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HUB-INTEGRITY POLYNOMIAL OF GRAPHS

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Abstract. Graph polynomials are polynomials assigned to graphs. Interestingly, they also arise in many areas outside graph theory as well. Many properties of graph polynomials have been widely studied. In this paper, we introduce a new graph polynomial. The hub-integrity polynomial of G is the polynomial

$$HI_s(G,x)=\sum_{i=h}^{p} h_i(G,i)x^i,$$

such that $h_i(G,i)$ is the number of HI-sets of G of size i, and h is the hub number of G. Some properties of $HI_s(G,x)$ and its coefficients are obtained. Also, the hub-integrity polynomial of some specific graphs is computed.

Keywords: Integrity, Hub set, Hub-integrity, HI-set of a graph G, HI_s-polynomials.

AMS Subject Classification: 05C40, 05C99.

1. Introduction

There are many polynomials associated with graphs. For example, domination polynomial, chromatic polynomial, clique polynomial, characteristic polynomial and Tutte polynomial, see [1, 5, 6, 16, 19]. Throughout this work, we consider a finite, undirected graph with neither loops nor multiple edges. For a graph G, we denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. We use p to denote the number of vertices and q to denote the number of edges of a graph G. The reader can follow [8], for graph-theoretical terminology and notation not defined here. The complement $\overline{G}$ of a graph G has $V(G)$ as its vertex set, two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G [8]. A firefly graph $F_{s,t,p-2s-2t-1}(s\geq 0, t\geq 0\ and\ p-2s-2t-1\geq 0)$ is a graph of order p that consists of s triangles, t pendant paths of length 2 and $p-2s-2t-1$ pendant edges sharing a common vertex [9]. A galaxy graph G_x is a forest in which each component is a star [18]. A friendship graph F_n is a graph which consists of n triangles with a common vertex, $\lceil x\rceil$ denotes the smallest integer number that is greater than or equal to x.

Networks appear in many different applications and settings. The most common networks are telecommunication networks, computer networks, the internet, road and rail

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networks and other logistic networks. In all applications, vulnerability and reliability are crucial and have important features. Network designers often build a network configuration around specific processing, performance and cost requirements. But there is little consideration given to the stability of the networks' communication structure when under the pressure of link or node loses. This lack of consideration makes the networks have low survivability. Therefore network design process must identify the critical points of failure and be able to modify the design to eliminate them [17].

Several vulnerability parameters were defined in graph theory to study the vulnerability of the networks. These parameters can be estimated by using the number of the elements that are not working, the number of the subnetworks, and the number of elements in the remaining largest network that can still mutually communicate. Connectivity, toughness, integrity, tenacity, rupture degree, and scattering number are some of the vulnerability parameters defined in graph theory. Some information about the vulnerability of the network modeled by graphs can be obtained by using these graph parameters. The concept of integrity was introduced as a measure of graph stability by Barefoot, Entringer and Swart [3], and defined as, $I(G) = \min_{S\subset V}\{|S| + m(G-S)\}$, where $m(G-S)$ denotes the order of the largest component of $G-S$. If the set S achieves the integrity, then it is called an I-set of G. That is, if $|S| + m(G-S) = I(G)$ for any set S, then S is called an I-set. For more details on the integrity see [2, 4, 7].

Suppose that $H \subseteq V(G)$ and let $x, y \in V(G)$. An H-path between x and y is a path where all intermediate vertices are from H. (This includes the degenerate cases where the path consists of the single edge xy or a single vertex x if $x = y$, call such an H-path trivial). A set $H \subseteq V(G)$ is a hub set of G if it has the property that, for any $x, y \in V(G) - H$, there is an H-path in G between x and y. The smallest size of a hub set in G is called a hub number of G and is denoted by $h(G)$ [20].

Sultan et al. [10] have introduced the concept of hub-integrity of a graph as a new measure of vulnerability which is defined as follows.

Definition 1.1. [10] *The hub-integrity of a graph G denoted by $HI(G)$ is defined by,*

$$HI(G) = \min\{|S| + m(G-S), S \text{ is a hub set of } G\},$$

where $m(G-S)$ is the order of a maximum component of $G-S$.

For more details on hub-integrity of graphs see [14, 11, 13, 12, 15].

