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CONVOLUTIONS OF A SUBCLASS OF HARMONIC UNIVALENT MAPPINGS

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Abstract. The main object of this paper is to investigate the convolution of a subclass of harmonic univalent mappings which is denoted by f_a and generalized harmonic univalent mapping which is denoted by P_c. We obtained $P_c * f_a$ is univalent and convex in the horizantal direction for $0 < c \leq \frac{2(1-a)}{1+a}$. In addition, we present an example and illustrate it graphically with the help of Maple to explain the behaviour of image domain.

Keywords:Harmonic mappings, harmonic convolution, univalence.

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1. Introduction

A continuous function $f = u + iv$ is a complex-valued harmonic function defined on the open unit disc $U = \{z \in C : |z| < 1\}$, where u and v are real harmonic functions in U. Such function can be expressed as $f = h + \overline{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=1}^{\infty} b_n z^n$$

are analytic in U. We call h and g the analytic part and co-analytic part of f, respectively. A necessary and sufficient condition for f to be locally univalent and sense-preserving in U is that the dilatation of f defined by $w(z) = g'(z)/h'(z)$, satisfies $|w(z)| < 1$ for all $z \in U$.

Denote by S_H the class of all harmonic, sense-preserving and univalent mappings $f = h + \overline{g}$ in U, which are normalized by the conditions $f(0) = f_z(0) - 1 = 0$. Let S_H^0 be the subset of all $f \in S_H$ in which $f_{\overline{z}}(0) = 0$. Further, let K_H, C_H (resp. K_H^0,C_H^0) be the subclass of S_H (resp. S_H^0) whose image domains are convex and close-to-convex domains. A domain $\Omega \in C$ is said to be convex in the horizontal direction (CHD) if every line parallel to the real axis has a connected intersection with Ω. For basic details of harmonic univalent functions, see [1, 2] .
Let

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$$f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n} \overline{z}^n$$

and

$$F(z) = H(z) + \overline{G(z)} = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B_n} \overline{z}^n$$

be harmonic univalent functions. The convolution of two harmonic univalent functions is defined by

$$(f * F)(z) = (h * H)(z) + \overline{(g * G)(z)} = z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{B_n} \overline{z}^n.$$

Let $f_a = h_a + \overline{g_a}$ which satisfy the conditions

$$h_a - g_a = (1+a)\frac{z}{1-z} \text{ with } w(z) = \frac{a+z}{1+az} \ (-1 < a < 1). \tag{1}$$

By using the technique of shear construction method (see [1]), we have

$$h_a(z) = \frac{1+a}{1-a}\left[\frac{z}{(1-z)^2} - \frac{1}{2}\frac{z^2}{(1-z)^2}\right] = \frac{1+a}{2(1-a)}\left[\frac{z}{(1-z)^2} + \frac{z}{1-z}\right]$$

and

$$g_a(z) = \frac{1+a}{1-a}\left[\frac{az}{(1-z)^2} + \frac{1-2a}{2}\frac{z^2}{(1-z)^2}\right] = \frac{1+a}{2(1-a)}\left[\frac{z}{(1-z)^2} - \frac{(1-2a)z}{1-z}\right]. \tag{2}$$

The image of $f_a(U)$ for $a = 0.5$ is shown in Figure 1.

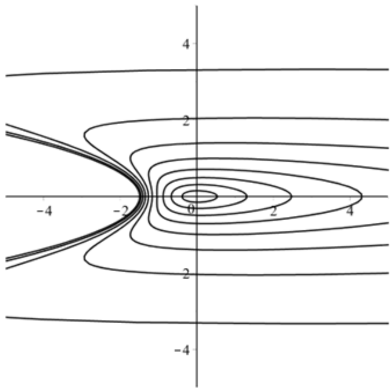

Fig. 1. Image of $f_a(U)$ for $a = 0.5$.

