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ADDITION THEOREM AND CERTAIN PROPERTIES OF k-BESSEL FUNCTION

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Abstract. Our purpose in this paper is to study certain basic properties of k-Bessel function, introduced by Gehlot [Nonl. Anal. Diff. Eq., 2(2)(2014), 61-67]. In this regard, we obtain addition theorem, expansions formula, integral representations and recurrence relation etc. Further, we also evaluate the relation between k-Bessel's function and Gauss hypergemetric function.

Keywords: k-Gamma function, k-Pochhammer symbol, k-Bessel function.

AMS Subject Classification: 33B15, 33C20, 34B30.

1. Introduction and preliminaries

Special functions are important in studying solutions of differential equations, and they are associated with a wide range of problems in many areas of mathematical physics, like problems of acoustics, radio physics, hydrodynamics, and atomic and nuclear physics. Particularly, Bessel functions and modified Bessel functions, are central to the electromagnetic wave transmission, analysis of microwave and optical transmission in waveguides, including coaxial and fiber, also in the study of the scattering of light and other electromagnetic radiation from cylindrical surfaces and rough surfaces. Modified Bessel functions appear in applications, such as transmission line studies, non-uniform beams, and the statistical treatment of a relativistic gas in statistical mechanics. It appears in the study of harmonic analysis on arithmetic and in ergodic theory. In physics, it also has application in arithmetic quantum chaos and in cosmology. For more detailed applications of Bessel functions one can refer [9, 2, 1, 7, 10]. These considerations have led various workers in the field of special functions for exploring the possible extensions and applications for the Bessel function. A useful generalization of the Bessel function called as k-Bessel function has been introduced and studied in [4] (see also, [5] and [6]). Here we aim at presenting the addition theorem and certain properties of k-Bessel's function.

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Throughout this paper, let $\mathbb{C}, \mathbb{R}, \mathbb{R}^+, \mathbb{Z}, \mathbb{Z}^-, \mathbb{N}$ be the sets of complex numbers, real numbers, positive real numbers, integers, negative integers, positive integers respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Diaz and Pariguan [3] introduced the generalized k-Gamma function $\Gamma_k(x)$ as follows:

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n! k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}} \tag{1}$$

$$(k > 0; x \in \mathbb{C}\backslash k\mathbb{Z}^-, k\mathbb{Z}^- = \left\{ kn |\, n \in \mathbb{Z}^- \right\}),$$

where $(x)_{n,k}$ is the k-Pochhammer symbol defined by

$$(x)_{n,k} = x(x+k)(x+2k)\cdots(x+(n-1)k) \ (x \in \mathbb{C}, k \in \mathbb{R}, n \in \mathbb{N}). \tag{2}$$

The integral form of the generalized k-Gamma function is given by,

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} dt \ (x \in \mathbb{C}; k \in \mathbb{R}; \Re(x) > 0). \tag{3}$$

It is easy to find the following relations:

$$\Gamma_k(x) = k^{\frac{x}{k}-1} \Gamma(\frac{x}{k}) \tag{4}$$

and

$$\Gamma_k(x+k) = x\Gamma_k(x), \tag{5}$$

where Γ is the familiar Gamma function.
We recall the Beta function ([8])

$$\int_0^1 t^{n-1}(1-t)^{m-1} dt = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)} \tag{6}$$

and k-Bessel function ([4] and [5]):

$$J^k_{\pm\vartheta}(z) = \sum_{r=0}^{\infty} \frac{(-1)^r (\frac{z}{2})^{2r\pm\frac{\vartheta}{k}}}{\Gamma_k(rk \pm \vartheta + k)(r!)}, \tag{7}$$

where $k \in \mathbb{R}^+, \vartheta \in \mathbb{Z}$ and $\vartheta > -k$.

