_{s-1}x_s}{\frac{1}{2}s(s-1)}, \\
&... \\
M_{s-1} &= \frac{x_1x_2...x_{s-1} + x_1x_2...x_{s-2}x_s + ... + x_2x_3...x_{s-1}x_s}{s}, \\
M_s &= x_1x_2...x_s.
\end{aligned}$$

(See details in [6].)

Lemma 2.3 (1). *Let $x_1, x_2, ..., x_s$ be real nonnegative numbers. Then*

$$M_1 \geq M_2^{\frac{1}{2}} \geq M_3^{\frac{1}{3}} \geq ... \geq M_s^{\frac{1}{s}}.$$

This equality gives if and only if $x_1 = x_2 = ... = x_s$.

Lemma 2.4 (12). *Suppose G is a graph and G has n vertices and m edges. Thus,*

$$\sum_{i=1}^{n} \mu_i^L = n(n-1) - 4m,$$

$$K = \sum_{i=1}^{n} (\mu_i^L)^2 = n^2(n-1) - 8m(n-1) + 4M_1(G)$$

where $M_1(G) = \sum_{i=1}^{n} d_i^2$ is called the first Zagreb index, for details see [2].

3. MAIN RESULTS

3.1. **On The Seidel Incidence Energy.** Let $S_I(G)$ be Seidel incidence matrix of G. Let $SIE(G)$ be Seidel incidence energy of $S(G)$ such that $SIE(G) = \sum_{i=1}^{n} s_i$. s_1,s_2,...,s_n are singular values of the Seidel incidence matrix and $s_i = \sqrt{\mu_i^{L+}}$. (see [7, 8, 9])

Lemma 3.1. *Let G be a graph with n vertices, m edges and $SIE(G)$ be incidence energy of the Seidel matrix of G. Then,*

$$SIE(G) \leq \sqrt{n(n-1) - 4m}.$$

Proof. Let $SL^{+}(G)$ be signless Laplacian matrix of G and $\mu_1^{L^+}, \mu_2^{L^+}, ..., \mu_n^{L^+}$ be eigenvalues of this matrix. Using the Cauchy-Schwarz inequality,

$$SIE^2(G) = (\sum_{i=1}^{n} s_i)^2 \leq \sum_{i=1}^{n} (s_i)^2 \tag{1}$$

$$= \sum_{i=1}^{n} (\sqrt{\mu_i^{L^+}})^2 \tag{2}$$

$$= \sum_{i=1}^{n} (\mu_i^{L^+}). \tag{3}$$

Since the Lemma 2.4. states that $trace[S_L(G)] = \sum_{i=1}^{n} (\mu_i^{L})$ and $trace[S_L(G)] = trace[S_{L^+}(G)]$ then $\sum_{i=1}^{n} (\mu_i^{L^+}) = n(n-1) - 4m$. Therefore,

$$SIE(G) \leq \sqrt{n(n-1) - 4m}.$$

□

Theorem 3.1. *Let G be a graph with m edges and n vertices. Thus, $SIE(G)$ is bounded with*

i)

$$SIE(G) \geq \frac{n(n-1) - 4m + n\sqrt{\dfrac{detS_{L^+(G)}}{\prod_{i=2}^{n-1} \mu_i^{L^+}}}}{(2\max_{i\in V}(2d_i))}. \tag{4}$$

ii)

$$SIE(G) \geq \sqrt{n(n(n-1) - 4m) - \gamma(n)(\sqrt{\Delta} - \sqrt{\delta})^2}. \tag{5}$$

Proof. i) Let s_1 be the smallest singular value and s_n be the largest singular value of $S(G)$. Suppose $a_i = 1$ and $b_i = s_i$, $1 \leq i \leq n$. Using Lemma 2.2 implies that

$$\sum_{i=1}^{n} s_i^2 + s_n s_1 \sum_{i=1}^{n} 1 \leq (s_n + s_1) \sum_{i=1}^{n} s_i. \tag{6}$$

Hence,

$$\sum_{i=1}^{n} (\sqrt{\mu_i^{L^+}})^2 + n(\sqrt{\mu_n^{L^+} \mu_1^{L^+}}) \leq (\sqrt{\mu_n^{L^+}} + \sqrt{\mu_1^{L^+}})(SIE(G)). \tag{7}$$

By Lemma 3.1, it is readily seen that,

$$n(n-1) - 4m + n\sqrt{\frac{detS_{L^+(G)}}{\prod_{i=2}^{n-1} \mu_i^{L^+}}} \leq (\sqrt{\mu_n^{L^+}} + \sqrt{\mu_1^{L^+}})(SIE(G)). \tag{8}$$

