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NEW CONCEPTS ON m-POLAR INTERVAL-VALUED INTUITIONISTIC FUZZY GRAPH

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Abstract. Theoretical concepts of graphs are highly utilized by computer science applications. Especially in research areas of computer science such as data mining, image segmentation, clustering, image capturing and networking. The concept of interval-valued intuitionistic fuzzy set was introduced by Atanassov [3]. Interval-valued intuitionistic fuzzy sets provide a more adequate description of uncertainly than the traditional fuzzy sets. It has many applications in fuzzy control and the most computationally intensive part of fuzzy control is defuzzification. In this paper the authors introduced the concepts of m-polar interval-valued intuitionistic fuzzy graph (IVIFG), edge regular m-polar IVIFG, totally edge regular m-polar IVIFG and highly irregular m-polar IVIFG.

Keywords: m-polar IVIFG, edge regular m-polar IVIFG, totally edge regular m-polar IVIFG, highly irregular m-polar IVIFG.

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1. Introduction

The fundamental characteristic of the IVIFS is that the values of its membership function and non-membership function are intervals rather than exact numbers. The concept of intuitionistic fuzzy sets (IFSs), as a generalization of fuzzy set was introduced by K. Atanassov [2] and defined new operations on intuitionistic fuzzy graphs. Later, K. Atanassov and G. Gargov [3] introduced the interval valued intuitionistic fuzzy sets (IVIFSs) theory, as a generalization of both interval valued fuzzy sets (IVFSs) and intuitionistic fuzzy sets (IFSs). Muhammad, Akram and Wieslaw A. Dudek [10] defined the interval-valued fuzzy graphs and a few operations on them. Mohamed Ismayil and Mohamed Ali [9] studied the strong IVIFGs.

Akram [1] introduced the notion of bipolar fuzzy graphs describing various methods of their construction as well as investigating some of their important properties. Bhutani [4] discussed automorphism of fuzzy graphs. Chen et al. [5] generalizad the concept of bipolar fuzzy set to obtain the notion of m-polar fuzzy set. The notion of m-polar fuzzy set is

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more advanced than fuzzy set and eliminates doubtfulness more obsolutely Ghorai and Pal [6, 7, 8] studied some operations and properties of m-polar fuzzy graphs. Rashmanlou et al. [12, 13, 14, 15, 16, 17, 18] discussed some properties of bipolar fuzzy graphs and interval-valued fuzzy graphs and some of its results are investigated. Ramprasad et al. [11] studied product m-polar fuzzy graph, product m-polar fuzzy intersection graph, and product m-polar fuzzy line graph.

In the paper, the authors introduce the concepts of m-polar IVIFG, edge regular m-polar IVIFG, totally edge regular m-polar IVIFG, and highly irregular m-polar IVIFG.

2. Preliminaries

Throughout this paper we assume $D[0,1]$ be the set of all closed sub-intervals of the interval $[0,1]$ and elements of this set are denoted by uppercase letters. If $\mu \in D[0,1]$ then this interval can be represented as $\mu = [\mu^L, \mu^U]$, where μ^l and μ^U are the lower and upper limits of μ when these sub-intervals are membership of the elements of any set A then the membership values are denoted by μ^A and by ν^A we mean the non-membership values.

Definition 2.1. *An interval-valued intuitionistic fuzzy graph (IVIFG) with underlying graph $G^* = (V, E)$ is defined to be a pair $G = (A, B)$, where*
(i) The function $\mu^A : V \to D[0,1]$ and $\nu^A : V \to D[0,1]$ denote the degree of membership and non-membership of the element respectively, such that $0 \le \mu^A + \nu^A \le 1$ for all $x \in V$.
(ii) The functions $\mu^B : E \subset V \times V \to D[0,1]$ and $\nu^B : E \subset V \times V \to D[0,1]$ are defined by

$$\mu^{BL}(xy) \le \min\{\mu^{AL}(x), \mu^{AL}(y)\}$$
$$\mu^{BU}(xy) \le \min\{\mu^{AU}(x), \mu^{AU}(y)\}$$
$$\nu^{BL}(xy) \ge \max\{\nu^{AL}(x), \nu^{AL}(y)\}$$
$$\nu^{BU}(xy) \ge \max\{\nu^{AU}(x), \nu^{AU}(y)\}$$

such that $0 \le \mu^{BU}(xy) + \nu^{BU}(xy) \le 1\ \forall xy \in E$.

Definition 2.2. *The interval-valued intuitionistic fuzzy graph is said to be strong if*

$$\mu^{BL}(xy) = \min\{\mu^{AL}(x), \mu^{AL}(y)\}$$
$$\mu^{BU}(xy) = \min\{\mu^{AU}(x), \mu^{AU}(y)\}$$
$$\nu^{BL}(xy) = \max\{\nu^{AL}(x), \nu^{AL}(y)\}$$
$$\nu^{BU}(xy) = \max\{\nu^{AU}(x), \nu^{AU}(y)\}$$

Definition 2.3. *Let $G = (A, B)$ be an IVIFG. Then the degree of a vertex x is defined by*

$$d_G(x) = \left\langle \left[\sum_{\substack{xy\in E\\ x\neq y}} \mu^{BL}(xy), \sum_{\substack{xy\in E\\ x\neq y}} \mu^{BU}(xy) \right], \left[\sum_{\substack{xy\in E\\ x\neq y}} \nu^{BL}(xy), \sum_{\substack{xy\in E\\ x\neq y}} \nu^{BU}(xy) \right] \right\rangle$$