Definition 1.2. *A subset S of $V(G)$ is said to be a HI-set, if $HI(G) = |S| + m(G-S)$.*

We use the following results for our later results.

Theorem 1.1. [20] *Let T be a tree with p vertices and l terminals. Then $h(G) = p - l$.*

Theorem 1.2. [10] *Let T be a tree with p vertices and l terminal vertices. Then $HI(G) = p - l + 1$.*

We introduce hub-integrity polynomial of a graph as a new polynomial in the field of hub set in graphs.

2. Hub-integrity polynomial of graphs

In this section, we define hub-integrity polynomial and obtain some of its properties.

Definition 2.1. *For any graph G of order p, the hub-integrity polynomial of G is the polynomial*

$$HI_s(G,x) = \sum_{i=h}^{p} h_i(G,i)x^i,$$

such that $h_i(G,i)$ is the number of HI-sets of G of size i, and h is the hub number of G. The roots of $HI_s(G,x)$ are called HI_s-roots and denoted by $R(HI_s(G,x))$. To show this polynomial, we discuss this example.

Example 2.1. *Let G be a graph as shown in Figure 1.*

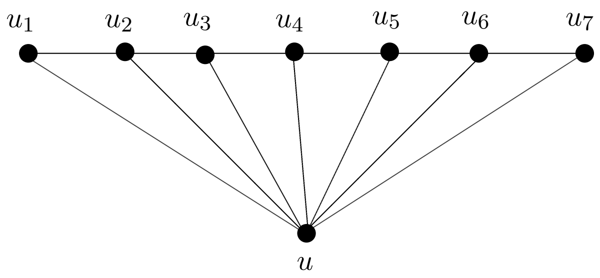

Figure 1

We have $S_1 = \{u,u_4\}, S_2 = \{u,u_3,u_5\}, S_3 = \{u,u_3,u_6\}, S_4 = \{u,u_2,u_5\},$ *and* $S_5 = \{u,u_2,u_4,u_6\}$ are HI-sets of G.
Then, $HI_s(G,x) = x^2+3x^3+x^4$, and $R(HI_s(G,x)) = \{0, -\frac{3}{2}+\frac{\sqrt{5}}{2}, -\frac{3}{2}-\frac{\sqrt{5}}{2}\}$.

Theorem 2.1. *For any complete graph K_p, $HI_s(K_p,x) = \sum_{k=0}^{p} \binom{p}{k} x^k$.*

Proof. Let $v_1, v_2, ..., v_p$ be vertices of K_p, we have p HI-sets of K_p of size one are $\{v_1\}, \{v_2\}$, $\{v_3\}, ..., \{v_p\}$. The number of HI-sets of size 2 can be selected in $\binom{p}{2}$ ways and the number of HI-sets of size 3 can be selected in $\binom{p}{3}$ ways, so the number of HI-sets of size i can be selected in $\binom{p}{i}$ ways.
Then $HI_s(K_p,x) = \binom{p}{0}x^0 + \binom{p}{1}x + \binom{p}{2}x^2 + ... + \binom{p}{p}x^p = \sum_{k=0}^{p} \binom{p}{k}$. □

Definition 2.2. [8] *The composition $G[H]$ of two graphs G and H has its vertex set $V(G) \times V(H)$, with (u_1,u_2) adjacent to (v_1,v_2) if either u_1 is adjacent to v_1 in G or $u_1 = v_1$ and u_2 is adjacent to v_2 in H.*

Lemma 2.1.

$$HI_s((P_p[K_2]),x) = \begin{cases} 1+4x+6x^2+4x^3+x^4, & \textit{if } p=2; \\ x^2, & \textit{if } p=3; \\ 2x^3+9x^4, & \textit{if } p=4. \end{cases}$$

Proof. Let P_p be a path with vertices $u_1, u_2, u_3, ..., u_p$ and K_2 be a complete graph with vertices v_1, v_2. For simplicity, denote (u_i,v_1) by $j_{i1}, 1 \le i \le p$ and (u_i,v_2) by $j_{i2}, 1 \le i \le p$. The graph $P_p[K_2]$ is shown in Figure 2.