Obviously, for $a = 0$, denote by $F_1(z) = H_1(z) + \overline{G_1(z)}$, which satisfy the conditions $H_1 - G_1 = z/(1-z)$ and $w_1(z) = z$ studied by Liu and Li [6] . They proved that $F_1(U) = \{u + iv : v^2 > -(u + (1/4))\}$ which implies that $F_1(z)$ is a CHD mapping (not a right half plane mapping). This result was also shown by Dorff and Suffridge [3]. Also, Wang et al. [7] studied convolutions of $F_1(z)$.

Denote by SH the class of harmonic, sense-preserving and univalent mappings $f_a = h_a + \overline{g_a}$ with $h_a - g_a = (1+a)\frac{z}{1-z}$ and $w(z) = \frac{a+z}{1+az}$ $(-1 < a < 1)$ in U, which are normalized by the condition $f_a(0) = 0$. Note that, S_H is subclass of SH.

Also, Liu and Li [6] introduced the following generalized harmonic univalent mappings

$$P_c(z) = H_c(z)+\overline{G_c(z)} = \frac{1}{1+c}\left[\frac{cz}{(1-z)^2}+\frac{z}{1-z}\right]+\frac{1}{1+c}\overline{\left[\frac{cz}{(1-z)^2}-\frac{z}{1-z}\right]} \quad (z \in U; c > 0). \tag{3}$$

Obviously, $P_0(z) = F_1(z)$. If $f = h + \overline{g} \in S_H$, then

$$P_c * f = \frac{czh' + h}{1+c} + \overline{\frac{czg' - g}{1+c}}. \tag{4}$$

There are several research papers in recent years which investigate the convolution of harmonic univalent functions, see [4-9]. In particular, Dorff and Dorff et al. studied the convolution of harmonic univalent mappings in the right half-plane [4, 5] . They proved that:

Theorem A ([4,Theorem 5]). Let $f_1 = h_1 + \overline{g_1}$, $f_2 = h_2 + \overline{g_2} \in S_H^0$ with $h_i + g_i = z/(1-z)$ for $i = 1, 2$. If $f_1 * f_2$ is locally univalent and sense-preserving, then $f_1 * f_2 \in S_H^0$ is convex in the horizontal direction.

Theorem B ([5,Theorem 3]). Let $f_n = h_n + \overline{g_n} \in S_H^0$ with $h + g = z/(1-z)$ and $w(z) = g'(z)/h'(z) = e^{i\theta}z^n(\theta \in R, n \in N^+)$ and $f_0 = h_0 + \overline{g_0}$ be the canonical right half plane mapping with the dilatation $w_0(z) = -z$. If $n = 1, 2$, then $f_0 * f_n \in S_H^0$ is convex in the horizontal direction.

An important tool to prove harmonic functions are locally univalent and sense-preserving is Cohn's Rule. This rule is given as follows

Cohn's Rule ([10,pp.375]). Given a polynomial

$$p(z) = p_0(z) = a_{n,0}z^n + a_{n-1,0}z^{n-1} + \cdots + a_{1,0}z + a_{0,0} \qquad (a_{n,0} \neq 0) \tag{5}$$

of degree n, let

$$p^*(z) = p_0^*(z) = z^n\overline{p(\frac{1}{\overline{z}})} = \overline{a_{n,0}} + \overline{a_{n-1,0}}z + \cdots + \overline{a_{1,0}}z^{n-1} + \overline{a_{0,0}}z^n. \tag{6}$$

Denote by r and s the number of zeros of $p(z)$ inside the unit circle and on it, respectively. If $|a_{0,0}| < |a_{n,0}|$, then $p_1(z) = \frac{\overline{a_{n,0}}p(z)-a_{0,0}p^*(z)}{z}$ is of degree $n-1$ with $r_1 = r-1$ and $s_1 = s$ the number of zeros of $p_1(z)$ inside the unit circle and on it, respectively.

In this paper, we investigate the convolution of the harmonic functions $f_a = h_a + \overline{g_a}$ which satisfy condition (1) and generalized harmonic univalent mappings which are given by (3). We obtain the condition for $P_c * f_a$ to be univalent and convex in the horizontal direction. Also, we present an example and illustrate it graphically with the help of Maple to explain the behaviour of the image domain.