Definition 2.4. *An interval-valued intuitionistic fuzzy graph G is said to be regular if the degree of each vertex of an IVIFG is constant. If the degree of each vertex is k, then we say the graph is k-regular IVIFG.*

Definition 2.5. *Let G be an IVIFG, then the order of G is defined to be*

$$O(G) = \left\langle \left[\sum_{x\in V} \mu^{AL}(x), \sum_{x\in V} \mu^{AU}(x) \right], \left[\sum_{x\in V} \nu^{AL}(x), \sum_{x\in V} \nu^{AU}(x) \right] \right\rangle$$

Definition 2.6. *Let G be an IVIFG, then the size of G is defined to be*

$$S(G) = \left\langle \left[\sum_{x \neq y} \mu^{BL}(xy), \sum_{x \neq y} \mu^{BU}(xy) \right], \left[\sum_{x \neq y} \nu^{BL}(xy), \sum_{x \neq y} \nu^{BU}(xy) \right] \right\rangle$$

Definition 2.7. *An m-polar interval-valued intuitionistic fuzzy set A on V is defined as*

$$A = \{ \langle [\mu_1^{AL}(x), \mu_1^{AU}(x)], \cdots, [\mu_m^{AL}(x), \mu_m^{AU}(x)], [\nu_1^{AL}(x), \nu_1^{AU}(x)], \cdots, [\nu_m^{AL}(x), \nu_m^{AU}(x)] \rangle \}$$

for all $x \in V$ and or shortly

$$A = \left\{ \left\langle [\mu_i^{AL}(x), \mu_i^{AU}(x)]_{i=1}^m, [\nu_i^{AL}(x), \nu_i^{AU}(x)]_{i=1}^m \right\rangle |x \in V, \ m \in \mathbb{N} \right\}$$

where the functions $\mu_i^A : V \to D[0,1]$ and $\nu_i^A : V \to D[0,1]$ denote the degree of m-polar memberships and m-polar non-memberships of the element respectively, such that

$$\begin{aligned} 0 &\leq \mu_i^{AL}(x) \leq \mu_i^{AU}(x) \leq 1 \\ 0 &\leq \nu_i^{AL}(x) \leq \nu_i^{AU}(x) \leq 1 \\ 0 &\leq \mu_i^{AL}(x) + \nu_i^{AU}(x) \leq 1, \ \forall x \in V \end{aligned}$$

Definition 2.8. *An m-polar interval-valued intuitionistic fuzzy graph with underlying graph $G^* = (V, E)$ is defined to be a pair $G = (V, A, B)$, where*
(i) A is an m-polar interval-valued intuitionistic fuzzy set on V

$$A = \left\{ \left\langle [\mu_i^{AL}(x), \mu_i^{AU}(x)]_{i=1}^m ; [\nu_i^{AL}(x), \nu_i^{AU}(x)]_{i=1}^m \right\rangle |x \in V, \ m \in \mathbb{N} \right\}$$

(ii) B is an m-polar interval-valued intuitionistic fuzzy relation on $V \times V$

$$B = \left\{ \left\langle [\mu_i^{BL}(xy), \mu_i^{BU}(xy)]_{i=1}^m ; [\nu_i^{BL}(xy), \nu_i^{BU}(xy)]_{i=1}^m \right\rangle |xy \in E, \ m \in \mathbb{N} \right\}$$

that the functions $\mu_i^B : E \subset V \times V \to D[0,1]$ and $\nu_i^B : E \subset V \times V \to D[0,1]$ are defined by

$$\begin{aligned} \mu_i^{BL}(xy) &\leq \min\{\mu_i^{AL}(x), \mu_i^{AL}(y)\} \\ \mu_i^{BU}(xy) &\leq \min\{\mu_i^{AU}(x), \mu_i^{AU}(y)\} \\ \nu_i^{BL}(xy) &\geq \max\{\nu_i^{AL}(x), \nu_i^{AL}(y)\} \\ \nu_i^{BU}(xy) &\geq \max\{\nu_i^{AU}(x), \nu_i^{AU}(y)\} \end{aligned}$$

such that $0 \leq \mu_i^{BU}(xy) + \nu_i^{BU}(xy) \leq 1$, $\forall xy \in E$ and $i = 1, 2, \cdots, m$.

Definition 2.9. *The m-polar interval-valued intuitionistic fuzzy graph is said to be strong if for $i = 1, 2, \cdots, m$*

$$\begin{aligned} \mu_i^{BL}(xy) &= \min\{\mu_i^{AL}(x), \mu_i^{AL}(y)\} \\ \mu_i^{BU}(xy) &= \min\{\mu_i^{AU}(x), \mu_i^{AU}(y)\} \\ \nu_i^{BL}(xy) &= \max\{\nu_i^{AL}(x), \nu_i^{AL}(y)\} \\ \nu_i^{BU}(xy) &= \max\{\nu_i^{AU}(x), \nu_i^{AU}(y)\} \end{aligned}$$

3. A New Theory of Regularity in m-polar IVIFGs

Definition 3.1. *Let $G = (V, A, B)$ be an m-polar IVIFG. Then the degree of a vertex x is defined as*

$$d_G(x) = \left\langle \left[d\mu_i^L(x), d\mu_i^U(x)\right]_{i=1}^m ; \left[d\nu_i^L(x), d\nu_i^U(x)\right]_{i=1}^m \right\rangle,$$

where for $i = 1, 2, \cdots, m$.