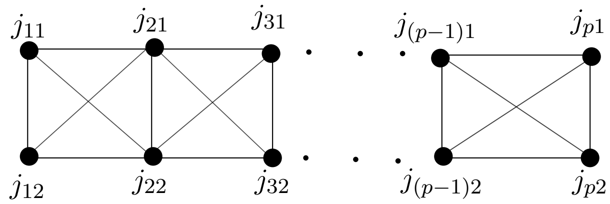

Figure 2: $P_p[K_2]$

Depending on the number of vertices we have the following cases:
Case 1: p = 2, then $P_2[K_2] \cong K_4$, so by Theorem 2.1, $HI_s((P_2[K_2]), x) = 1 + 4x + 6x^2 + 4x^3 + x^4$.
Case 2: p = 3, since $\{j_{21}, j_{22}\}$ is the only HI-set, we have $HI_s((P_3[K_2]), x) = x^2$.
Case 3: p = 4, we have two HI-sets of size 3, namely, $S_1 = \{j_{21}, j_{22}, j_{31}\}, S_2 = \{j_{21}, j_{22}, j_{32}\}$, also we have nine HI-sets of size 4, namely, $S_3 = \{j_{21}, j_{22}, j_{31}, j_{32}\}$, $S_4 = \{j_{21}, j_{22}, j_{41}, j_{42}\}$, $S_5 = \{j_{11}, j_{12}, j_{31}, j_{32}\}$, $S_6 = \{j_{21}, j_{22}, j_{31}, j_{42}\}$, $S_7 = \{j_{21}, j_{22}, j_{32}, j_{41}\}$, $S_8 = \{j_{21}, j_{22}, j_{31}, j_{41}\}$, $S_9 = \{j_{31}, j_{32}, j_{11}, j_{21}\}$, $S_{10} = \{j_{31}, j_{32}, j_{12}, j_{22}\}$ and $S_{11} = \{j_{21}, j_{22}, j_{32}, j_{42}\}$. So $HI_s((P_4[K_2]), x) = 2x^3 + 9x^4$. □

Theorem 2.2.

$$HI_s((P_p[K_2]), x) = \begin{cases} 2x^{5+4i}, & \textit{if } p=5+3i; \\ 2x^{6+4i}, & \textit{if } p=6+3i; \\ 2x^{7+4i} + 2x^{8+4i}, & \textit{if } p=7+3i, \end{cases}$$

where $i \in Z^+ \cup \{0\}$.

Proof. Three Cases are discussed.

Case 1: $p = 5 + 3i$, where $i = 0, 1, 2, \cdots$. We consider $S_1 = \{j_{(2+3k)1}, j_{(2+3k)2}/0 \leq k \leq i\} \cup \{j_{(3k)2}, j_{(3k+1)2}/1 \leq k \leq i+1\} \cup \{j_{(p-1)1}\}$ and $S_2 = \{j_{(2+3k)1}, j_{(2+3k)2}/0 \leq k \leq i\} \cup \{j_{(3k)1}, j_{(3k+1)1}/1 \leq k \leq i+1\} \cup \{j_{(p-1)2}\}$. Then there exist two HI-sets of size $5+4i$. Thus $HI_s((P_p[K_2]), x) = 2x^{5+4i}$.
Case 2: $p = 6 + 3i$, $i \in Z^+ \cup \{0\}$. We have $S_1 = \{j_{(2+3k)1}, j_{(2+3k)2}/0 \leq k \leq i+1\} \cup \{j_{(3k)2}, j_{(3k+1)2}/1 \leq k \leq i+1\}$ and $S_2 = \{j_{(2+3k)1}, j_{(2+3k)2}/0 \leq k \leq i+1\} \cup \{j_{(3k)1}, j_{(3k+1)1}/1 \leq k \leq i+1\}$. Then we have two HI-sets of size $6+4i$. Hence, $HI_s((P_p[K_2]), x) = 2x^{6+4i}$.
Case 3: $p = 7 + 3i$, $i = 0, 1, 2, ...$, we have two HI-sets of size $7 + 4i$ as follows:
$S_1 = \{j_{(2+3k)1}, j_{(2+3k)2}, 0 \leq k \leq i+1\} \cup \{j_{(3k)2}, j_{(3k+1)2}, 1 \leq k \leq i+2\}$, $S_2 = \{j_{(2+3k)1}, j_{(2+3k)2}, 0 \leq k \leq i+1\} \cup \{j_{(3k)1}, j_{(3k+1)1}, 1 \leq k \leq i+2\}$, also we have two HI-sets of size $8 + 4i$ as follows:
$S_3 = \{j_{(2+3k)1}, j_{(2+3k)2}, 0 \leq k \leq i+1\} \cup \{j_{(3k)2}/1 \leq k \leq i+2\} \cup \{j_{(3k+1)2}, 1 \leq k \leq i+1\}$, and $S_4 = \{j_{(2+3k)1}, j_{(2+3k)2}, 0 \leq k \leq i+1\} \cup \{j_{(3k)1}/1 \leq k \leq i+2\} \cup \{j_{(3k+1)1}, 1 \leq k \leq i+1\}$. Therefore, $HI_s((P_p[K_2]), x) = 2x^{7+4i} + 2x^{8+4i}$. □

