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COMPUTING INTEGER POWERS FOR A CERTAIN FAMILY OF SKEW CIRCULANT MATRICES

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Abstract. In this paper, we derive a formula for the entries of the integer powers of a certain type of skew circulant matrices of odd and even order in terms of the Chebyshev polynomials of the first and second kind. Finally, we give a Maple procedure along with some numerical examples in order to verify our calculation.

Keywords: Skew circulant matrix, eigenvalues, eigenvectors, Jordan's form, Chebyshev polynomial.

AMS Subject Classification: 15Bxx, 15A18, 12E10

1. Introduction

A certain type of transformation of a set of numbers can be represented as the multiplication of a vector by a square matrix. Repetition of the operation is equivalent to multiplying the original vector by a power of the matrix. Solving some difference equations, differential and delay differential equations and boundary value problems, we need to compute the arbitrary integer powers of a square matrix. Properties of powers of matrices are thus of considerable importance [1, 2, 3].

One can find in [4] the rth power ($r \in \mathbb{N}$) of an $n \times n$ matrix A_n using the well-known expression

$$A_n^r = P_n J_n^r P_n^{-1}, \tag{1}$$

where J_n is the Jordan's form of A_n, and P_n is the transforming matrix. Matrices J_n and P_n can be found with the help of eigenvalues and eigenvectors of the matrix A_n. If the matix A_n is invertible then the expression given by (1) is also valid for negative integers.

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An $n \times n$ skew circulant matrix $S_n := \text{scirc}_n(s_0, s_1, \ldots, s_{n-1})$ is a square matrix having the form

$$S_n := \begin{bmatrix} s_0 & s_1 & s_2 & \cdots & s_{n-2} & s_{n-1} \\ -s_{n-1} & s_0 & s_1 & \ddots & & s_{n-2} \\ -s_{n-2} & -s_{n-1} & s_0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & s_2 \\ -s_2 & & \ddots & \ddots & \ddots & s_1 \\ -s_1 & -s_2 & \cdots & -s_{n-2} & -s_{n-1} & s_0 \end{bmatrix},$$

where each row is a cyclic shift of the row above it.

As it is well-known, skew circulant and circulant matrices have a wide range of applications such as in graph theory, mechanics, mathematical chemistry, signal processing, coding theory and image processing, etc. They arise in applications involving the discrete Fourier transform and the study of cyclic codes for error correction. They also play a crucial role for solving various differential equations. Numerical solutions of certain types of elliptic and parabolic partial differential equations with periodic boundary conditions often involve linear systems associated with circulant matrices [5, 6, 7, 8].

In recent years, computing the integer powers of circulant matrices has been a very popular problem by using equation (1). For instance, Rimas derived a general expression for the entries of the rth power ($r \in \mathbb{N}$) of the $n \times n$ real symmetric circulant $\text{circ}_n(0, 1, 0, \ldots, 0, 1)$ depending on the Chebyshev polynomials (see [9] or [10] for the odd case and [11] or [12] for the even case).

In [13], Gutiérrez derived a single formula by generalizing the results obtained [9] and [10] for the entries of the positive integer powers of complex symmetric circulant matrix of odd and even order given as

$$\begin{array}{ll} \text{circ}_n\left(b_0, b_1, \ldots, b_{\frac{n-1}{2}}, b_{\frac{n-1}{2}}, \ldots, b_1\right)^T; & \text{if } n \text{ is odd}, \\ \text{circ}_n\left(b_0, b_1, \ldots, b_{\frac{n}{2}-1}, b_{\frac{n}{2}}, b_{\frac{n}{2}-1}, \ldots, b_1\right)^T; & \text{if } n \text{ is even}, \end{array}$$

and in [14], he also derived two separate formulas for the entries of the positive integer powers of complex skew-symmetric circulant matrix of odd and even order given as

$$\begin{array}{ll} \text{circ}_n\left(0, b_1, \ldots, b_{\frac{n-1}{2}}, -b_{\frac{n-1}{2}}, \ldots, -b_1\right)^T; & \text{if } n \text{ is odd}, \\ \text{circ}_n\left(0, b_1, \ldots, b_{\frac{n}{2}-1}, 0, -b_{\frac{n}{2}-1}, \ldots, -b_1\right)^T; & \text{if } n \text{ is even}. \end{array}$$

In [15], Köken and Bozkurt derived a general expression for the entries of the rth power ($r \in \mathbb{N}$) of the circulant matrix $\text{circ}_n(0, a, 0, \ldots, 0, b)$ of odd order depending on the Chebyshev polynomials.