$$d\mu_i^L(x) = \sum_{\substack{xy \in E \\ x \neq y}} \mu_i^{BL}(xy), \; d\mu_i^U(x) = \sum_{\substack{xy \in E \\ x \neq y}} \mu_i^{BU}(xy)$$

$$d\nu_i^L(x) = \sum_{\substack{xy \in E \\ x \neq y}} \nu_i^{BL}(xy), \; d\nu_i^U(x) = \sum_{\substack{xy \in E \\ x \neq y}} \nu_i^{BU}(xy)$$

Definition 3.2. *The degree of an edge $xy \in E$ in an m-polar IVIFG $G = (V, A, B)$ is defined as*

$$d_G(xy) = \left\langle \left[d\mu_i^L(xy), d\mu_i^U(xy)\right]_{i=1}^m ; \left[d\nu_i^L(xy), d\nu_i^U(xy)\right]_{i=1}^m \right\rangle,$$

where for $i = 1, 2, \cdots, m$

$$\begin{aligned} d\mu_i^L(xy) &= d\mu_i^L(x) + d\mu_i^L(y) - 2d\mu_i^{BL}(xy) \\ d\mu_i^U(xy) &= d\mu_i^U(x) + d\mu_i^U(y) - 2d\mu_i^{BU}(xy) \\ d\nu_i^L(xy) &= d\nu_i^L(x) + d\nu_i^L(y) - 2d\nu_i^{BL}(xy) \\ d\nu_i^U(xy) &= d\nu_i^U(x) + d\nu_i^U(y) - 2d\nu_i^{BU}(xy) \end{aligned}$$

Definition 3.3. *The total degree of an edge $xy \in E$ in an m-polar IVIFG $G = (V, A, B)$ is defined as*

$$td_G(xy) = \left\langle \left[td\mu_i^L(xy), td\mu_i^U(xy)\right]_{i=1}^m ; \left[td\nu_i^L(xy), td\nu_i^U(xy)\right]_{i=1}^m \right\rangle,$$

where for $i = 1, 2, \cdots, m$

$$\begin{aligned} td\mu_i^L(xy) &= td\mu_i^L(x) + td\mu_i^L(y) - td\mu_i^{BL}(xy) \\ td\mu_i^U(xy) &= td\mu_i^U(x) + td\mu_i^U(y) - td\mu_i^{BU}(xy) \\ td\nu_i^L(xy) &= td\nu_i^L(x) + td\nu_i^L(y) - td\nu_i^{BL}(xy) \\ td\nu_i^U(xy) &= td\nu_i^U(x) + td\nu_i^U(y) - td\nu_i^{BU}(xy) \end{aligned}$$

This is equivalent to

$$td_G(xy) = \Big\langle \left[d\mu_i^L(xy) + \mu_i^{BL}(xy), d\mu_i^U(xy) + \mu_i^{BU}(xy)\right]_{i=1}^m ; [d\nu_i^L(xy) + \nu_i^{BL}(xy),$$
$$d\nu_i^U(xy) + \nu_i^{BU}(xy)]_{i=1}^m \Big\rangle,$$

Definition 3.4. *The degree of an edge $xy \in E$ in a crisp graph G^* is $d_{G^*}(xy) = d_{G^*}(x) + d_{G^*}(y) - 2$.*

Example 3.1. *Consider an m-polar IVIFG $G = (V, A, B)$ of $G^* = (V, E)$ as in Figure 1 Then, we have*

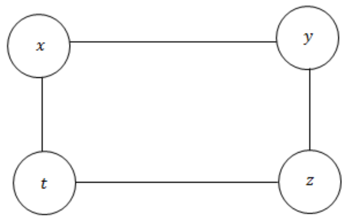

FIGURE 1. The 2-polar IVIFG

TABLE 1. The degree of membership and non-membership of the vertices and edges in 2-polar IVIFG G

A	x	$\langle[0.2, 0.5], [0.6, 0.7]; [0.3, 0.4], [0.1, 0.2]\rangle$
	y	$\langle[0.1, 0.4], [0.2, 0.4]; [0.3, 0.5], [0.4, 0.5]\rangle$
	z	$\langle[0.2, 0.5], [0.5, 0.7]; [0.1, 0.4], [0.1, 0.2]\rangle$
	t	$\langle[0.3, 0.4], [0.2, 0.3]; [0.5, 0.6], [0.1, 0.5]\rangle$
B	xy	$\langle[0.1, 0.3], [0.1, 0.4]; [0.4, 0.6], [0.5, 0.6]\rangle$
	yz	$\langle[0.2, 0.3], [0.1, 0.2]; [0.4, 0.6], [0.5, 0.7]\rangle$
	zt	$\langle[0.1, 0.2], [0.2, 0.3]; [0.6, 0.7], [0.3, 0.6]\rangle$
	xt	$\langle[0.1, 0.2], [0.2, 0.3]; [0.6, 0.7], [0.2, 0.6]\rangle$