Definition 2.3. [8] *The (Cartesian)product $G \times H$ of graphs G and H has $V(G) \times V(H)$ as its vertex set and (u_1, u_2) is adjacent to (v_1, v_2) if either $u_1 = v_1$ and u_2 is adjacent to v_2 or $u_2 = v_2$ and u_1 is adjacent to v_1 .*

Lemma 2.2.

$$HI_s((K_2 \times P_p), x) = \begin{cases} x^2 + 2x^3, & \textit{if } p=3; \\ 3x^4 + 2x^5, & \textit{if } p=4. \end{cases}$$

Proof. Let $V(P_p) = \{v_1, v_2, ..., v_p\}$ and $V(K_2) = \{u_1, u_2\}$. We denote (u_1, v_i) by $j_{1i}, 1 \leq i \leq p$ and (u_2, v_i) by $j_{2i}, 1 \leq i \leq p$, we show the graph $K_2 \times P_p$ in Figure 3.

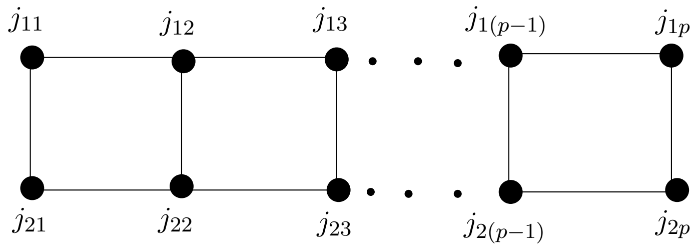

Figure 3: $K_2 \times P_p$

When $p = 3$, we can select one HI-set of size 2 and two HI-sets of size 3 of $K_2 \times P_3$ as follows: $S_1 = \{j_{12}, j_{22}\}, S_2 = \{j_{11}, j_{22}, j_{13}\}$ and $S_3 = \{j_{21}, j_{12}, j_{23}\}$. Therefore, $HI_s((K_2 \times P_3), x) = x^2 + 2x^3$.
When $p = 4$, the sets $S_1 = \{j_{12}, j_{22}, j_{13}, j_{23}\}, S_2 = \{j_{11}, j_{21}, j_{13}, j_{23}\}, S_3 = \{j_{12}, j_{22}, j_{14}, j_{24}\}, S_4 = \{j_{11}, j_{12}, j_{22}, j_{13}, j_{24}\}$ and $S_5 = \{j_{21}, j_{12}, j_{23}, j_{14}, j_{22}\}$ are HI-sets of $K_2 \times P_3$. Then $HI((K_2 \times P_3), x) = 3x^4 + 2x^5$. □

Theorem 2.3.

$$HI_s((K_2 \times P_p), x) = \begin{cases} 6x^{5+4i}, & \textit{if p=5+3i;} \\ 2x^{6+4i}, & \textit{if p=6+3i;} \\ 4x^{8+4i}, & \textit{if p=7+3i,} \end{cases}$$

where $i \in Z^+ \cup \{0\}$.