In [16], the authors derived a single formula for the entries of the rth power of the circulant matrix $\text{circ}_n(a_0, a_1, 0, \ldots, 0, a_{-1})$ of odd and even order depending on the Chebyshev polynomials.

In [17], Köken derived two separate formulas for the entries of the positive integer powers for the skew circulant matrix $\text{scirc}_n(0, a, 0, \ldots, 0, -b)$ of odd and even order depending on the Chebyshev polynomials.

In this paper, we derive a single formula for the entries of the rth power ($r \in \mathbb{Z}$) for the skew circulant matrix having the form

$$B_n := \operatorname{scirc}_n(a, b, 0, \ldots, 0, c) := \begin{bmatrix} a & b & 0 & \ldots & 0 & c \\ -c & a & b & \ddots & & 0 \\ 0 & -c & a & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & \ddots & b \\ -b & 0 & \ldots & 0 & -c & a \end{bmatrix},$$

of odd and even order depending on the Chebyshev polynomials. This study is an extension of the results obtained in [17] for the positive integer powers of the skew circulant matrix $\operatorname{scirc}_n(0, a, 0, \ldots, 0, -b)$ with ($n \in \mathbb{N}$). Finally, the paper finishes a Maple 13 procedures along with some numerical examples in order to verify our calculation.

2. Main Results

The eigenvalue decomposition of an $n \times n$ skew circulant matrix (see [5]) is that

$$\operatorname{scirc}_n(s_0, s_1, \ldots, s_{n-1}) = G_n D_n G_n^*, \tag{2}$$

where $*$ denotes conjugate transpose (i.e $G_n^* = \overline{G}_n^T$), G_n is the $n \times n$ square matrix with the entries

$$[G_n]_{jk} = \frac{1}{\sqrt{n}} \mathrm{e}^{\frac{\pi(2k-1)(j-1)}{n}\mathbf{i}}, \quad 1 \le j, k \le n$$

and $D_n = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with

$$\lambda_k = \sum_{q=1}^{n} s_{q-1} \mathrm{e}^{\frac{\pi(2k-1)(q-1)}{n}\mathbf{i}}, \quad 1 \le k \le n, \tag{3}$$

where $\mathbf{i} = \sqrt{-1}$.

Let $U_m(x)$ be the mth degree Chebyshev polynomial of the second kind with $m \in \mathbb{N} \cup \{-1, 0\}$ [18]:

$$U_m(x) = \frac{\sin((m+1)\arccos x)}{\sin \arccos x}, \quad -1 \le x \le 1 \tag{4}$$

and $T_m(x)$ is the mth degree Chebyshev polynomial of the first kind, with $m \in \mathbb{N} \cup \{0\}$ [18]:

$$T_m(x) = \cos(m \arccos x), \quad -1 \le x \le 1. \tag{5}$$

Theorem 2.1. *Let $B_n = \operatorname{scirc}_n(a, b, 0, \ldots, 0, c)$ be an $n \times n$ invertible skew circulant matrix for $3 \le n \in \mathbb{N}$ and $\alpha_h = \cos \frac{\pi(2h-1)}{n}$ with $1 \le h \le n$. Then (j, k)th entry of B_n^r is given by:*

$$[B_n^r]_{jk} = \frac{1}{n}(L_1 + L_2)$$

for all $r \in \mathbb{Z}$ and $1 \le j, k \le n$, where L_1, L_2 are respectively

$$L_1 = \sum_{h=1}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \left[\left(a + (b-c)\alpha_h + \mathbf{i}(b+c)\operatorname{sign}(n+1-2h)\sqrt{1-\alpha_h^2} \right)^r \times \left(T_{|j-k|}(\alpha_h) + \mathbf{i}\operatorname{sign}(j-k)\operatorname{sign}(n+1-2h)\sqrt{1-\alpha_h^2}\, U_{|j-k|-1}(\alpha_h) \right) \right]$$

and

$$L_2 = \sum_{h=1}^{\lfloor \frac{n}{2} \rfloor} \left[\left(a + (b-c)\,\alpha_h - \mathbf{i}\,(b+c)\,\mathrm{sign}\,(n+1-2h)\,\sqrt{1-\alpha_h^2} \right)^r \times \right.$$
$$\left. \left(T_{|j-k|}\,(\alpha_h) - \mathbf{i}\mathrm{sign}\,(j-k)\,\mathrm{sign}\,(n+1-2h)\,\sqrt{1-\alpha_h^2} U_{|j-k|-1}\,(\alpha_h) \right) \right].$$