TABLE 2. The degree of vertices in 2-polar IVIFG G

$d_G(x)$	$\langle[0.2, 0.5], [0.3, 0.7]; [1, 1.3], [0.7, 1.2]\rangle$
$d_G(y)$	$\langle[0.3, 0.6], [0.2, 0.6]; [0.8, 1.2], [1, 1.3]\rangle$
$d_G(z)$	$\langle[0.3, 0.5], [0.3, 0.5]; [1, 1.3], [0.8, 1.3]\rangle$
$d_G(t)$	$\langle[0.2, 0.4], [0.4, 0.6]; [1.2, 1.4], [0.5, 1.2]\rangle$

TABLE 3. The degree of edges in 2-polar IVIFG G

$d_G(xy)$	$\langle[0.3, 0.8], [0.3, 0.5]; [1, 1.9], [0.7, 1.3]\rangle$
$d_G(yz)$	$\langle[0.2, 0.8], [0.3, 0.5]; [1, 1.3], [0.8, 1.2]\rangle$
$d_G(zt)$	$\langle[0.3, 0.5], [0.3, 0.5]; [1, 1.3], [0.7, 1.3]\rangle$
$d_G(xt)$	$\langle[0.2, 0.5], [0.3, 0.7]; [1, 1.3], [0.8, 1.2]\rangle$

TABLE 4. The total degree of edges in 2-polar

$td_G(xy)$	$\langle[0.4, 0.8], [0.4, 0.9]; [1.4, 1.9], [1.2, 1.9]\rangle$
$td_G(yz)$	$\langle[0.4, 0.8], [0.4, 0.9]; [1.4, 1.9], [1.3, 1.9]\rangle$
$td_G(zt)$	$\langle[0.4, 0.7], [0.5, 0.8]; [1.6, 2], [1, 1.9]\rangle$
$td_G(xt)$	$\langle[0.3, 0.7], [0.5, 1]; [1.6, 2], [1, 1.8]\rangle$

Definition 3.5. *If every vertex in an m-polar IVIFG $G = (V, A, B)$ has the same degree $\langle[a_i, b_i]_{i=1}^m; [c_i, d_i]_{i=1}^m\rangle$, then $G = (V, A, B)$ is called regular m-polar IVIFG or m-polar IVIFG of degree $\langle[a_i, b_i]_{i=1}^m; [c_i, d_i]_{i=1}^m\rangle$.*

Definition 3.6. *If every edge in an m-polar IVIFG* $G = (V, A, B)$ *has the same degree* $\langle [a_i, b_i]_{i=1}^m; [c_i, d_i]_{i=1}^m \rangle$, *then* $G = (V, A, B)$ *is called an edge regular m-polar IVIFG.*

Definition 3.7. *If every edge in an m-polar IVIFG* $G = (V, A, B)$ *has the same total degree* $\langle [a_i, b_i]_{i=1}^m; [c_i, d_i]_{i=1}^m \rangle$, *then* $G = (V, A, B)$ *is called totally edge regular m-polar IVIFG.*

Definition 3.8. *Consider an m-polar fuzzy graph* $G = (V, A, B)$ *of* $G^* = (V, E)$, *we have*

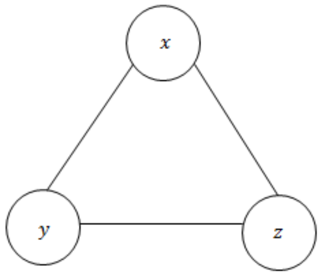

FIGURE 2. An edge regular m-polar IVIFG G

TABLE 5. The degree of membership and non-membership of the vertices and edges in edge regular m-polar IVIFG G

A	x	$\langle [0.2, 0.3], [0.3, 0.4], [0.4, 0.5], [0.2, 0.3] \rangle$
	y	$\langle [0.1, 0.2], [0.2, 0.3], [0.4, 0.6], [0.2, 0.5] \rangle$
	z	$\langle [0.1, 0.2], [0.2, 0.3], [0.4, 0.6], [0.3, 0.5] \rangle$
B	xy	$\langle [0.1, 0.2], [0.2, 0.3], [0.4, 0.6], [0.2, 0.5] \rangle$
	xz	$\langle [0.1, 0.2], [0.2, 0.3], [0.4, 0.6], [0.2, 0.5] \rangle$
	yz	$\langle [0.1, 0.2], [0.2, 0.3], [0.4, 0.6], [0.2, 0.5] \rangle$

Then, we have $d_G(xy) = d_G(xz) = d_G(yz) = \langle [0.2, 0.4], [0.4, 0.6];]0.8, 1.2], [0.4, 1] \rangle$.

Theorem 3.1. *Let* $G = (V, A, B)$ *be an m-polar IVIFG on a cycle* $G^* = (V, E)$. *Then*

$$\sum_{x_j \in V} d_G(x_j) = \sum_{j=1}^{n} d_G(x_j x_{j+1})$$

Proof. Suppose that $G = (V, A, B)$ is an m-polar IVIFG and G^* is a cycle $x_1 x_2 x_3 \cdots x_n x_1$. Now, we get for $i = 1, 2, \cdots, m$

$$\sum_{j=1}^{n} d_G(x_j x_{j+1}) = d_G(x_1 x_2) + d_G(x_2 x_3) + \cdots + d_G(x_n x_1), \text{ where } x_{n+1} = x_1$$