Proof. We consider three cases.
Case 1: $p = 5 + 3i$, where $i = 0, 1, 2, ...,$ $S_1 = \{j_{1(2+3k)}, j_{2(2+3k)}/0 \leq k \leq i\} \cup \{j_{2(3k)}, j_{2(3k+1)}/1 \leq k \leq i+1\} \cup \{j_{1p}\}$, $S_2 = \{j_{1(2+3k)}, j_{2(2+3k)}/0 \leq k \leq i\} \cup \{j_{1(3k)}, j_{1(3k+1)}/1 \leq k \leq i+1\} \cup \{j_{2p}\}$, $S_3 = \{j_{1(4+3k)}, j_{2(4+3k)}/0 \leq k \leq i\} \cup \{j_{2(3k-1)}, j_{2(3k)}/1 \leq k \leq i+1\} \cup \{j_{11}\}$, $S_4 = \{j_{1(4+3k)}, j_{2(4+3k)}/0 \leq k \leq i\} \cup \{j_{1(3k-1)}, j_{1(3k)}/1 \leq k \leq i+1\} \cup \{j_{21}\}$, $S_5 = \{j_{1(2+3k)}, j_{2(2+3k)}/0 \leq k \leq i\} \cup \{j_{2(3k)}/1 \leq k \leq i+1\} \cup \{j_{2(3k+1)}/1 \leq k \leq i \textit{ and } i \geq 1\} \cup \{j_{1(p-1)}, j_{2(p-1)}\}$ and $S_6 = \{j_{1(2+3k)}, j_{2(2+3k)}/0 \leq k \leq i\} \cup \{j_{1(3k)}/1 \leq k \leq i+1\} \cup \{j_{1(3k+1)}/1 \leq k \leq i \textit{ and } i \geq 1\} \cup \{j_{1(p-1)}, j_{2(p-1)}\}$, all these sets are HI-sets of $K_2 \times P_p$ graph. So we can select these sets of size $5 + 4i$ in six ways and hence $HI_s((K_2 \times P_p), x) = 6x^{5+4i}$.
Case 2: $p = 6 + 3i$, where $i \in Z^+ \cup \{0\}$. We consider $S_1 = \{j_{1(2+3k)}, j_{2(2+3k)}/0 \leq k \leq i+1\} \cup \{j_{2(3k)}, j_{2(3k+1)}/1 \leq k \leq i+1\}$ and $S_2 = \{j_{1(2+3k)}, j_{2(2+3k)}/0 \leq k \leq i+1\} \cup \{j_{1(3k)}, j_{1(3k+1)}/1 \leq k \leq i+1\}$. They satisfy condition of hub-integrity of $K_2 \times P_p$. Therefore, $HI_s((K_2 \times P_p), x) = 2x^{6+4i}$.
Case 3: $p = 7 + 3i$, where $i = 0, 1, 2, ...$. There are 4 ways to select HI-sets of $K_2 \times P_p$ graph of size $8 + 4i$, and these sets are given as follows: $S_1 = \{j_{1(2+3k)}, j_{2(2+3k)}/0 \leq k \leq i+1\} \cup \{j_{2(3k)}, j_{2(3k+1)}/1 \leq k \leq i+2\}$, $S_2 = \{j_{1(2+3k)}, j_{2(2+3k)}/0 \leq k \leq i+1\} \cup \{j_{1(3k)}, j_{1(3k+1)}/1 \leq k \leq i+2\}$, $S_3 = \{j_{1(2+3k)}, j_{2(2+3k)}/0 \leq k \leq i+1\} \cup \{j_{2(3k+1)}, j_{2(3k+2)}/1 \leq k \leq i+1\} \cup \{j_{21}, j_{22}\}$, and $S_4 = \{j_{1(3+3k)}, j_{2(3+3k)}/0 \leq k \leq i+1\} \cup \{j_{1(3k+1)}, j_{1(3k+2)}/1 \leq k \leq i+1\} \cup \{j_{11}, j_{12}\}$. Thus $HI_s((K_2 \times P_p), x) = 4x^{8+4i}$. □

Definition 2.4. [8] *For a simple connected graph G the square of G denoted by G^2, is defined as the graph with the same vertex set as of G and two vertices are adjacent in G^2 if they are at a distance 1 or 2 in G .*

Lemma 2.3.

$$HI_s(P_p^2,x) = \begin{cases} 2x + x^2 + 1, & \text{if } p=2; \\ 3x + 3x^2 + x^3 + 1, & \text{if } p=3. \end{cases}$$

Proof. Since $P_2^2 \cong K_2$ and $P_3^2 \cong K_3$, the result comes from Theorem 2.1. □

Theorem 2.4.