Since $\mu_1^{L^+} \geq \mu_2^{L^+} \geq, ..., \geq \mu_n^{L^+}$ and $\mu_1^{L^+} \leq \max_{i\in V}(2d_i)$, then

$$n(n-1) - 4m + n\sqrt{\frac{detS_{L^+(G)}}{\prod_{i=2}^{n-1}\mu_i^{L^+}}} \leq (2\max_{i\in V}(2d_i))(SIE(G)). \tag{9}$$

Therefore, it is expressed that

$$SIE(G) \geq \frac{n(n-1) - 4m + n\sqrt{\frac{detS_{L^+(G)}}{\prod_{i=2}^{n-1}\mu_i^{L^+}}}}{(2\max_{i\in V}(2d_i))}. \tag{10}$$

ii) Setting $k_i = s_i$, $t_i = s_i$, $k = t = s_n$ and $K = T = s_1$, $i = 1, 2, ..., n$. Using Lemma 2.1 the inequality becomes

$$|n\sum_{i=1}^{n}(s_i)^2 - (\sum_{i=1}^{n}s_i)^2| \leq \gamma(n)(s_1 - s_n)^2 \tag{11}$$

$$|n\sum_{i=1}^{n}\mu_i^{L^+} - (SIE(G))^2| \leq \gamma(n)(\sqrt{\mu_1^{L^+}} - \sqrt{\mu_n^{L^+}})^2. \tag{12}$$

By Lemma 3.1, it is seen that

$$SIE(G) \geq \sqrt{n(n(n-1) - 4m) - \gamma(n)(\sqrt{\mu_1^{L^+}} - \sqrt{\mu_n^{L^+}})^2}. \tag{13}$$

Since $\mu_n^{L^+} \leq \mu_1^{L^+} \leq \max_{i\in V}(2d_i)$, then $\mu_1^{L^+} = \Delta$ and $\mu_n^{L^+} = \delta$. Thus, it can be expressed that

$$SIE(G) \geq \sqrt{n(n(n-1) - 4m) - \gamma(n)(\sqrt{\Delta} - \sqrt{\delta})^2}. \tag{14}$$

□

3.2. **On The Seidel Laplacian Energy.** A lot of results have been improved for the Seidel Laplacian energy. Knowing the value of the previous studies, the bounds can be sharpened. Before stating the theorems, several lemmas in terms of n, m and degrees are given in this paper. Then, different bounds are found for the Seidel Laplacian energy concerned with the Zagreb index. In addition, the complement of a graph is considered and an equation between the Seidel Laplacian energy of graph and its complement is obtained.

Theorem 3.2. *Let G be a graph and $SLE(G)$ be an Seidel Laplacian energy of G. Let G has n vertices with m edges. Thus,*

$$SLE(G) \geq \sqrt{\frac{2n}{n+1}(K - (2n-1)((n-1) - \frac{4m}{n})^2)}. \tag{15}$$

Proof. Assume that $s = n$, $x_i = |T_i|$ and $t = n(n-1)$ for $i = 1, 2, ..., n$. According to Lemma 2.3, M_1 equals to

$$M_1 = \frac{\sum_{i=1}^{n}|T_i|}{n}. \tag{16}$$

For $j \neq i$,

$$M_2 = \frac{2}{t}\sum_{i=1}^{n}\sum_{j=1}^{n}|T_i||T_j| \quad (17)$$

$$=\frac{2}{t}((\sum_{i=1}^{n}|T_i|)^2 - (\sum_{j=1}^{n}|T_j|^2)) \quad (18)$$

$$=\frac{2}{t}(SLE(G)^2 - \sum_{j=1}^{n}|\mu_j^L - ((n-1) - \frac{4m}{n})|^2) \quad (19)$$

$$=\frac{2}{t}(SLE(G)^2 - \sum_{j=1}^{n}(\mu_j^L)^2 + 2((n-1) - \frac{4m}{n})(\sum_{j=1}^{n}\mu_j^L) \quad (20)$$

$$-((n-1) - \frac{4m}{n})^2. \quad (21)$$

The Lemma 2.4 implies that

$$M_2 = \frac{2}{t}(SLE(G)^2 - K + 2((n-1) - \frac{4m}{n})(n(n-1) - 4m) \quad (22)$$

$$-((n-1) - \frac{4m}{n})^2 \quad (23)$$

$$=\frac{2}{t}(SLE(G)^2 - K + (2n-1)((n-1) - \frac{4m}{n})^2. \quad (24)$$

$$(25)$$

It is known that $M_1 \geq M_2^{\frac{1}{2}}$. Therefore,

$$\frac{\sum_{i=1}^{n}|T_i|}{n} \geq \sqrt{\frac{2}{t}(SLE(G)^2 - K + (2n-1)((n-1) - \frac{4m}{n})^2)} \quad (26)$$

$$\frac{SLE(G)^2}{n^2} \geq \frac{2}{t}(SLE(G)^2 - K + (2n-1)((n-1) - \frac{4m}{n})^2) \quad (27)$$

$$SLE(G)^2 \geq \frac{2n}{n+1}(K - (2n-1)((n-1) - \frac{4m}{n})^2) \quad (28)$$

Hence,

$$SLE(G) \geq \sqrt{\frac{2n}{n+1}(K - (2n-1)((n-1) - \frac{4m}{n})^2)}. \quad (29)$$