$$\sum_{j=1}^{n} d\mu_i^L(x_jx_{j+1}) = d\mu_i^L(x_1x_2) + d\mu_i^L(x_2x_3) + \cdots + d\mu_i^L(x_nx_1)$$
$$= d\mu_i^L(x_1) + d\mu_i^L(x_2) - 2\mu_i^{BL}(x_1x_2) + d\mu_i^L(x_2) + d\mu_i^L(x_3) - 2\mu_i^{BL}(x_2x_3)$$
$$+ \cdots + d\mu_i^L(x_n) + d\mu_i^L(x_1) - 2\mu_i^{BL}(x_nx_1)$$
$$= \sum_{j=1}^{n} d\mu_i^L(x_j) - 2\sum_{j=1}^{n} \mu_i^{BL}(x_jx_{j+1})$$
$$= \sum_{x_j \in V} d\mu_i^L(x_j) + \sum_{x_j \in V} d\mu_i^L(x_j) - 2\sum_{j=1}^{n} \mu_i^{BL}(x_jx_{j+1})$$
$$= \sum_{x_j \in V} d\mu_i^L(x_j) + 2\sum_{j=1}^{n} \mu_i^{BL}(x_jx_{j+1}) - 2\sum_{j=1}^{n} \mu_i^{BL}(x_jx_{j+1})$$
$$= \sum_{x_j \in V} d\mu_i^L(x_j)$$

Similarly, in other bounds. Thus

$$\sum_{j=1}^{n} d_G(x_jx_{j+1}) = \left\langle \left[\sum_{x_j \in V} d\mu_i^L(x_j), \sum_{x_j \in V} d\mu_i^U(x_j) \right]_{i=1}^{m}; \left[\sum_{x_j \in V} d\nu_i^L(x_j), \sum_{x_j \in V} d\nu_i^U(x_j) \right]_{i=1}^{m} \right\rangle$$
$$= \sum_{x_j \in V} d_G(x_j)$$

□

Remark 3.1. *Let $G = (V, A, B)$ be an m-polar IVIFG on a crisp graph G^*. Then*

$$\sum_{xy \in E} d_G(xy) = \left\langle \left[\sum_{xy \in E} d_{G^*}(xy)\mu_i^{BL}(xy), \sum_{xy \in E} d_{G^*}(xy)\mu_i^{BU}(xy) \right]_{i=1}^{m}; \right.$$
$$\left. \left[\sum_{xy \in E} d_{G^*}(xy)\nu_i^{BL}(xy), \sum_{xy \in E} d_{G^*}(xy)\nu_i^{BU}(xy) \right]_{i=1}^{m} \right\rangle$$

where $d_{G^*}(xy) = d_{G^*}(x) + d_{G^*}(y) - 2$, for all $xy \in E$.

Definition 3.9. *In any m-polar IVIFG*

$$\sum_{x \in V} d_G(x) = 2\left\langle \left[\sum_{x \neq y} \mu_i^{BL}(xy), \sum_{x \neq y} \mu_i^{BU}(xy) \right]_{i=1}^{m}; \left[\sum_{x \neq y} \nu_i^{BL}(xy), \sum_{x \neq y} \nu_i^{BU}(xy) \right]_{i=1}^{m} \right\rangle$$

So, $\sum_{x \in V} d_G(x) = 2S(G)$.

Theorem 3.2. *Let $G = (V, A, B)$ be an m-polar IVIFG on a c-regular crisp graph G^*. Then*

$$\sum_{xy \in E} d_G(xy) = (c-1)\sum_{x \in V} d_G(x)$$

Proof. From Remark 3.1, we have

$$\sum_{xy\in E} d_G(xy) = \Bigg\langle \Bigg[\sum_{xy\in E} d_{G^*}(xy)\mu_i^{BL}(xy), \sum_{xy\in E} d_{G^*}(xy)\mu_i^{BU}(xy)\Bigg]_{i=1}^{m};$$
$$\Bigg[\sum_{xy\in E} d_{G^*}(xy)\nu_i^{BL}(xy), \sum_{xy\in E} d_{G^*}(xy)\nu_i^{BU}(xy)\Bigg]_{i=1}^{m}\Bigg\rangle$$
$$= \Bigg\langle \Bigg[\sum_{xy\in E} (d_{G^*}(x)+d_{G^*}(y)-2)\mu_i^{BL}(xy), \sum_{xy\in E} (d_{G^*}(x)+d_{G^*}(y)-2)\mu_i^{BU}(xy)\Bigg]_{i=1}^{m};$$
$$\Bigg[\sum_{xy\in E} (d_{G^*}(x)+d_{G^*}(y)-2)\nu_i^{BL}(xy), \sum_{xy\in E} (d_{G^*}(x)+d_{G^*}(y)-2)\nu_i^{BU}(xy)\Bigg]_{i=1}^{m}\Bigg\rangle$$

Since G^* is a regular crisp, we have the degree of every vertex in G^* as c.

$$\sum_{xy\in E} d_G(xy) = \Bigg\langle \Bigg[(c+c-2)\sum_{xy\in E})\mu_i^{BL}(xy), (c+c-2)\sum_{xy\in E} \mu_i^{BU}(xy)\Bigg]_{i=1}^{m};$$
$$\Bigg[(c+c-2)\sum_{xy\in E} \nu_i^{BL}(xy), (c+c-2)\sum_{xy\in E} \nu_i^{BU}(xy)\Bigg]_{i=1}^{m}\Bigg\rangle$$
$$= 2(c-1)\Bigg\langle \Bigg[\sum_{xy\in E} \mu_i^{BL}(xy), \sum_{xy\in E} \mu_i^{BU}(xy)\Bigg]_{i=1}^{m};$$
$$\Bigg[\sum_{xy\in E} \nu_i^{BL}(xy), \sum_{xy\in E} \nu_i^{BU}(xy)\Bigg]_{i=1}^{m}\Bigg\rangle$$
$$= (c-1)\sum_{x\in V} d_G(x)$$