$$HI_s(P_p^2,x) = \begin{cases} x^2, & \text{if } p=4; \\ 2x^2 + x^3, & \text{if } p=5. \end{cases}$$

Proof. In case $p = 4$ we have just one HI-set is $S = \{v_2, v_3\}$ of size 2, hence $HI_s(P_4^2, x) = x^2$. If $p = 5$ we have two HI-sets of size 2 and one set of size 3 as follows: $S_1 = \{v_3, v_4\}, S_2 = \{v_2, v_3\}$ and $S_3 = \{v_2, v_3, v_4\}$. Then $HI_s(P_5^2, x) = 2x^2 + x^3$. □

Theorem 2.5.

$$HI_s(P_p^2,x) = \begin{cases} 3x^3, & \text{if } p=7; \\ 3x^4, & \text{if } p=8. \end{cases}$$

Proof. There are three HI-sets of P_7^2 of size 3, that are $\{v_3, v_4, v_6\}, \{v_2, v_4, v_5\}$ and $\{v_3, v_4, v_5\}$. Then $HI_s(P_7^2, x) = 3x^3$. Also there are three HI-sets of P_8^2 of size 4, that are $\{v_3, v_4, v_6, v_7\}$, $\{v_2, v_3, v_5, v_6\}$ and $\{v_3, v_4, v_5, v_6\}$. Thus $HI_s(P_8^2, x) = 3x^4$. □

Theorem 2.6.

$$HI_s(P_p^2,x) = \begin{cases} x^{2\frac{p}{3}-2}, & \text{if } p \geq 6 \text{ and } p \equiv 0 \ (mod\ 3); \\ 2x^{2\frac{p-1}{3}-1}, & \text{if } p \geq 10 \text{ and } p \equiv 1 \ (mod\ 3); \\ 2x^{2\frac{p+2}{3}-2}, & \text{if } p \geq 11 \text{ and } p \equiv 2 \ (mod\ 3). \end{cases}$$

Proof. The three cases are considered:
Case 1: $p \equiv 0 \ (mod\ 3)$, then $p = 3k$ and $k \geq 2$. Since $S = \{v_{3+3j}, v_{4+3j}/0 \leq j \leq k-2\}$ is HI-set of P_p^2 such that $|S| = 2(k-1)$. Then $h_i(P_p^2, i) = \frac{2p}{3} - 2$. Hence, $HI_s(P_p^2, x) = x^{\frac{2p}{3}-2}$.
Case 2: $p \equiv 1 \ (mod\ 3)$, then $p = 3k+1$ and $k \geq 3$. We consider $S_1 = \{v_{3+3j}, v_{4+3j}/0 \leq j \leq k-2\} \cup \{v_{p-1}\}$ and $S_2 = \{v_{4+3j}, v_{5+3j}/0 \leq j \leq k-2\} \cup \{v_2\}$. They are HI-sets of P_p^2 such that $|S_1| = |S_2| = \frac{2(p-1)}{3} - 1$. Then we have $HI_s(P_p^2, x) = 2x^{\frac{2(p-1)}{3}-1}$.
Case 3: $p \equiv 2 \ (mod\ 3)$, then $p = 3k-1$ and $k \geq 4$. We have $S_1 = \{v_{3+3j}, v_{4+3j}/0 \leq j \leq k-2\}$ and $S_2 = \{v_{2+3j}, v_{3+3j}/0 \leq j \leq k-2\}$ are HI-sets of P_p^2 of size $2k-1$, and this means we have two sets of size $\frac{2p+2}{3} - 2$. Therefore, $HI_s(P_p^2, x) = 2x^{\frac{2p+2}{3}-2}$. □

Proposition 2.1. *$h_i(P_p^2, i) = \phi$ if and only if $i > p$ or $i < \lceil \frac{p}{3} \rceil$.*

Lemma 2.4. (1) *If $h_i(P_{p-1}^2, i-1) = h_i(P_{p-3}^2, i-3) = \phi$, then $h_i(P_{p-2}^2, i-2) = \phi$.*
(2) *If $h_i(P_{p-1}^2, i-1) \neq \phi$, $h_i(P_{p-3}^2, i-3) \neq \phi$, then $h_i(P_{p-2}^2, i-2) \neq \phi$.*