Here $\lfloor x \rfloor$ *is the largest integer less than or equal to* x*,* sign *denotes the signum function*

$$\mathrm{sign}\,(x) = \begin{cases} 1; & \text{if } x > 0, \\ 0; & \text{if } x = 0, \\ -1; & \text{if } x < 0. \end{cases}$$

Proof. By using (2) we can write the (j,k)th entry of B_n^r as

$$[B_n^r]_{jk} = [(G_n D_n G_n^*)^r]_{jk} = [G_n D_n^r G_n^*]_{jk} = \sum_{h=1}^{n} [G_n]_{jh}\,[D_n^r G_n^*]_{hk}$$
$$= \sum_{h=1}^{n} [G_n]_{jh}\,\lambda_h^r \overline{[G_n]_{kh}} = \frac{1}{n} \sum_{h=1}^{n} \lambda_h^r \mathrm{e}^{\frac{\pi(2h-1)(j-1)}{n}\mathbf{i}} \mathrm{e}^{-\frac{\pi(2h-1)(k-1)}{n}\mathbf{i}}.$$

From the last equation we have

$$[B_n^r]_{jk} = \frac{1}{n} \sum_{h=1}^{n} \lambda_h^r \mathrm{e}^{\frac{\pi(2h-1)(j-k)}{n}\mathbf{i}}. \tag{6}$$

From (3) we can write λ_h as

$$\lambda_h = \sum_{q=1}^{n} [B_n]_{1q}\,\mathrm{e}^{\frac{\pi(2h-1)(q-1)}{n}\mathbf{i}}$$
$$= a\mathrm{e}^{\frac{\pi(2h-1)0}{n}\mathbf{i}} + b\mathrm{e}^{\frac{\pi(2h-1)1}{n}\mathbf{i}} + c\mathrm{e}^{\frac{\pi(2h-1)(n-1)}{n}\mathbf{i}}.$$

Since

$$\mathrm{e}^{\frac{\pi(2h-1)(n-1)}{n}\mathbf{i}} = \mathrm{e}^{\pi(2h-1)\mathbf{i} - \frac{\pi(2h-1)}{n}\mathbf{i}} = -\mathrm{e}^{-\frac{\pi(2h-1)}{n}\mathbf{i}},$$

then we get λ_h as

$$\lambda_h = a + b\mathrm{e}^{\frac{\pi(2h-1)}{n}\mathbf{i}} - c\mathrm{e}^{-\frac{\pi(2h-1)}{n}\mathbf{i}}.$$

If $\beta \in \mathbb{R}$ then

$$\mathrm{e}^{\beta\mathbf{i}} = \cos\beta + \mathbf{i}\sin\beta.$$

Therefore,

$$\lambda_h = a + (b-c)\cos\frac{\pi\,(2h-1)}{n} + \mathbf{i}\,(b+c)\sin\frac{\pi\,(2h-1)}{n}. \tag{7}$$

Furthermore, $\alpha_h = \cos\frac{\pi(2h-1)}{n}$ and

$$\sin\frac{\pi\,(2h-1)}{n} = \begin{cases} \sqrt{1-\alpha_h^2}; & \text{if } n+1-2h > 0, \\ -\sqrt{1-\alpha_h^2}; & \text{if } n+1-2h < 0. \end{cases} \tag{8}$$

Consequently, λ_h can be obtained as

$$\lambda_h = a + (b-c)\,\alpha_h + \mathbf{i}\,(b+c)\,\mathrm{sign}\,(n+1-2h)\,\sqrt{1-\alpha_h^2}. \tag{9}$$

Thus, $[B_n^r]_{jk}$ can be expressed from (6) as

$$[B_n^r]_{jk} = \begin{cases} \frac{1}{n}\left[\sum_{h=1}^{\frac{n+1}{2}} \lambda_h^r e^{\frac{\pi(2h-1)(j-k)}{n}\mathbf{i}} + \sum_{h=\frac{n+3}{2}}^{n} \lambda_h^r e^{\frac{\pi(2h-1)(j-k)}{n}\mathbf{i}}\right]; \text{ if } n \text{ is odd,} \\ \frac{1}{n}\left[\sum_{h=1}^{\frac{n}{2}} \lambda_h^r e^{\frac{\pi(2h-1)(j-k)}{n}\mathbf{i}} + \sum_{h=\frac{n}{2}+1}^{n} \lambda_h^r e^{\frac{\pi(2h-1)(j-k)}{n}\mathbf{i}}\right]; \text{ if } n \text{ is even,} \end{cases}$$