□

Theorem 3.3. *Let $G=(V,A,B)$ be an m-polar IVIFG. Then the function $\Big\langle \Big[\mu_i^{BL}, \mu_i^{BU}\Big]_{i=1}^{m}; \Big[\nu_i^{BL}, \nu_i^{BU}\Big]_{i=1}^{m}\Big\rangle$ is a constant function if and only if the following conditions are equivalent.*
(i) G is an edge regular m-polar IVIFG.
(ii) G is a totally edge regular m-polar IVIFG.

Proof. Suppose that $\Big\langle \Big[\mu_i^{BL}, \mu_i^{BU}\Big]_{i=1}^{m}; \Big[\nu_i^{BL}, \nu_i^{BU}\Big]_{i=1}^{m}\Big\rangle$ is a constant function. Then

$$\Big\langle \Big[\mu_i^{BL}, \mu_i^{BU}\Big]_{i=1}^{m}; \Big[\nu_i^{BL}, \nu_i^{BU}\Big]_{i=1}^{m}\Big\rangle = \Big\langle \Big[a_i, b_i\Big]_{i=1}^{m}; \Big[c_i, d_i\Big]_{i=1}^{m}\Big\rangle, \ \forall xy \in E$$

where a_i, b_i, c_i, d_i are constant and $a_i, b_i, c_i, d_i \in [0,1]$, for all $i = 1, 2, \cdots, m$. Let G be an edge regular m-polar IVIFG. Then, for all $xy \in E$, $d_G(xy) = \Big\langle \Big[e_i, f_i\Big]_{i=1}^{m}; \Big[g_i, h_i\Big]_{i=1}^{m}\Big\rangle$

Now we have to show that G is a totally edge regular m-polar IVIFG. Now for all $i = 1, 2, \cdots, m$

$$td_G(xy) = \left\langle \left[d\mu_i^L(xy) + \mu_i^{BL}(xy), d\mu_i^U(xy) + \mu_i^{BU}(xy)\right]_{i=1}^m; \left[d\nu_i^L(xy) + \nu_i^{BL}(xy), d\nu_i^U(xy) + \nu_i^{BU}(xy)\right]_{i=1}^m \right\rangle$$

$$td_G(xy) = \left\langle \left[e_i + a_i, f_i + b_i\right]_{i=1}^m; \left[g_i + c_i, h_i + d_i\right]_{i=1}^m \right\rangle, \ \forall xy \in E$$

Thus G is a totally edge regular m-polar IVIFG.

Now let G be a $\left\langle \left[k_i, l_i\right]_{i=1}^m; \left[p_i, q_i\right]_{i=1}^m \right\rangle$-totally edge regular m-polar IVIFG. Then

$$td_G(xy) = \left\langle \left[d\mu_i^L(xy) + a_i, d\mu_i^U(xy) + b_i\right]_{i=1}^m; \left[d\nu_i^L(xy) + c_i, d\nu_i^U(xy) + d_i\right]_{i=1}^m \right\rangle = \left\langle \left[k_i, l_i\right]_{i=1}^m; \left[p_i, q_i\right]_{i=1}^m \right\rangle$$

So, we have

$$\begin{aligned} d\mu_i^L(xy) + a_i = k_i &\implies d\mu_i^L(xy) = k_i - a_i \\ d\mu_i^U(xy) + b_i = l_i &\implies d\mu_i^L(xy) = l_i - b_i \\ d\nu_i^L(xy) + c_i = p_i &\implies d\nu_i^L(xy) = p_i - c_i \\ d\nu_i^U(xy) + d_i = q_i &\implies d\nu_i^U(xy) = q_i - d_i \end{aligned}$$

Hence

$$d_G(xy) = \left\langle \left[k_i - a_i, l_i - b_i\right]_{i=1}^m; \left[p_i - c_i, q_i - d_i\right]_{i=1}^m \right\rangle$$

Then G is an $\left\langle \left[k_i - a_i, l_i - b_i\right]_{i=1}^m; \left[p_i - c_i, q_i - d_i\right]_{i=1}^m \right\rangle$-edge regular m-polar IVIFG.

Conversely, suppose that G is an edge regular m-polar IVIFG and G is a totally edge regular m-polar IVIFG which are equivalent. We have to prove that $\left\langle \left[\mu_i^{BL}, \mu_i^{BU}\right]_{i=1}^m; \left[\nu_i^{BL}, \nu_i^{BU}\right]_{i=1}^m \right\rangle$ is a constant function. In the contrary way, we suppose that $\left\langle \left[\mu_i^{BL}, \mu_i^{BU}\right]_{i=1}^m; \left[\nu_i^{BL}, \nu_i^{BU}\right]_{i=1}^m \right\rangle$ is not a constant function. Then,

$$\left\langle \left[\mu_i^{BL}(x_jx_k), \mu_i^{BU}(x_jx_k)\right]_{i=1}^m; \left[\nu_i^{BL}(x_jx_k), \nu_i^{BU}(x_jx_k)\right]_{i=1}^m \right\rangle \neq \left\langle \left[\mu_i^{BL}(x_rx_s), \mu_i^{BU}(x_rx_s)\right]_{i=1}^m; \left[\nu_i^{BL}(x_rx_s), \nu_i^{BU}(x_rx_s)\right]_{i=1}^m \right\rangle$$