2. Main results

In this section, we evaluate addition theorem for the k-Bessel funation, expansion of z^ϑ in a series of k-Bessel's function, three different integral representation, recurrence relation and relation between k-Bessel's function and Hypergeometric function. The main results are given as the following theorems:

Theorem 2.1. *For $k \in \mathbb{R}^+, \vartheta \in \mathbb{Z}$ and $\vartheta > -k$ when ϑ is integral*

$$J^k_{\vartheta k}(a+b) = \sum_{r=-\infty}^{\infty} J^k_{rk}(a) J^k_{(\vartheta-r)k}(b) \tag{8}$$

and if ϑ is a positive integer, then

$$J^k_{\vartheta k}(a+b) = \sum_{r=0}^{\vartheta} J^k_{rk}(a) J^k_{(\vartheta-r)k}(b) + \sum_{r=1}^{\infty} (-k)^r [J^k_{rk}(a) J^k_{(\vartheta+r)k}(b) + J^k_{(\vartheta+r)k}(a) J^k_{rk}(b)]. \tag{9}$$

Proof. Considering the generating function of k-Beassel function (cf.[4]), we have

$$\sum_{\vartheta=-\infty}^{\infty} x^{\vartheta} J^k_{\vartheta k}(z) = e^{\frac{z}{2\sqrt{k}}\left(\frac{x}{\sqrt{k}}-\frac{\sqrt{k}}{x}\right)}. \tag{10}$$

Therefore,

$$\sum_{\vartheta=-\infty}^{\infty} x^{\vartheta} J^k_{\vartheta k}(a+b) = e^{\frac{a+b}{2\sqrt{k}}\left(\frac{x}{\sqrt{k}}-\frac{\sqrt{k}}{x}\right)}$$

$$\sum_{\vartheta=-\infty}^{\infty} x^{\vartheta} J^k_{\vartheta k}(a+b) = e^{\frac{a}{2\sqrt{k}}\left(\frac{x}{\sqrt{k}}-\frac{\sqrt{k}}{x}\right)} e^{\frac{b}{2\sqrt{k}}\left(\frac{x}{\sqrt{k}}-\frac{\sqrt{k}}{x}\right)}.$$

On using (10), we obtain

$$\sum_{\vartheta=-\infty}^{\infty} x^{\vartheta} J^k_{\vartheta k}(a+b) = \sum_{r=-\infty}^{\infty} x^{r} J^k_{rk}(a) \sum_{s=-\infty}^{\infty} x^{s} J^k_{sk}(b).$$

Now on equating the coefficients of x^{ϑ} on both side, we obtain the desired result

$$J^k_{\vartheta k}(a+b) = \sum_{r=-\infty}^{\infty} J^k_{rk}(a) J^k_{(\vartheta-r)k}(b).$$

If ϑ is a positive integer, then

$$\begin{aligned} J^k_{\vartheta k}(a+b) = \cdots + J^k_{-3k}(a)J^k_{(\vartheta+3)k}(b) + J^k_{-2k}(a)J^k_{(\vartheta+2)k}(b) + J^k_{-k}(a)J^k_{(\vartheta+1)k}(b) + J^k_{0}(a)J^k_{(\vartheta)k}(b) \\ + J^k_{k}(a)J^k_{(\vartheta-1)k}(b) + J^k_{2k}(a)J^k_{(\vartheta-2)k}(b) + \cdots + J^k_{\vartheta k}(a)J^k_{0}(b) + J^k_{(\vartheta+1)k}(a)J^k_{-k}(b) \\ + J^k_{(\vartheta+2)k}(a)J^k_{-2k}(b) + J^k_{(\vartheta+3)k}(a)J^k_{-3k}(b) + \cdots, \end{aligned}$$

which on making use of relation ([4], eqn. (20)), yields to

$$\begin{aligned} J^k_{\vartheta k}(a+b) = \sum_{r=0}^{\vartheta} J^k_{rk}(a)J^k_{(\vartheta-r)k}(b) + (-k)\{J^k_{k}(a)J^k_{(\vartheta+1)k}(b) + J^k_{(\vartheta+1)k}(a)J^k_{k}(b)\} \\ + (-k)^2\{J^k_{2k}(a)J^k_{(\vartheta+2)k}(b) + J^k_{(\vartheta+2)k}(a)J^k_{2k}(b)\} \\ + (-k)^3\{J^k_{3k}(a)J^k_{(\vartheta+3)k}(b) + J^k_{(\vartheta+3)k}(a)J^k_{3k}(b)\} + \cdots, \end{aligned}$$

$$J^k_{\vartheta k}(a+b) = \sum_{r=0}^{\vartheta} J^k_{rk}(a)J^k_{(\vartheta-r)k}(b) + \sum_{r=1}^{\infty}(-k)^r[J^k_{rk}(a)J^k_{(\vartheta+r)k}(b) + J^k_{(\vartheta+r)k}(a)J^k_{rk}(b)].$$