$$= \begin{cases} \frac{1}{n}\left[\sum_{h=1}^{\frac{n+1}{2}} \lambda_h^r e^{\frac{\pi(2h-1)(j-k)}{n}\mathbf{i}} + \sum_{h=1}^{\frac{n-1}{2}} \lambda_{n+1-h}^r e^{\frac{\pi(2(n+1-h)-1)(j-k)}{n}\mathbf{i}}\right]; \text{ if } n \text{ is odd,} \\ \frac{1}{n}\left[\sum_{h=1}^{\frac{n}{2}} \lambda_h^r e^{\frac{\pi(2h-1)(j-k)}{n}\mathbf{i}} + \sum_{h=1}^{\frac{n}{2}} \lambda_{n+1-h}^r e^{\frac{\pi(2(n+1-h)-1)(j-k)}{n}\mathbf{i}}\right]; \text{ if } n \text{ is even.} \end{cases}$$

Notice that

$$\cos\frac{\pi\left(2\left(n+1-h\right)-1\right)}{n} = \cos\left(2\pi - \frac{\pi\left(2h-1\right)}{n}\right) = \cos\frac{\pi\left(2h-1\right)}{n} \tag{10}$$

and

$$\sin\frac{\pi\left(2\left(n+1-h\right)-1\right)}{n} = \sin\left(2\pi - \frac{\pi\left(2h-1\right)}{n}\right) = -\sin\frac{\pi\left(2h-1\right)}{n}. \tag{11}$$

Taking into account (7), (10) and (11) we obtain the following fact for the eigenvalues of B_n

$$\lambda_{n+1-h} = \overline{\lambda_h} \tag{12}$$

for $1 \le h \le n$. Morever,

$$e^{\frac{\pi(2(n+1-h)-1)(j-k)}{n}\mathbf{i}} = e^{2\pi(j-k)\mathbf{i} - \frac{\pi(2h-1)(j-k)}{n}\mathbf{i}} = e^{-\frac{\pi(2h-1)(j-k)}{n}\mathbf{i}}$$

for $1 \le h \le n$. Consequently, $[B_n^r]_{jk}$ can be deduced as

$$[B_n^r]_{jk} = \begin{cases} \frac{1}{n}\left[\sum_{h=1}^{\frac{n+1}{2}} \lambda_h^r e^{\frac{\pi(2h-1)(j-k)}{n}\mathbf{i}} + \sum_{h=1}^{\frac{n-1}{2}} \overline{\lambda_h}^r e^{-\frac{\pi(2h-1)(j-k)}{n}\mathbf{i}}\right]; \text{ if } n \text{ is odd,} \\ \frac{1}{n}\left[\sum_{h=1}^{\frac{n}{2}} \lambda_h^r e^{\frac{\pi(2h-1)(j-k)}{n}\mathbf{i}} + \sum_{h=1}^{\frac{n}{2}} \overline{\lambda_h}^r e^{-\frac{\pi(2h-1)(j-k)}{n}\mathbf{i}}\right]; \text{ if } n \text{ is even.} \end{cases}$$

Furthermore,

$$\begin{array}{ll} \left\lfloor \frac{n+1}{2} \right\rfloor = \frac{n+1}{2} \text{ and } \left\lfloor \frac{n}{2} \right\rfloor = \frac{n-1}{2}; & \text{if } n \text{ is odd,} \\ \left\lfloor \frac{n+1}{2} \right\rfloor = \frac{n}{2} \text{ and } \left\lfloor \frac{n}{2} \right\rfloor = \frac{n}{2}; & \text{if } n \text{ is even.} \end{array}$$

Thus,

$$[B_n^r]_{jk} = \frac{1}{n}\left[\sum_{h=1}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \lambda_h^r e^{\frac{\pi(2h-1)(j-k)}{n}\mathbf{i}} + \sum_{h=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \overline{\lambda_h}^r e^{-\frac{\pi(2h-1)(j-k)}{n}\mathbf{i}}\right] = \frac{1}{n}\left[L_1 + L_2\right],$$

where

$$L_1 = \sum_{h=1}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \lambda_h^r e^{\frac{\pi(2h-1)(j-k)}{n}\mathbf{i}},$$

and

$$L_2 = \sum_{h=1}^{\lfloor \frac{n}{2} \rfloor} \overline{\lambda_h}^r \mathrm{e}^{-\frac{\pi(2h-1)(j-k)}{n}\mathbf{i}}.$$