2. Main results

Lemma 2.1. *([6], Lemma 2) Let $P_c = H_c(z)+\overline{G_c(z)}$ be defined by (3) and $f = h+\overline{g} \in S_H^0$ which satisfy the conditions $h - g = z/(1-z)$ and $w(z) = \frac{g'(z)}{h'(z)}(h'(z) \neq 0, z \in U)$. Then $\widetilde{w}_1$ the dilatation of $P_c * f$, is given by*

$$\widetilde{w}_1(z) = \frac{[(c-1)+(c+1)z]w(1-w)+cw'z(1-z)}{[(c+1)+(c-1)z](1-w)+cw'z(1-z)}. \tag{7}$$

Lemma 2.2. *([6], Lemma 3) Let* $P_c = H_c(z)+\overline{G_c(z)}$ *be defined by (3) and* $f = h+\overline{g} \in S_H^0$ *which satisfy the conditions* $h-g = z/(1-z)$. *If* $P_c * f$ *is locally univalent, then* $P_c * f \in S_H^0$ *and is convex in the horizontal direction (CHD).*

Theorem 2.1. $P_c = H_c(z) + \overline{G_c(z)}$ *be given by (3). If* $f_a = h_a + \overline{g_a} \in SH$ *which satisfy the conditions* $h_a - g_a = (1+a)\frac{z}{1-z}$ *and dilatation* $w_a(z) = \frac{a+z}{1+az}$ $(-1 < a < 1)$, *then* $P_c * f_a \in SH$ *and is convex in the horizontal direction (CHD) for* $0 < c \leq \frac{2(1-a)}{1+a}$.

Proof. In view of Lemma 2.2, it suffices to show that $P_c * f_a$ is locally univalent and sense-preserving. Substituting $w(z) = w_a(z) = (a+z)/(1+az)$ into (7), we have

$$\begin{aligned} \widetilde{w}_1(z) &= \frac{[(c-1)+z(1+c)]\frac{a+z}{1+az}(1-\frac{a+z}{1+az}) + c\frac{1-a^2}{(1+az)^2}z(1-z)}{[(c+1)+(c-1)z](1-\frac{a+z}{1+az}) + c\frac{1-a^2}{(1+az)^2}z(1-z)} \\ &= -\frac{z^3 - \frac{2-a-c-2ac}{1+c}z^2 + \frac{1-2a-2c-ac}{1+c}z - \frac{a(c-1)}{1+c}}{1 - \frac{2-a-c-2ac}{1+c}z + \frac{1-2a-2c-ac}{1+c}z^2 - \frac{a(c-1)}{1+c}z^3}. \end{aligned} \tag{8}$$

Next we just need to show that $|\widetilde{w}_1(z)| < 1$ for $0 < c \leq \frac{2(1-a)}{1+a}$, where $-1 < a < 1$. We need to consider the following two cases:

Case 1. Suppose that $a = 0$. Then substituting $a = 0$ into (8) yields

$$\widetilde{w}_1(z) = -z\frac{z^2 - \frac{2-c}{1+c}z + \frac{1-2c}{1+c}}{1 - \frac{2-c}{1+c}z + \frac{1-2c}{1+c}z^2} = -z\frac{(z-1)\left(z - \frac{1-2c}{1+c}\right)}{(1-z)\left(1 - \frac{1-2c}{1+c}z\right)}.$$

Then two zeros of the above numerator are $z_1 = 1$ and $z_2 = \frac{1-2c}{1+c}$ which lie in or on the unit circle for all $0 < c \leq 2$. Thus, $|\widetilde{w}_1(z)| < 1$.