This completes the proof of result (10). □

Theorem 2.2. *Expansion of z^{ϑ}, in a series of k-Bessel's function is given by*

$$z^{\vartheta} = (2k)^{\vartheta} \sum_{r=0}^{\infty} \frac{k^r(\vartheta+2r)\Gamma(\vartheta+r)}{r!} J^k_{(\vartheta+2r)k}(z), \tag{11}$$

where $k \in \mathbb{R}^+, \vartheta \in \mathbb{Z}$ and $\vartheta > -k$.

Proof. Considering the function,

$$f(z) = \left(\frac{2k}{z}\right)^{\vartheta} \sum_{r=0}^{\infty} \frac{k^r(\vartheta+2r)\Gamma(\vartheta+r)}{r!} J^k_{(\vartheta+2r)k}(z), \tag{12}$$

and differenciting with respect to z, we obtain

$$f'(z) = (\frac{2k}{z})^{\vartheta} \sum_{r=0}^{\infty} \frac{k^r(\vartheta+2r)\Gamma(\vartheta+r)}{r!} \dot{J}^k_{(\vartheta+2r)k}(z)$$

$$-\vartheta(2k)^{\vartheta} z^{-\vartheta-1} \sum_{r=0}^{\infty} \frac{k^r(\vartheta+2r)\Gamma(\vartheta+r)}{r!} J^k_{(\vartheta+2r)k}(z).$$

Now, on substituting the value of $\dot{J}^k_{(\vartheta+2r)k}(z)$ and $J^k_{(\vartheta+2r)k}(z)$ from ([5], eqn. (2.4) and (2.5) respectively), we obtain

$$f'(z) = (\frac{2k}{z})^{\vartheta} \sum_{r=0}^{\infty} \frac{k^r \Gamma(\vartheta+r)}{r!} [\frac{r}{k} J^k_{(\vartheta+2r-1)k}(z) - (\vartheta+r) J^k_{(\vartheta+2r+1)k}(z)]$$

$$f'(z) = (\frac{2k}{z})^{\vartheta} [\sum_{r=1}^{\infty} \frac{k^{r-1}\Gamma(\vartheta+r)}{(r-1)!} J^k_{(\vartheta+2r-1)k}(z) - \sum_{r=0}^{\infty} \frac{k^r \Gamma(\vartheta+r+1)}{(r)!} J^k_{(\vartheta+2r+1)k}(z)].$$

Further, on replacing r by $r+1$ in the first summation above, get

$$f'(z) = 0 \Rightarrow f(z) = c(constant).$$

On choosing $z = 0$ in (12), we have $f(z) = 1 \Rightarrow c = 1$, substitute in (12), we obtain the desired result. □

Theorem 2.3. *The following integral holds:*

$$\int_0^1 y^{\frac{\vartheta}{k}+1} J^k_{\vartheta}(yz)dy = \frac{k}{z} J^k_{\vartheta+k}(z), \tag{13}$$

where $k \in \mathbb{R}^+, \vartheta \in \mathbb{Z}$ *and* $\vartheta > -k$.

Proof. Consider the left hand side (say A) of integral (13), we have

$$A \equiv \int_0^1 y^{\frac{\vartheta}{k}+1} J^k_{\vartheta}(yz)dy.$$

On using (7) and integrating, we obtain

$$A \equiv \frac{k}{z} \sum_{r=0}^{\infty} \frac{(-1)^r (\frac{z}{2})^{2r+\frac{\vartheta+k}{k}}}{(rk+\vartheta+k)\Gamma_k(rk+\vartheta+k)(r!)},$$

which on using (5), reduces to

$$A \equiv \frac{k}{z} \sum_{r=0}^{\infty} \frac{(-1)^r (\frac{z}{2})^{2r+\frac{\vartheta+k}{k}}}{\Gamma_k(rk+(\vartheta+k)+k)(r!)}.$$

This completes the proof. □

Theorem 2.4. *If* $\vartheta > m > -1$, *then*

$$J^k_{\vartheta}(z) = \frac{(\frac{z}{2})^{\frac{\vartheta-m}{k}}}{k\Gamma_k(\vartheta-m)} \int_0^1 u^{\frac{m}{2k}} (1-u)^{\frac{\vartheta}{k}-\frac{m}{k}-1} J^k_m(z\sqrt{u})du, \tag{14}$$

where $k \in \mathbb{R}^+$ *and* $\vartheta \in \mathbb{Z}$.