If $\beta \in \mathbb{R}$ then

$$\mathrm{e}^{\beta\mathbf{i}} = \cos\beta + \mathbf{i}\sin\beta.$$

Therefore, we obtain that

$$L_1 = \sum_{h=1}^{\lfloor \frac{n+1}{2} \rfloor} \lambda_h^r \left(\cos\frac{\pi(2h-1)(j-k)}{n} + \mathbf{i}\sin\frac{\pi(2h-1)(j-k)}{n} \right),$$

and

$$L_2 = \sum_{h=1}^{\lfloor \frac{n}{2} \rfloor} \overline{\lambda_h}^r \left(\cos\frac{\pi(2h-1)(j-k)}{n} - \mathbf{i}\sin\frac{\pi(2h-1)(j-k)}{n} \right).$$

Observe that from (4) and (5)

$$T_{|j-k|}(\alpha_h) = T_{|j-k|}\left(\cos\frac{\pi(2h-1)}{n}\right) = \cos\frac{\pi(2h-1)|j-k|}{n} = \cos\frac{\pi(2h-1)(j-k)}{n},$$

and

$$\begin{aligned} U_{|j-k|-1}(\alpha_h) = U_{|j-k|-1}\left(\cos\frac{\pi(2h-1)}{n}\right) &= \frac{\sin\frac{\pi(2h-1)|j-k|}{n}}{\sin\frac{\pi(2h-1)}{n}} \\ &= \operatorname{sign}(j-k)\frac{\sin\frac{\pi(2h-1)(j-k)}{n}}{\sin\frac{\pi(2h-1)}{n}}. \end{aligned}$$

Consequently,

$$L_1 = \sum_{h=1}^{\lfloor \frac{n+1}{2} \rfloor} \lambda_h^r \left(T_{|j-k|}(\alpha_h) + \mathbf{i}\operatorname{sign}(j-k)\sin\frac{\pi(2h-1)}{n} U_{|j-k|-1}(\alpha_h) \right), \tag{13}$$

and

$$L_2 = \sum_{h=1}^{\lfloor \frac{n}{2} \rfloor} \overline{\lambda_h}^r \left(T_{|j-k|}(\alpha_h) - \mathbf{i}\operatorname{sign}(j-k)\sin\frac{\pi(2h-1)}{n} U_{|j-k|-1}(\alpha_h) \right). \tag{14}$$

Then the theorem follows by substituting (8) and (9) into (13) and (14). □

From Theorem 2.1 we can easily derive the expressions given by [17, Theorem 2.1] when the order is even and [17, Theorem 2.2] when the order is odd for the entries of the powers of the matrix $\operatorname{scirc}_n(0, a, 0, \ldots, 0, -b)$.

Remark 2.1. *We notice that from (12) the diagonal matrix D_n has the following form*

$$D_n = \begin{cases} \operatorname{diag}\left(\lambda_1, \lambda_2, \ldots, \lambda_{\frac{n-1}{2}}, \lambda_{\frac{n+1}{2}}, \overline{\lambda_{\frac{n-1}{2}}}, \ldots, \overline{\lambda_2}, \overline{\lambda_1}\right); & \text{if } n \text{ is odd,} \\ \operatorname{diag}\left(\lambda_1, \lambda_2, \ldots, \lambda_{\frac{n}{2}}, \overline{\lambda_{\frac{n}{2}}}, \ldots, \overline{\lambda_2}, \overline{\lambda_1}\right); & \text{if } n \text{ is even.} \end{cases}$$

Corollary 2.1. *Let* $B_n = \mathrm{scirc}_n\,(a, b, 0, \ldots, 0, -b)$ *be an* $n \times n$ *invertible skew-symmetric circulant matrix for* $3 \leq n \in \mathbb{N}$ *and* $\alpha_h = \cos\frac{\pi(2h-1)}{n}$ *with* $1 \leq h \leq n$*. Then* (j,k)*th entry of* B_n^r *is given by:*

$$[B_n^r]_{jk} = \frac{1}{n}\sum_{h=1}^{\lfloor\frac{n+1}{2}\rfloor} l_{n-2h+1}\left((a+2b)\,\alpha_h\right)^r T_{|j-k|}\left(\alpha_h\right) \tag{15}$$

for all $r \in \mathbb{Z}$ *and* $1 \leq j, k \leq n$*, where* $\lfloor x \rfloor$ *is the largest integer less than or equal to* x *and*

$$l_s = \begin{cases} 1; & \text{if } s = 0, \\ 2; & \text{otherwise.} \end{cases}$$