Proof. (1) From given, $h_i(P_{p-1}^2, i-1) = h_i(P_{p-3}^2, i-3) = \phi$, then by Proposition 2.1, $i-1 > p-1$ or $i-1 < \lceil \frac{p-3}{3} \rceil$, thus $i-1 > p-2$ or $i-1 < \lceil \frac{p-2}{3} \rceil$, hence $h_i(P_{p-2}^2, i-2) = \phi$.
(2) Assume $h_i(P_{p-2}^2, i-2) = \phi$, from Proposition 2.1, $i-1 > p-2$ or $i-1 < \lceil \frac{p-2}{3} \rceil$. Now if $i-1 > p-2$, it follows that $i-1 > p-3$. Then $h_i(P_{p-3}^2, i-3) = \phi$, a contradiction, hence we get the result. □

Proposition 2.2. *For any path P_p, $p \geq 3$,*

$$HI_s(P_p,x) = \begin{cases} x, & \text{if } p = 3; \\ 3x^2, & \text{if } p = 4; \\ 6x^3, & \text{if } p = 5; \\ (p+2)x^{p-2}, & \text{if } p \geq 6. \end{cases}$$

Proof. Let $V(P_p) = \{v_1, v_2, ..., v_p\}$, and we have the following cases:
Case 1: $p = 3$. Since $\{v_2\}$ is HI-set of P_3, $HI_s(P_3, x) = x$.
Case 2: $p = 4$. We have three HI-sets of P_4 of size 2 as follows: $S_1 = \{v_2, v_3\}, S_2 = \{v_1, v_3\}$, and $S_3 = \{v_2, v_4\}$. Hence, $HI_s(P_4, x) = 3x^2$.
Case 3: $p = 5$. We can choose HI-sets of P_5 of size 3 in six ways and these sets are $S_1 = \{v_2, v_3, v_4\}, S_2 = \{v_1, v_3, v_5\}, S_3 = \{v_1, v_2, v_4\}, S_4 = \{v_2, v_4, v_5\}, S_5 = \{v_1, v_3, v_4\}$, and $S_6 = \{v_2, v_3, v_5\}$. Then $HI_s(P_5, x) = 6x^3$.
Case 4: $p \geq 6$. Since $h(P_p) = p - 2, p \geq 6$, there exist $p + 2$ ways to find HI-sets of P_p of size $p - 2$. In addition, there does not exist any HI-set of other order satisfying $HI(P_p)$. So $HI_s(P_p, x) = (p + 2)x^{p-2}, p \geq 6$. □

Theorem 2.7. *For any tree $T \neq P_p$ with p vertices,*

$$HI_s(T, x) = \begin{cases} 2x^{p-l}, & \text{if } S \text{ contains a terminal vertex of } T; \\ x^{p-l}, & \text{otherwise,} \end{cases}$$

where S is any HI-set of T.

Proof. Suppose that T is a tree with p vertices and l terminal vertices such that $T \neq P_p$. Let S be HI-set with $|S| + m(T - S) = HI(T)$. By Theorem 1.1, $h(T) = p - l$ and by Theorem 1.2, $HI(T) = p - l + 1$ and $|S| = p - l$. If one terminal vertex belongs to HI-set, then we have two ways to choose the set S of size $p - l$. The first way is that we can choose all internal vertices as HI-set, and the second way is that we choose S such that there exists at least one terminal vertex in HI-set. Thus, $HI_s(T, x) = 2x^{p-l}$. □

By Theorem 2.7, the proof of the following result is straightforward.

Proposition 2.3. (1) *For the star $K_{1,p-1}, HI_s(K_{1,p-1}, x) = x$.*
(2) *For the double star $S_{n,m}, HI_s(S_{n,m}, x) = x^2$.*

Theorem 2.8. *Let T be a tree of order p, then $h_i(K_{1,p-1}, i) \leq h_i(T, i) \leq h_i(P_p, i)$, for $i = 1, 2, ..., p - 2$.*

Proof. Since $HI_s(K_{1,p-1}, x) = x$, $h_i(K_{1,p-1}, 2) = ... = h_i(K_{1,p-1}, p-2) = 0$ and $h_i(K_{1,p-1}, 1) = 1$. It is clear that, $h_i(K_{1,p-1}, i) \leq h_i(T, i)$ for $i = 1, 2, ..., p - 2$. We get $h_i(T, i) \leq h_i(P_p, i)$, for $i = 1, 2, ..., p - 2$, from Proposition 2.2 and Theorem 2.7. □