□

Lemma 3.2. *Let T_1 be $\mu_1^L - (n-1) + \frac{4m}{n}$ with $m \neq 0$. Then,*

$$T_1 \geq \Delta + n + 2 + \frac{4m}{n}. \quad (30)$$

Proof. It is known that $\mu_1^L \geq \Delta + 1$ in [4]. From here, $T_1 \geq \Delta + n + 2 + \frac{4m}{n}$. □

Lemma 3.3. *Suppose G is a graph and the edge of G is m ($m \neq 0$). Also, let $P = (K_1 - (T_1)^2)(K_2 - (T_1)^2)$. Then,*

$$P \leq 16M_1(G_1)M_1(G_2) \quad (31)$$

$$+4N(M_1(G_1) + M_1(G_2)) + N^2 \quad (32)$$

where $K_1 = \sum_{i=1}^{n}(T_i)^2$ *and* $K_2 = \sum_{i=1}^{n}(T_j)^2$.

Proof. Let $N = -(\frac{4\Delta m + 16m + 16m^2}{n} - \frac{16m^2}{n^2} + 8m + (\Delta - 2)^2 - n(2\Delta + 5))$. Since $T_1 \geq \Delta + 2 + n + \frac{4m}{n}$, the result follows that

$$P \geq (n(n-1)) + 4M_1(G_1) - \frac{16m^2}{n} - (\Delta + 2 + n + \frac{4m}{n})^2 \quad (33)$$

$$(n(n-1)) + 4M_1(G_2) - \frac{16m^2}{n} - (\Delta + 2 + n + \frac{4m}{n})^2 \quad (34)$$

$$= 16M_1(G_1)M_1(G_2) + 4N(M_1(G_1) + M_1(G_2)) + N^2. \quad (35)$$

□

Theorem 3.3. *If G is a graph of order n with m $(m \neq 0)$ edges then,*

$$(SLE(G_1)) + (SLE(G_2)) \leq \frac{8m}{n} + 4 + 2n + 2\Delta \quad (36)$$

$$+\sqrt{4M_1(G_1 + G_2) - 8n^3 - 32mn + 2N + X} \quad (37)$$

where $X = \sqrt{16M_1(G_1) + M_1(G_2) + 4N(M_1(G_1) + M_1(G_2) - 2n^3 - 8mn) + N^2}$.

Proof. Let $T_i = \mu_i^L - (n-1) + \frac{4m}{n}$ and $T_j = \mu_j^L - (n-1) + \frac{4m}{n}$ then

$$\sum_{i=2}^{n}(|T_i| + |T_j|)^2 \leq \sum_{i=2}^{n}|T_i|^2 + \sum_{i=2}^{n}|T_j|^2 + 2\sqrt{\sum_{i=2}^{n}|T_i|^2 \sum_{i=2}^{n}|T_j|^2} \quad (38)$$

$$\leq \sum_{i=1}^{n}|T_i|^2 + \sum_{i=1}^{n}|T_j|^2 - 2|T_1|^2 \quad (39)$$

$$+2\sqrt{(\sum_{i=1}^{n}|T_i|^2 - |T_1|^2)(\sum_{i=1}^{n}|T_j|^2 - |T_1|^2)} \quad (40)$$

$$= K_1 + K_2 - 2|T_1|^2 + 2\sqrt{(K_1 - |T_1|^2)(K_2 - |T_1|^2)} \quad (41)$$

$$= K_1 + K_2 - 2|T_1|^2 + 2\sqrt{P}. \quad (42)$$

By Lemma 3.3 and Lemma 3.2,

$$\sum_{i=2}^{n}(|T_i| + |T_j|)^2 \leq K_1 + K_2 - 2|T_1|^2 \quad (43)$$

$$+2\sqrt{16M_1(G_1)M_1(G_2) + 4N(M_1(G_1) + M_1(G_2)) + N^2} \quad (44)$$

$$\leq K_1 + K_2 - 2(\Delta + n + 2 + \frac{4m}{n})^2 \quad (45)$$

$$+2\sqrt{16M_1(G_1)M_1(G_2) + 4N(M_1(G_1) + M_1(G_2)) + N^2}. \quad (46)$$

Since, $(M_1(G_1) + M_1(G_2)) = M_1(G_1 + G_2) - 2n^3 - 8mn$, then

$$\sum_{i=2}^{n}(|T_i| + |T_j|)^2 \leq 4M_1(G_1 + G_2) - 8n^3 - 32mn + 2N \quad (47)$$

$$+\sqrt{16M_1(G_1) + M_1(G_2) + 4N(M_1(G_1) + M_1(G_2) - 2n^3 - 8mn) + N^2}. \quad (48)$$