Proof. From Theorem 2.1, we have

$$[B_n^r]_{jk} = \frac{1}{n}\left(L_1 + L_2\right)$$

for all $r \in \mathbb{Z}$ and $1 \leq j, k \leq n$, where L_1, L_2 are respectively

$$L_1 = \sum_{h=1}^{\lfloor\frac{n+1}{2}\rfloor}\left(a + 2b\alpha_h\right)^r \Big(T_{|j-k|}\left(\alpha_h\right) + \mathbf{i}\mathrm{sign}\left(j-k\right)\mathrm{sign}\left(n+1-2h\right) \times \sqrt{1-\alpha_h^2}\,U_{|j-k|-1}\left(\alpha_h\right)\Big)$$

and

$$L_2 = \sum_{h=1}^{\lfloor\frac{n}{2}\rfloor}\left(a + 2b\alpha_h\right)^r \Big(T_{|j-k|}\left(\alpha_h\right) - \mathbf{i}\mathrm{sign}\left(j-k\right)\mathrm{sign}\left(n+1-2h\right) \times \sqrt{1-\alpha_h^2}\,U_{|j-k|-1}\left(\alpha_h\right)\Big).$$

Since

$$\begin{array}{ll} \lfloor\frac{n-1}{2}\rfloor = \frac{n-1}{2} = \lfloor\frac{n}{2}\rfloor; & \text{if } n \text{ is odd,} \\ \lfloor\frac{n+1}{2}\rfloor = \frac{n}{2} = \lfloor\frac{n}{2}\rfloor; & \text{if } n \text{ is even,} \end{array}$$

then if n is odd,

$$[B_n^r]_{jk} = \frac{2}{n}\sum_{h=1}^{\frac{n-1}{2}}\left(a + 2b\alpha_h\right)^r T_{|j-k|}\left(\alpha_h\right) + \frac{1}{n}\left(a + 2b\alpha_{\frac{n+1}{2}}\right) T_{|j-k|}\left(\alpha_{\frac{n+1}{2}}\right), \tag{16}$$

and if n is even,

$$[B_n^r]_{jk} = \frac{2}{n}\sum_{h=1}^{\frac{n}{2}}\left(a + 2b\alpha_h\right)^r T_{|j-k|}\left(\alpha_h\right). \tag{17}$$

Consequently, we get (15) by using properties of the floor function in the equations (16) and (17). So the theorem is proved. □

Now let us consider the tridiagonal matrix $A_n := \text{tridiag}_n(-c, a, b)$ as

$$A_n := \begin{bmatrix} a & b & 0 & \dots & 0 & 0 \\ -c & a & b & \ddots & & 0 \\ 0 & -c & a & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & \ddots & b \\ 0 & 0 & \dots & 0 & -c & a \end{bmatrix}.$$

Then the following theorem gives the relationship between determinants of the matrices $A_n = \text{tridiag}_n(-c, a, b)$ and $B_n = \text{scirc}_n(a, b, 0, \dots, 0, c)$.

Theorem 2.2. *Let* $A_n = \text{tridiag}_n(-c, a, b)$ *and* $B_n = \text{scirc}_n(a, b, 0, \dots, 0, c)$. *Then*

$$|B_n| = a\,|A_{n-1}| + 2bc\,|A_{n-2}| + (-1)^n b^n + c^n.$$

Proof. By applying the Laplace expansion according to the first row of B_n, we get

$$\begin{aligned} |B_n| &= a \begin{vmatrix} a & b & 0 & \dots & 0 & 0 \\ -c & a & b & \ddots & & 0 \\ 0 & -c & a & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & \ddots & b \\ 0 & 0 & \dots & 0 & -c & a \end{vmatrix}_{n-1} - b \begin{vmatrix} -c & b & 0 & \dots & 0 & 0 \\ 0 & a & b & \ddots & & 0 \\ 0 & -c & a & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & \ddots & b \\ -b & 0 & \dots & 0 & -c & a \end{vmatrix}_{n-1} \\ &+ (-1)^{n+1} c \begin{vmatrix} -c & a & b & \dots & 0 & 0 \\ 0 & -c & a & \ddots & & 0 \\ 0 & 0 & -c & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & b \\ 0 & & \ddots & \ddots & \ddots & a \\ -b & 0 & \dots & 0 & 0 & -c \end{vmatrix}_{n-1}. \end{aligned}$$

