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ABSOLUTE INDEX NÖRLUND SUMMABILITY OF IMPROPER INTEGRALS

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Abstract. Özgen (Int. J. Anal. Appl., 11 (2016), 19-22) introduced $|C,1|_k$ $(k \geq 1)$ integrability of improper integrals. In this paper, we have introduced the notion of $|N,p_n,\delta|_k$-summability of improper integrals and established a new result which generalizes some existing results under suitable conditions.

Keywords: Cesàro summability, Nörlund summability, improper integral, infinite series.

AMS Subject Classification: 40G05.

1. Introduction

Let $\sum a_n$ be a given infinite series with sequence of partial sums (s_n). Let

$$\sigma_n = \frac{1}{n}\sum_{k=1}^{n} s_k. \tag{1}$$

The series $\sum a_n$ is said to be $(C,1)$ summable, if

$$\lim_{n\to\infty} \sigma_n = s. \tag{2}$$

Moreover, $\sum a_n$ is said to be $|C,1|_k$ $(k \geq 1)$ summable, if

$$\sum_{n=1}^{\infty} n^{k-1}|\sigma_n - \sigma_{n-1}|^k < \infty. \tag{3}$$

Let f be a real valued continuous function defined in the interval $[0,\infty)$ and $s(x) = \int_0^x f(t)dt$. We recall that, the Cesàro mean of $s(x)$ denoted by $\tau(x)$ is given by

$$\tau(x) = \frac{1}{x}\int_0^x s(t)dt. \tag{4}$$

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Now substituting $s(t)$ and simplifying, we have

$$\tau(x) = \frac{1}{x}\int_0^x (x-t)f(t)dt. \tag{5}$$

The integral $\int_0^\infty f(t)dt$ is said to be $|C,1|$-summable, if

$$\int_0^\infty |\tau'(x)|dx < \infty \tag{6}$$

and also $|C,1|_k$ $(k \geq 1)$ summable, if

$$\int_0^\infty x^{k-1}|\tau'(x)|^k dx < \infty. \tag{7}$$

Let $p(x)$ be a real valued continuous function defined in the interval $[0,\infty)$ and $P(x) = \int_0^x p(t)dt$. We recall the Nörlund (N,p_n) mean of $s(x)$ as a function $t(x)$ of the form

$$t(x) = \frac{1}{P(x)}\int_0^x p(t)s(t)dt. \tag{8}$$

The integral $\int_0^\infty f(t)dt$ is said to be $|N,p_n|$-summable, if

$$\int_0^\infty |t'(x)|dx < \infty \tag{9}$$

and also $|N,p_n|_k$ $(k \geq 1)$ summable, if

$$\int_0^\infty \left(\frac{P(x)}{p(x)}\right)^{k-1} |t'(x)|^k dx < \infty. \tag{10}$$

Similarly, it is $|N,p_n,\delta|_k$ $(\delta \geq 0; \delta k \leq 1)$ summable, if

$$\int_0^\infty \left(\frac{P(x)}{p(x)}\right)^{\delta k + k - 1} |t'(x)|^k dx < \infty. \tag{11}$$

Now, we have

$$s(x) - t(x) = \frac{1}{P(x)}\int_0^x P(t)f(t)dt.$$

Setting

$$s(x) - t(x) = \nu(x), \tag{12}$$

(10) can be written as

$$\int_0^\infty \frac{p(x)}{P(x)}\left(\frac{P'(x)}{p(x)}\right)^k |\nu(x)|^k dx < \infty. \tag{13}$$

Also, (11) can be written as

$$\int_0^\infty \left(\frac{P(x)}{p(x)}\right)^{\delta k - 1}\left(\frac{P'(x)}{p(x)}\right)^k |\nu(x)|^k dx < \infty. \tag{14}$$

In the year 2016, Özgen [5] introduced $|C,1|_k$ $(k \geq 1)$ integrability of improper integrals. Again, he established a result on equivalence of two integrability methods (see [6]) via Riesz means. Subsequently, Sonker and Munjal [7] demonstrated absolute summability factor of improper integrals via $|N,p_n|_k$ mean.

2. Known Results

Concerning absolute Cesàro summability factors of integrals via $|C,1|_k$ mean, Özgen [5] obtained the following result.

Theorem 2.1. *Let $\gamma(x)$ be a positive monotonic non-decreasing function such that*

$$\lambda(x)\gamma(x) = O(1), \text{ as } x \to \infty,$$

$$\int_0^x u|\lambda''(u)|\ \gamma(u)du = O(1),$$

and

$$\int_0^x \frac{|\nu(u)|^k}{u}\ du = O(\gamma(x)),\ \text{ as } x \to \infty.$$

Then the integral $\int_0^\infty f(t)dt$ is summable $|C,1|_k,\ k \geq 1$.

Subsequently, dealing with Nörlund summability of improper integrals, Sonker and Munjal [7] established the following result.