for at least one pair of edges $x_jx_k, x_rx_s \in E$. Let G be an $\left\langle \left[e_i, f_i\right]_{i=1}^m; \left[g_i, h_i\right]_{i=1}^m \right\rangle$-edge regular m-polar IVIFG. Then $d_G(x_jx_k) = d_G(x_rx_s) = \left\langle \left[e_i, f_i\right]_{i=1}^m; \left[g_i, h_i\right]_{i=1}^m \right\rangle$. Hence, for every $x_jx_k \in E$ and for every $x_rx_s \in E$,

$$td_G(x_jx_k) = \left\langle \left[d\mu_i^L(x_jx_k) + \mu_i^{BL}(x_jx_k), d\mu_i^U(x_jx_k) + \mu_i^{BU}(x_jx_k)\right]; \left[d\nu_i^L(x_jx_k) + \nu_i^{BL}(x_jx_k), d\nu_i^U(x_jx_k) + \nu_i^{BU}(x_jx_k)\right]\right\rangle$$

$$= \left\langle \left[e_i + \mu_i^{BL}(x_jx_k), f_i + \mu_i^{BU}(x_jx_k)\right]; \left[g_i + \nu_i^{BL}(x_jx_k), h_i + \nu_i^{BU}(x_jx_k)\right]\right\rangle$$

$$td_G(x_rx_s) = \left\langle \left[d\mu_i^L(x_rx_s) + \mu_i^{BL}(x_rx_s), d\mu_i^U(x_rx_s) + \mu_i^{BU}(x_rx_s)\right]; \left[d\nu_i^L(x_rx_s) + \nu_i^{BL}(x_rx_s), d\nu_i^U(x_rx_s) + \nu_i^{BU}(x_rx_s)\right]\right\rangle$$

$$= \left\langle \left[e_i + \mu_i^{BL}(x_rx_s), f_i + \mu_i^{BU}(x_rx_s)\right]; \left[g_i + \nu_i^{BL}(x_rx_s), h_i + \nu_i^{BU}(x_rx_s)\right]\right\rangle$$

Since

$$\left\langle \left[\mu_i^{BL}(x_jx_k), \mu_i^{BU}(x_jx_k)\right]_{i=1}^m; \left[\nu_i^{BL}(x_jx_k), \nu_i^{BU}(x_jx_k)\right]_{i=1}^m\right\rangle \neq \left\langle \left[\mu_i^{BL}(x_rx_s), \mu_i^{BU}(x_rx_s)\right]_{i=1}^m; \left[\nu_i^{BL}(x_rx_s), \nu_i^{BU}(x_rx_s)\right]_{i=1}^m\right\rangle$$

we have $td_G(x_jx_k) \neq td_G(x_rx_s)$. Hence, G is not a totally edge regular m-polar IVIFG. This is a contradiction to our assumption. Hence, $\left\langle \left[\mu_i^{BL}, \mu_i^{BU}\right]_{i=1}^m; \left[\nu_i^{BL}, \nu_i^{BU}\right]_{i=1}^m\right\rangle$ is a constant function. In the same way, we can prove that $\left\langle \left[\mu_i^{BL}, \mu_i^{BU}\right]_{i=1}^m; \left[\nu_i^{BL}, \nu_i^{BU}\right]_{i=1}^m\right\rangle$ is a constant function, when G is a totally edge regular m-polar IVIFG. □

Theorem 3.4. *Let G^* be a k-regular crisp graph and $G = (V, A, B)$ be an m-polar IVIFG on G^*. Then $\left\langle \left[\mu_i^{BL}, \mu_i^{BU}\right]_{i=1}^m; \left[\nu_i^{BL}, \nu_i^{BU}\right]_{i=1}^m\right\rangle$ is a constant function if and only if $G$is both regular m-polar IVIFG and totally edge regular m-polar IVIFG.*

Proof. Let $G = (V, A, B)$ be an m-polar IVIFG on G^* and let G^* be a k-regular crisp graph. Assume that $\left\langle \left[\mu_i^{BL}, \mu_i^{BU}\right]_{i=1}^m; \left[\nu_i^{BL}, \nu_i^{BU}\right]_{i=1}^m\right\rangle$ is a constant function. Then

$$\left\langle \left[\mu_i^{BL}(xy), \mu_i^{BU}(xy)\right]_{i=1}^m; \left[\nu_i^{BL}(xy), \nu_i^{BU}(xy)\right]_{i=1}^m\right\rangle = \left\langle \left[a_i, b_i\right]_{i=1}^m; \left[c_i, d_i\right]_{i=1}^m\right\rangle, \ \forall xy \in E$$

where a_i, b_i, c_i, d_i are constants and $a_i, b_i, c_i, d_i \in [0, 1]$ for $i = 1, 2, \cdots, m$. From the definition of degree of a vertex, we get

$$d_G(x) = \left\langle \left[\sum_{\substack{xy\in E\\x\neq y}} \mu_i^{BL}(xy), \sum_{\substack{xy\in E\\x\neq y}} \mu_i^{BU}(xy)\right]_{i=1}^m; \left[\sum_{\substack{xy\in E\\x\neq y}} \nu_i^{BL}(xy), \sum_{\substack{xy\in E\\x\neq y}} \nu_i^{BU}(xy)\right]_{i=1}^m\right\rangle$$

$$= \left\langle \left[\sum_{\substack{xy\in E\\x\neq y}} a_i, \sum_{\substack{xy\in E\\x\neq y}} b_i\right]_{i=1}^m; \left[\sum_{\substack{xy\in E\\x\neq y}} c_i, \sum_{\substack{xy\in E\\x\neq y}} d_i\right]_{i=1}^m\right\rangle$$

$$= \left\langle \left[ka_i, kb_i\right]_{i=1}^m; \left[kc_i, kd_i\right]_{i=1}^m\right\rangle, \ for\ every\ x \in V.$$