Since

$$\begin{vmatrix} -c & b & 0 & \dots & 0 & 0 \\ 0 & a & b & \ddots & & 0 \\ 0 & -c & a & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & \ddots & b \\ -b & 0 & \dots & 0 & -c & a \end{vmatrix}_{n-1} = -c \begin{vmatrix} a & b & 0 & \dots & 0 & 0 \\ -c & a & b & \ddots & & 0 \\ 0 & -c & a & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & \ddots & b \\ 0 & 0 & \dots & 0 & -c & a \end{vmatrix}_{n-2} - b(-1)^n \begin{vmatrix} b & 0 & 0 & \dots & 0 & 0 \\ a & b & 0 & \ddots & & 0 \\ -c & a & b & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & \ddots & 0 \\ -b & 0 & \dots & -c & a & b \end{vmatrix}_{n-2}$$

and

$$\begin{vmatrix} -c & a & b & \dots & 0 & 0 \\ 0 & -c & a & \ddots & & 0 \\ 0 & 0 & -c & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & b \\ 0 & & \ddots & \ddots & \ddots & a \\ -b & 0 & \dots & 0 & 0 & -c \end{vmatrix}_{n-1} = -c \begin{vmatrix} -c & b & 0 & \dots & 0 & 0 \\ 0 & -c & b & \ddots & & 0 \\ 0 & 0 & -c & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & \ddots & b \\ 0 & 0 & \dots & 0 & 0 & -c \end{vmatrix}_{n-2} -b(-1)^n \begin{vmatrix} a & b & 0 & \dots & 0 & 0 \\ -c & a & b & \ddots & & 0 \\ 0 & -c & a & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & \ddots & b \\ 0 & 0 & \dots & 0 & -c & a \end{vmatrix}_{n-2},$$

then

$$\begin{aligned} |B_n| &= a \begin{vmatrix} a & b & 0 & \dots & 0 & 0 \\ -c & a & b & \ddots & & 0 \\ 0 & -c & a & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & \ddots & b \\ 0 & 0 & \dots & 0 & -c & a \end{vmatrix}_{n-1} + 2bc \begin{vmatrix} a & b & 0 & \dots & 0 & 0 \\ -c & a & b & \ddots & & 0 \\ 0 & -c & a & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & \ddots & b \\ 0 & 0 & \dots & 0 & -c & a \end{vmatrix}_{n-2} + (-1)^n b^n + c^n \\ &= a\,|A_{n-1}| + 2bc\,|A_{n-2}| + (-1)^n b^n + c^n, \end{aligned}$$

which is desired. □

3. Numerical example

In this section, we give two examples. One of them is 4×4 and the other is 5×5 skew circulant matrix. We calculated 5th and -3th powers of these matrices, respectively. These examples can be verified by using Maple procedure given by Appendix.

Let B_n be an $n \times n$ skew circulant matrix. Consequently, B_n^r is also skew circulant matrix with $r \in \mathbb{N}$ [5].

Example 3.1. *Let* $B_4 = \mathrm{scirc}_4(-2, 1, 0, 3)$ *be the* 4×4 *skew circulant matrix. From Theorem 2.1,* (j, k)*th entry of* B_4^5 *is*

$$\left[B_4^5\right]_{jk} = \frac{1}{4}(L_1 + L_2) \tag{18}$$

for $1 \le j, k \le 4$*, where*

$$L_1 = \sum_{h=1}^{2} \lambda_h^5 \left(T_{|j-k|}(\alpha_h) + \mathbf{i}\mathrm{sign}(j-k)\,\mathrm{sign}(5-2h)\sqrt{1-\alpha_h^2}\,U_{|j-k|-1}(\alpha_h) \right),$$

and

$$L_2 = \sum_{h=1}^{2} \overline{\lambda_h}^5 \left(T_{|j-k|}(\alpha_h) - \mathbf{i}\mathrm{sign}(j-k)\,\mathrm{sign}(5-2h)\sqrt{1-\alpha_h^2}\,U_{|j-k|-1}(\alpha_h) \right)$$

with $\alpha_h = \cos\frac{\pi(2h-1)}{4}$ *and* $\mathbf{i} = \sqrt{-1}$*. By using (9) we have*

$$\begin{aligned} \lambda_1 &= & -3.4142 + 2.8284\mathbf{i}, \\ \lambda_2 &= & 0.5857 + 2.8284\mathbf{i}, \\ \lambda_3 &= \overline{\lambda_2} = & -0.5857 - 2.8284\mathbf{i}, \\ \lambda_4 &= \overline{\lambda_1} = & -3.4142 - 2.8284\mathbf{i}. \end{aligned} \tag{19}$$