Case 2. Suppose that $a \neq 0$. From (8), we have

$$\begin{aligned} \widetilde{w}_1(z) &= -\frac{z^3 - \frac{2-a-c-2ac}{1+c}z^2 + \frac{1-2a-2c-ac}{1+c}z - \frac{a(c-1)}{1+c}}{1 - \frac{2-a-c-2ac}{1+c}z + \frac{1-2a-2c-ac}{1+c}z^2 - \frac{a(c-1)}{1+c}z^3} \\ &= -\frac{p(z)}{p^*(z)} = -\frac{(z-A)(z-B)(z-C)}{(1-\overline{A}z)(1-\overline{B}z)(1-\overline{C}z)}. \end{aligned}$$

We will show that $A, B, C \in \overline{U}$ for $0 < c \leq \frac{2(1-a)}{(1+a)}$. Applying Cohn's Rule to

$$p(z) = z^3 - \frac{2-a-c-2ac}{1+c}z^2 + \frac{1-2a-2c-ac}{1+c}z - \frac{a(c-1)}{1+c},$$

note that $\left|\frac{a(c-1)}{1+c}\right| < 1$ for $c > 0$ and $-1 < a < 1$, we get

$$\begin{aligned}
p_1(z) &= \frac{\overline{a_3}p(z) - a_0p^*(z)}{z} = \frac{p(z) + \frac{a(c-1)}{1+c}p^*(z)}{z} \\
&= \frac{(1+c+a-ac)(1+c-a+ac)}{(1+c)^2}z^2 + \frac{-2-c+6ac+c^2+2a^2-a^2c-a^2c^2}{(1+c)^2}z \\
&\quad + \frac{1-c-6ac-2c^2-a^2-a^2c+2a^2c^2}{(1+c)^2} \\
&= \frac{(1+c+a-ac)(1+c-a+ac)}{(1+c)^2}\left(z^2 + \frac{-2+c+2a+ac}{1+c-a+ac}z + \frac{1-2c-a-2ac}{1+c-a+ac}\right) \\
&= \frac{(1+c+a-ac)(1+c-a+ac)}{(1+c)^2}(z-1)\left(z - \frac{1-a-2c(1+a)}{1-a+c(1+a)}\right).
\end{aligned}$$

So $p_1(z)$ has two zeros $z_1^* = 1$ and $z_2^* = \frac{1-a-2c(1+a)}{1-a+c(1+a)}$ which are in or on the unit circle for $0 < c \le \frac{2(1-a)}{1+a}$. Thus, by Cohn's Rule, all zeros of $p(z)$ lie on $\overline{U}$, that is $A, B, C \in \overline{U}$ and so $|\widetilde{w}_1(z))| < 1$ for all $z \in U$.

Theorem 2.2. *Let $P_c = H_c(z) + \overline{G_c(z)}$ be given by (3). If $F_1 = h + g = z/(1-z)$ and dilatation $w(z) = e^{i\theta}z^n(\theta \in R, n \in N^+)$. Then $P_c * F_1 \in S_H^0$ and is convex in the horizontal direction (CHD) for $0 < c \le 2/n$.*

Proof. Proof of Theorem 2.2 is similar to the proof of Theorem 7 in [6] .

Example 2.1. In Theorem 2.1, by (1) and (4), we have

$$\begin{aligned}
P_c * f_a &= \frac{1}{1+c}\left[czh_a'(z) + h_a(z)\right] + \frac{1}{1+c}\overline{[czg_a'(z) - g_a(z)]} \\
&= \frac{1}{1+c}\left[cz\left(\frac{1+a}{1-a}\right)\frac{1}{(1-z)^3} + \left(\frac{1+a}{1-a}\right)\left(\frac{z-\frac{1}{2}z^2}{(1-z)^2}\right)\right] \\
&\quad + \frac{1}{1+c}\left[\left(\overline{cz\left(\frac{1+a}{1-a}\right)\left(\frac{a+z-az}{(1-z)^3}\right)}\right) - \left(\overline{\left(\frac{1+a}{1-a}\right)\left(\frac{az+\frac{1-2a}{2}z^2}{(1-z)^2}\right)}\right)\right] \\
&= Re\left\{\frac{1+a}{(1+c)(1-a)(1-z)^3}\left[cz(1+a+z(1-a))\right] + \frac{(1+a)z}{(1+c)(1-z)}\right\} \\
&\quad + iIm\left\{\frac{(1+a)cz}{(1+c)(1-z)^2} + \frac{1+a}{(1+c)(1-a)(1-z)^2}\left[z+az-az^2\right]\right\}.
\end{aligned}$$

Now, if we set the parameters a and c, by Theorem 2.1, we can know that $P_c * f_a$ is univalent or not. If we take $a = 0.5$ and $c = 0.1$, we have $0 < c \le \frac{2(1-a)}{1+a}$ and hence $P_c * f_a$ is univalent and CHD (see Fig. 2).