Proposition 2.4. *For any totally disconnected graph $\overline{K_p}$, $HI_s(\overline{K_p}, x) = x^p$.*

Proof. Since $h(\overline{K_p}) = p$, we have only one HI-set of size p, so the result. □

Proposition 2.5.

$$HI_s(K_{n,m}, x) = \begin{cases} x^n, & \text{if } n < m; \\ 2x^n, & \text{if } n = m. \end{cases}$$

Proof. Let $V(K_{n,m}) = \{u_1, u_2, ..., u_n, v_1, v_2, ..., v_m\}$, depending on the number of vertices of $K_{n,m}$, we consider two cases:
Case 1: $n < m$, the hub number is n and $S = \{u_1, u_2, ..., u_n\}$ is HI-set of $K_{n,m}$, then we have only one HI-set of size n hence, $HI_s(K_{n,m}, x) = x^n$.
Case 2: $n = m$, $h(K_{n,n}) = n$ and we have two HI-sets, namely, $S_1 = \{u_1, u_2, ..., u_n\}$ and $S_2 = \{v_1, v_2, ..., v_n\}$. Therefore, $HI_s(K_{n,m}, x) = 2x^n$. □

Theorem 2.9. *For any graph G, $HI_s(G, x) = \sum_{k=0}^{p} \binom{p}{k} x^k$ if and only if $G \cong K_p$.*

Proof. If $G \cong K_p$, then by Theorem 2.1, we get the proof.
Now, if $HI_s(G,x) = \sum_{k=0}^{p} \binom{p}{k} x^k$, it follows that $HI_s(G,x) = \binom{p}{0} x^0 + \binom{p}{1} x + \binom{p}{2} x^2 + ... + \binom{p}{p} x^p$, this means that any set with at least one vertex of the graph G is HI-set and has one HI-set of size p, the complete graph K_p only achieves these properties, this completes the proof.

□

In a polynomial $P(x) = a_0 + a_1 x + a_2 x^2 + ... + a_p x^p$, the coefficients a_0 and a_p are called the constant and leading coefficients of $P(x)$, respectively, and the greatest exponent of x is called the degree of $P(x)$ and denote by $deg(P(x))$.

Observation 2.1. *For any graph G,*
(a) $deg(HI_s(G,x)) = max|S_i|, S_i$ *is HI-set of G.*
(b) $deg(HI_s(G,x)) = p$ *if and only if* $G \cong K_p$, *or* $\overline{K_p}$.

Proposition 2.6. *Let G be any graph and $G \neq K_p$, and $HI_s(G,x) = a_0 + a_1 x + a_2 x^2 + ... + a_p x^p$. Then*

(1) $a_0 = 0$,
(2) $a_p = 1$ *or* $a_p = 0$,
(3) *If G is connected, then zero is a root of $HI_s(G,x)$ with multiplicity $h(G)$.*

Proof. (1) Since $h(G) \geq 1$ for any graph G except K_p, as a result G has at least one nonempty HI-set. So $a_0 = 0$.
(2) Since HI-set of size p for G is unique if it found, then the result.
(3) From (1), we have $HI_s(G,x) = 0$, implying $x = 0$. Then 0 is the root of polynomial $HI_s(G,x)$, it is clear $h(G)$ is the least power of x in $HI_s(G,x)$. Hence $h(G)$ is multiplicity of the root 0. □

Remark 2.1. *If $HI_s(G_1,x) = HI_s(G_2,x)$, then it is not necessary $HI(G_1) = HI(G_2)$, for example, $G_1 \cong K_{1,p-1}$ and $G_2 \cong F_n$ such that $HI_s(K_{1,p-1},x) = HI_s(F_n,x) = x$. But $HI(K_{1,p-1}) = 2$ and $HI(F_n) = 3$.*

Proposition 2.7. *$HI_s(G,x)$ is linear if and only if $G \cong F_{s,0,p-2s-1}, s \geq 2$, $G \cong F_n, n \geq 2$ or $G \cong K_{1,p-1}, p \geq 3$.*

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