Proof. Consider the right hand side integral (say B), then

$$B \equiv \int_0^1 u^{\frac{m}{2k}}(1-u)^{\frac{\vartheta}{k}-\frac{m}{k}-1} J_m^k(z\sqrt{u})du,$$

and on using (7), we have

$$B \equiv \sum_{r=0}^{\infty} \frac{(-1)^r (\frac{z}{2})^{2r+\frac{m}{k}}}{\Gamma_k(rk+m+k)(r!)} \int_0^1 u^{\frac{m}{k}+r}(1-u)^{\frac{\vartheta}{k}-\frac{m}{k}-1}du.$$

Further, on making use of (6) and (4), we get

$$B \equiv \frac{k\Gamma_k(\vartheta - m)}{(\frac{z}{2})^{\frac{\vartheta-m}{k}}} J_\vartheta^k(z).$$

□

Theorem 2.5. *The following result holds true*

$$\int_0^\infty e^{-t^2} t^{2a-\frac{\vartheta}{k}-1} J_\vartheta^k(zt)dt = \frac{(\frac{z}{2k})^{\frac{\vartheta}{k}}\Gamma(a)}{2\Gamma(\frac{\vartheta}{k}+1)} {}_1F_1(a; \frac{\vartheta}{k}+1; -\frac{z^2}{4k}), \tag{15}$$

provided $k \in \mathbb{R}^+, \vartheta \in \mathbb{Z}$ *and* $\vartheta > -k$.

Proof. Consider the left hand side integral (say C), then

$$C \equiv \int_0^\infty e^{-t^2} t^{2a-\frac{\vartheta}{k}-1} J_\vartheta^k(zt)dt,$$

which on using (7) yields to

$$C \equiv \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma_k(rk+\vartheta+k)(r!)} \int_0^\infty e^{-t^2} t^{2a-\frac{\vartheta}{k}-1} (\frac{zt}{2})^{2r+\frac{\vartheta}{k}} dt.$$

Substituting $t^2 = u$ and using (3), for $k = 1$, we obtain

$$C \equiv \sum_{r=0}^{\infty} \frac{(-1)^r (\frac{z}{2})^{2r+\frac{\vartheta}{k}}}{\Gamma_k(rk+\vartheta+k)(r!)} \frac{\Gamma(a+r)}{2}.$$

Now, on using (4) and rearranging the terms, we get

$$C \equiv \frac{(\frac{z}{2k})^{\frac{\vartheta}{k}}\Gamma(a)}{2\Gamma(\frac{\vartheta}{k}+1)} \sum_{r=0}^{\infty} \frac{(a)_r(-\frac{z^2}{4k})^r}{(r!)(\frac{\vartheta}{k}+1)_r},$$

which further arrive at the result (15). □

Theorem 2.6. *Suppose that* $k \in \mathbb{R}^+, \vartheta \in \mathbb{Z}$ *and* $\vartheta > -k$, *then the k-Bessel function satisfies the following recurrence relation*

$$\dot{J}_\vartheta^k(z) J_{-\vartheta}^k(z) - \dot{J}_{-\vartheta}^k(z) J_\vartheta^k(z) = \frac{2}{\pi z}\sin(\frac{\pi\vartheta}{k}). \tag{16}$$

Proof. Following the k-Bessel's differential equation cf. ([4], equation (7)), given by

$$\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + \frac{1}{k^2}(k - \frac{\vartheta^2}{z^2})y = 0. \tag{17}$$

Since $J_\vartheta^k(z)$ and $J_{-\vartheta}^k(z)$ are solutions of equation (17), therefore,

$$\ddot{J}_\vartheta^k(z) + \frac{1}{z}\dot{J}_\vartheta^k(z) + \frac{1}{k^2}(k - \frac{\vartheta^2}{z^2})J_\vartheta^k(z) = 0 \tag{18}$$

and

$$\ddot{J}^{k}_{-\vartheta}(z) + \frac{1}{z}\dot{J}^{k}_{-\vartheta}(z) + \frac{1}{k^2}(k - \frac{\vartheta^2}{z^2})J^{k}_{-\vartheta}(z) = 0. \tag{19}$$