So, G is regular m-polar IVIFG. Now

$$td_G(xy) = \left\langle \left[td\mu_i^L(xy), td\mu_i^U(xy)\right]_{i=1}^m ; \left[td\nu_i^L(xy), td\nu_i^U(xy)\right]_{i=1}^m \right\rangle$$

where, for $i = 1, 2, \cdots, m$

$$td\mu_i^L(xy) = d\mu_i^L(x) + d\mu_i^L(y) - \mu_i^{BL}(xy) = \sum_{\substack{xy \in E \\ x \neq y}} \mu_i^{BL}(xy) + \sum_{\substack{yz \in E \\ y \neq z}} \mu_i^{BL}(yz) - \mu_i^{BL}(xy)$$

$$= ka_i + ka_i - a_i = (2k-1)a_i$$

Similarly,

$$td\mu_i^U(xy) = (2k-1)b_i$$
$$td\nu_i^L(xy) = (2k-1)c_i$$
$$td\nu_i^U(xy) = (2k-1)d_i$$

So, $td_G(xy) = \left\langle \left[(2k-1)a_i, (2k-1)b_i\right]_{i=1}^m ; \left[(2k-1)c_i, (2k-1)d_i\right]_{i=1}^m \right\rangle$, $\forall xy \in E$. Hence, G is also a totally edge regular m-polar IVIFG.

Conversely, assume that G is both regular and totally edge regular m-polar IVIFG. Now we have to prove that $\left\langle \left[\mu_i^{BL}, \mu_i^{BU}\right]_{i=1}^m ; \left[\nu_i^{BL}, \nu_i^{BU}\right]_{i=1}^m \right\rangle$ is a constant function. Since G is regular $d_G(x) = \left\langle \left[a_i, b_i\right]_{i=1}^m ; \left[c_i, d_i\right]_{i=1}^m \right\rangle$ for all $x \in V$. Also G is totally edge regular. Hence, $td_G(x, y) = \left\langle \left[p_i, q_i\right]_{i=1}^m ; \left[r_i, s_i\right]_{i=1}^m \right\rangle$ for all $xy \in E$. From the definition of totall edge degree, we get for $i = 1, 2, \cdots, m$ and $\forall xy \in E$

$$td\mu_i^L(xy) = d\mu_i^L(x) + d\mu_i^L(y) - \mu_i^{BL}(xy) \Longrightarrow p_i = a_i + a_i - \mu_i^{BL}(xy) \Longrightarrow \mu_i^{BL}(xy) = 2a_i - p_i$$
$$td\mu_i^U(xy) = d\mu_i^U(x) + d\mu_i^U(y) - \mu_i^{BU}(xy) \Longrightarrow q_i = b_i + b_i - \mu_i^{BU}(xy) \Longrightarrow \mu_i^{BU}(xy) = 2b_i - q_i$$
$$td\nu_i^L(xy) = d\nu_i^L(x) + d\nu_i^L(y) - \nu_i^{BL}(xy) \Longrightarrow r_i = c_i + c_i - \nu_i^{BL}(xy) \Longrightarrow \nu_i^{BL}(xy) = 2c_i - r_i$$
$$td\nu_i^U(xy) = d\nu_i^U(x) + d\nu_i^U(y) - \nu_i^{BU}(xy) \Longrightarrow s_i = d_i + d_i - \nu_i^{BU}(xy) \Longrightarrow \nu_i^{BU}(xy) = 2d_i - s_i$$

So, for all $xy \in E$

$$\left\langle \left[\mu_i^{BL}(xy), \mu_i^{BU}(xy)\right]_{i=1}^m ; \left[\nu_i^{BL}(xy), \nu_i^{BU}(xy)\right]_{i=1}^m \right\rangle = \left\langle \left[2a_i - p_i, 2b_i - q_i\right]_{i=1}^m ; \left[2c_i - r_i, 2d_i - s_i\right]_{i=1}^m \right\rangle$$

Hence $\left\langle \left[\mu_i^{BL}, \mu_i^{BU}\right]_{i=1}^m ; \left[\nu_i^{BL}, \nu_i^{BU}\right]_{i=1}^m \right\rangle$ is a constant function. □

4. Conclusion

Any dissimilar fuzzy graph hypothesis needs large data for training to be able to help in decision-making which is crucial to utilitarian research in science and technology. A regular m-polar IVIFG has numerous application in the modeling of real life systems where the level of information inherited in the system varies with respect to time and have a different level of precision and hesitation. The concept of m-polar IVIFGs, regular m-polar IVIFGs, highly irregular m-polar IVIFGs is discussed in this paper. In our future work we will study on f-morphism in m-polar IVIFGs and investigate some of its results.

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