By substituting (19) into (18) we get $B_4^5 = \mathrm{scirc}_4(728, 484, -320, -788)$*.*

Example 3.2. *Let* $B_5 = \mathrm{scirc}_5\,(4,-3,0,0,5)$ *be the* 5×5 *invertible skew circulant matrix. From Theorem 2.1,* (j,k)*th entry of* B_5^{-3} *is*

$$\left[B_5^{-3}\right]_{jk} = \frac{1}{5}\left(L_1+L_2\right) \tag{20}$$

for $1\le j,k\le 5$*, where*

$$L_1=\sum_{h=1}^{3}\lambda_h^{-3}\left(T_{|j-k|}\left(\alpha_h\right)+\mathbf{i}\mathrm{sign}\left(j-k\right)\mathrm{sign}\left(6-2h\right)\sqrt{1-\alpha_h^2}U_{|j-k|-1}\left(\alpha_h\right)\right),$$

and

$$L_2=\sum_{h=1}^{2}\overline{\lambda_h}^{-3}\left(T_{|j-k|}\left(\alpha_h\right)-\mathbf{i}\mathrm{sign}\left(j-k\right)\mathrm{sign}\left(6-2h\right)\sqrt{1-\alpha_h^2}U_{|j-k|-1}\left(\alpha_h\right)\right)$$

with $\alpha_h=\cos\frac{\pi(2h-1)}{5}$ *and* $\mathbf{i}=\sqrt{-1}$*. By using (9) we have*

$$\begin{aligned}
\lambda_1 &= & -2.4721+1.1755\mathbf{i},\\
\lambda_2 &= & 6.4721+1.9021\mathbf{i},\\
\lambda_3 &= & 12,\\
\lambda_4 &= \overline{\lambda_2} = & 6.4721-1.9021\mathbf{i},\\
\lambda_5 &= \overline{\lambda_1} = & -2.4721-1.1755\mathbf{i}.
\end{aligned} \tag{21}$$

By substituting (21) into (20) we get $B_5^{-3}=\mathrm{scirc}_5\left(-0.0036,-0.0161,-0.0194,-0.0154,-0.0079\right)$.

4. Conclusions

Skew circulant and circulant matrices have so many applications in mathematics and engineering. For example, graph theory, signal processing, coding theory, image processing, boundary value problems, parallel computing, telecommunication system analysis, and so on. There is a vast literature concerned with the powers of skew circulant and circulant matrices, in terms of Chebyshev polynomials. In this paper, we introduced and studied a single formula for the entries of the integer powers of a certain type of skew circulant matrix of odd and even order in terms of the Chebyshev polynomials of the first and second kind.

Appendix. The following Maple procedure calculates the rth power ($r\in\mathbb{Z}$) of the invertible skew circulant matrix $B_n=\mathrm{scirc}_n\left(a,b,0,\ldots,0,c\right)$.

```
restart:
with(LinearAlgebra):
power:=proc(n,r,a,b,c)
local f,B,alpha,p,lambda,P;
f:=(j,k)->piecewise(k-j=0,a,k-j=1,b,k-j=n-1,c,k-j=-1,-c,k-j=-n+1,-b,0);
B:=Matrix(n,n,f):
alpha:=h->cos(Pi*(2*h-1)/n);
lambda:=(h)->evalf(a+(b-c)*alpha(h)+(b+c)*I*sign(n+1-2*h)*
sqrt(1-alpha(h)^2));
p:=(j,k)->evalf((1/n)*(sum(((a+(b-c)*alpha(h)+(b+c)*I*
sign(n+1-2*h)*sqrt(1-alpha(h)^2))^r)*(ChebyshevT(abs(j-k),
alpha(h))+sign(j-k)*I*sign(n+1-2*h)*sqrt(1-(alpha(h))^2)*
ChebyshevU(abs(j-k)-1,alpha(h))),h=1..floor((n+1)/2))+
sum((a+(b-c)*alpha(h)-(b+c)*I*sign(n+1-2*h)*sqrt(1-
alpha(h)^2))^r*(ChebyshevT(abs(j-k),alpha(h))-sign(j-k)*
```

```
I*sign(n+1-2*h)*sqrt(1-(alpha(h))^2)*ChebyshevU(abs(j-k)-1,
alpha(h))),h=1..floor(n/2))));
P:=Matrix(n,n,p);
print(B);
print(P);
end proc:
power(n,r,a,b,c);
```

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