On multiplying (18) by $J^{k}_{-\vartheta}(z)$ and (19) by $J^{k}_{\vartheta}(z)$ and subtracting, we obtain

$$[\ddot{J}^{k}_{\vartheta}(z)J^{k}_{-\vartheta}(z) - \ddot{J}^{k}_{-\vartheta}(z)J^{k}_{\vartheta}(z)] + \frac{1}{z}[\dot{J}^{k}_{\vartheta}(z)J^{k}_{-\vartheta}(z) - \dot{J}^{k}_{-\vartheta}(z)J^{k}_{\vartheta}(z)] = 0. \tag{20}$$

Let $u = \dot{J}^{k}_{\vartheta}(z)J^{k}_{-\vartheta}(z) - \dot{J}^{k}_{-\vartheta}(z)J^{k}_{\vartheta}(z)$, then from (20), we have

$$\frac{\dot{u}}{u} = -\frac{1}{z},$$

which on integrating yields to

$$u = \frac{c}{z},$$

where c is constant, hence

$$\dot{J}^{k}_{\vartheta}(z)J^{k}_{-\vartheta}(z) - \dot{J}^{k}_{-\vartheta}(z)J^{k}_{\vartheta}(z) = \frac{c}{z}. \tag{21}$$

Now, on substituting the values of $\dot{J}^{k}_{\vartheta}(z), J^{k}_{-\vartheta}(z), \dot{J}^{k}_{-\vartheta}(z)$ and $J^{k}_{\vartheta}(z)$ from (7), and comparing the coefficient of $\frac{1}{z}$ on both side, we obtain

$$c = \frac{2(\frac{\vartheta}{k})}{\Gamma_k(\vartheta + k)\Gamma_k(k - \vartheta)},$$

and then using (4), we get

$$c = \frac{2}{\Gamma(\frac{\vartheta}{k})\Gamma(1 - \frac{\vartheta}{k})}.$$

Further, on using the relation ([4], page 21), we obtain

$$c = \frac{2}{\pi}\sin(\frac{\pi\vartheta}{k}).$$

On putting the value of c in (21), we obtain the final result. □

Theorem 2.7. *For $k \in \mathbb{R}^+, \vartheta \in \mathbb{Z}$ and $\vartheta > -k$, the relation between k-Bessel's function and hypergeometric function is given by*

$$J^{k}_{\vartheta}(z) = \frac{(\frac{z}{2k})^{\frac{\vartheta}{k}}}{\Gamma(\frac{\vartheta}{k} + 1)}\, {}_0F_1(-; \frac{\vartheta}{k} + 1; -\frac{z^2}{4k}). \tag{22}$$

Proof. Following the definition of k-Bessel's function, i.e. equation (7), we have

$$J^{k}_{\vartheta}(z) = \sum_{r=0}^{\infty} \frac{(-1)^r(\frac{z}{2})^{2r+\frac{\vartheta}{k}}}{\Gamma_k(rk + \vartheta + k)(r!)}.$$

On making use of (4), we easily obtain

$$J^{k}_{\vartheta}(z) = \frac{(\frac{z}{2k})^{\frac{\vartheta}{k}}}{\Gamma(\frac{\vartheta}{k} + 1)} \sum_{r=0}^{\infty} \frac{(-\frac{z^2}{4k})^r}{(\frac{\vartheta}{k} + 1)_r(r!)},$$

which completes the proof.

□

We conclude this paper with the remark that the addition theorem, expansions formula, integral representations and recurrence relation are investigated in this paper, can provide number of new results involving variety of Bessel functions. These results may be useful in a wide range of problems in many areas of mathematical physics.

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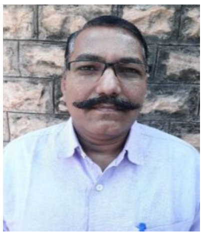

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Sunil Dutt Purohit for the photography and short autobiography, see TWMS J. App. Eng. Math., V.7, N.1, 2017.

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