Hence,

$$(SLE(G_1)) + (SLE(G_2)) \leq \frac{8m}{n} + 4 + 2n + 2\Delta \quad (49)$$

$$+\sqrt{4M_1(G_1+G_2) - 8n^3 - 32mn + 2N + X}. \quad (50)$$

□

Proposition 3.1. *If μ_i^L, ${}^c\mu_i^L$ are the Seidel Laplacian eigenvalues of G and ${}^cG$, respectively then $SLE(G) = SLE({}^cG)$ with $i = 1, 2, ..., n$.*

Proof. If μ_i^L is the eigenvalue of G, then $-\mu_i^L$ is the eigenvalue of the complement of $G({}^cG)$. Since ${}^cT_i = -\mu_i^L - (n-1) + \frac{4\bar{m}}{n}$, then

$$T_i +^c T_i = \mu_i^L - (n-1) + \frac{4m}{n} + -\mu_i^L - (n-1) + \frac{4\bar{m}}{n}. \quad (51)$$

Since $m + \bar{m} = \frac{n(n-1)}{2}$, then $T_i +^c T_i = 0$. Thus, $SLE(G) = SLE({}^cG)$. □

4. Conclusions

In this study, some results on bounds are focused for Seidel incidence energy of graphs with some fixed parameters. Also, Seidel signless Laplacian energy of graphs is improved. In addition, Seidel Laplacian energy is investigated and some results are obtained using the vertices, the edges and the degrees.

References

[1] Biler, P. and Witkowski, A., (1990), Problems in Mathematical Analysis, Marcel Dekker, New York.

[2] Borovićanin, B., Das, K. C., Furtula, B. and Gutman, I., (2017), Bounds for Zagreb indices, MATCH Commun. Math. Comput. Chem., 78, pp.17-100.

[3] Büyükköse, S., Mutlu, N. and Kaya Gok, G., (2018), A note on the spectral radius of weighted signless laplacian matrix, Advances in Linear Algebra Matrix Theory, 8, pp.53-63.

[4] Das, K. C. and Mojallal, S.A., (2014), On laplacian energy of graphs. MATCH Commun. Math. Comput. Chem., 325, pp.52-64.

[5] Das, K. C., Elumalai, S. and Gutman, I., (2017), On ABC index of graphs, MATCH Commun. Math. Comput. Chem., 78, pp.459-468.

[6] Fan, K., (1949), On a theorem Weyl concerning eigenvalues of linear transformation, I. Proc. Natl. Acad. Sci. USA, 35,pp.652-655.

[7] Gutman, I., Kiani, D. and Mirzakhah, M., (2009), On incidence energy of graphs, MATCH Commun. Math. Comput. Chem., 62, pp.573-580.

[8] Gutman, I., Li, X. and Zhang, J., (2009), Graph Energy, in: Dehmer, M.,Emmert, F.- Streib(Eds.), Analysis of Complex Networks, From Biology to Linguistics, Wiley-VCH, Weinheim, pp.145-174.

[9] Joojandeh, M. R., Kiani, D. and Mirzakhah, M., (2009), Incidence energy of a graph, MATCH Commun. Math. Comput. Chem., 62, pp.561-572.

[10] Milovanovć, I.Z., Milovanovć, E.I. and Zakić, A., (2014), A short note on graph energy, MATCH Commun. Math. Comput. Chem., 72, pp.179-182.

[11] O. Mohammad R., (2016), Energy and seidel energy of graphs, MATCH Commun. Math. Comput. Chem., 75, pp.291-303.

[12] Ramane, H.S. and Jummannaver, R.B., (2017), Seidel laplacian energy of graphs, International Journal of Applied Graph Theory, 1,2, pp.74-82.

Gülistan Kaya Gök received her BSc (Maths) degrees from Abant Izzet Baysal University, Bolu, Turkey in 2005, her MSc (Maths) degrees from Karadeniz Technical University, Trabzon, Turkey in 2008 and her PhD (Maths) degrees from Cukurova University, Adana, Turkey in 2014. She is still working as an assistant professor in the Department of Mathematics Education at Hakkari University (Turkey). Her interest area includes Graph Theory, Energy, Bounds and Lie algebra.