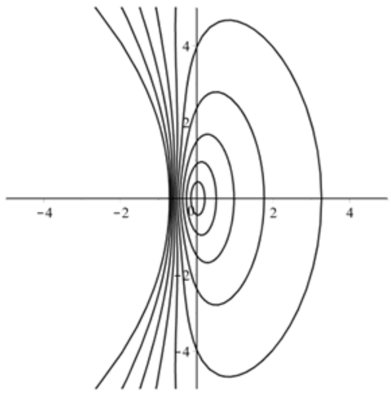

Fig 2. Image of $P_c * f_a(U)$ for $a = 0.5$ and $c = 0.1$.

If we take $a = 0.1$, $c = 2$ or $a = 0.5, c = 2$ then $P_c * f_a$ is not univalent (see Figs. 3-4).

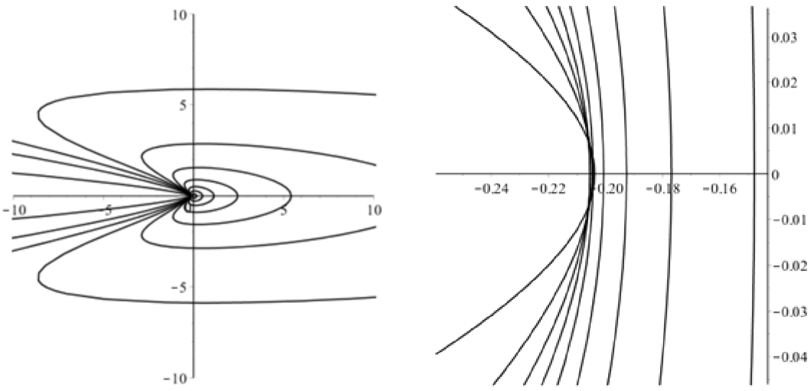

Fig 3. Image of $P_c * f_a(U)$ for $a = 0.1$ and $c = 2$.

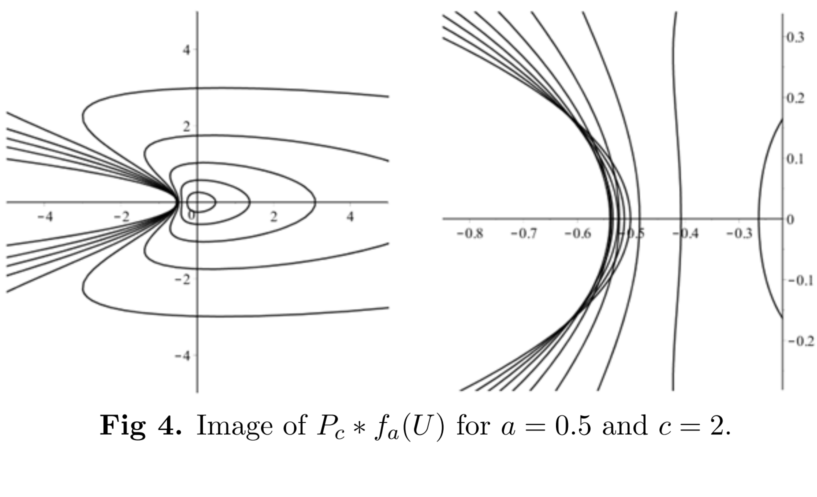

Fig 4. Image of $P_c * f_a(U)$ for $a = 0.5$ and $c = 2$.

Area For further Investigation: Let $f_a = h_a + \overline{g_a} \in SH$ which satisfy the conditions $h_a - g_a = (1 + a)\frac{z}{1-z}$ and dilatation $w_a(z) = \frac{a+z}{1+az}$ $(-1 < a < 1)$. Determine other values of $a \in U$ for which the result of Theorem 2.1 holds.

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