Theorem 2.2. *Let $p(0) > 0,\ p(x) \geq 0$ and $p(x)$ be a non-increasing function. Let $\chi(x)$ be a positive non-decreasing function and there be two functions $\beta(x)$ and $\varepsilon(x)$ such that*

$$|\varepsilon'(x)| \leq \beta(x),$$

$$\beta(x) \to 0,\ \text{ as } x \to \infty,$$

$$|\varepsilon(x)|\ \chi(x) = O(1),$$

$$\frac{1}{(p(x))^{k-1}} \int_0^x P(u)\ |\beta'(u)|\ \chi(u)du = O(1),\ \text{ as } x \to \infty,$$

$$\frac{1}{(p(x))^{k-1}} \int_0^x P'(u)\ |\beta(u)|\ \chi(u)du = O(1),\ \text{ as } x \to \infty,$$

and

$$\int_0^x \frac{p(t)|P'(t)|^k}{P(t)(p(t))^k}|\nu(t)|^k = O(\chi(x)),\ \text{ as } x \to \infty.$$

Then the integral $\int_0^\infty \varepsilon(t)f(t)dt$ is $|N,p_n|_k$-summable for $k \geq 1$.

Motivated essentially by the above-mentioned results, in this paper, we introduce the notion of $|N,p_n,\delta|_k$-summability of improper integrals and accordingly establish a new result. Also, our result is a non-trivial extension of some existing results which were established earlier. Moreover, matrix summability or matrix transformation is very important in the study of summability theory in the sense that it generalizes different summability methods like Cesàro summability, Nörlund summability, Riesz summability etc. For more current works in this direction, see [3], [4], [8], [9] and [10], and for some preliminaries and related works, see [1] and [2].

3. Main Theorem

Based on our proposed $|N,p_n,\delta|_k$-summability mean, here we establish the following theorem.

Theorem 3.1. *Let $\chi(x)$ be a positive non-decreasing function and there be two functions $\beta(x)$ and $\varepsilon(x)$ such that*

$$|\varepsilon'(x)| \leq \beta(x), \tag{15}$$

$$\beta(x) \to 0, \text{ as } x \to \infty, \tag{16}$$

$$|\varepsilon(x)|\ \chi(x) = O(1), \tag{17}$$

$$\left(\frac{P(x)}{p(x)}\right)^{\delta k} \frac{1}{(p(x))^{k-1}} \int_0^x P(u)\ |\beta'(u)|\ \chi(u)du = O(1), \text{ as } x \to \infty, \tag{18}$$

$$\left(\frac{P(x)}{p(x)}\right)^{\delta k} \frac{1}{(p(x))^{k-1}} \int_0^x P'(u)\ |\beta(u)|\ \chi(u)du = O(1), \text{ as } x \to \infty, \tag{19}$$

and

$$\int_0^x \left(\frac{P(x)}{p(x)}\right)^{\delta k-1} \left(\frac{P'(x)}{p(x)}\right)^k |\nu(x)|^k dx = O(\chi(x)), \text{ as } x \to \infty. \tag{20}$$

Then the integral $\int_0^\infty \varepsilon(t)f(t)dt$ is $|N, p, \delta|_k$-summable for $k \geq 1$.

Remark 3.1. *The above theorem can be proved by using the fact that $\int_0^\infty x|\beta'(x)|\chi(x)dx < \infty$ is weaker than $\int_0^\infty x|\varepsilon''(x)|\chi(x)dx < \infty$, and hence the introduction of the function $\{\beta(x)\}$ is justified.*

It may be possible to choose the function $\beta(x)$ such that

$$|\varepsilon'(x)| \leq \beta(x). \tag{21}$$

When $\varepsilon'(x)$ oscillates, $\beta(x)$ may be chosen such that $|\beta(x)| < |\varepsilon''(x)|$.

Hence, $\beta'(x) < |\varepsilon''(x)|$, so that $\int_0^\infty x|\beta'(x)|\chi(x)dx < \infty$ is a weaker requirement than $\int_0^\infty x|\varepsilon''(x)|\chi(x)dx < \infty$.

4. Proof of the Theorem 3.1

Proof. Let $T(x)$ be the (N, p_n) mean of the integral $\int_0^\infty \varepsilon(t)f(t)dt$. The integral $\int_0^\infty \varepsilon(t)f(t)dt$ is $|N, p_n, \delta|_k$ summable, if

$$\int_0^x \left(\frac{P(x)}{p(x)}\right)^{\delta k+k-1} |T'(t)|dt = O(1), \text{ as} \to \infty, \tag{22}$$

where $T(x)$ is given by

$$\begin{aligned}
T(x) &= \frac{1}{P(x)} \int_0^x p(t) \left(\int_0^t \varepsilon(u)f(u)du\right) dt \\
&= \frac{1}{P(x)} \int_0^x \varepsilon(u)f(u)du \int_u^x p(t)dt \\
&= \frac{1}{P(x)} \int_0^x (P(x) - P(u))\, \varepsilon(u)f(u)du \\
&= \int_0^x \left(1 - \frac{P(u)}{P(x)}\right) \varepsilon(u)f(u)du.
\end{aligned}$$

Now, differentiating both sides of (22) with respect to x, we get

$$\begin{aligned}
T'(x) &= \frac{1}{(P(x))^2}\int_0^x P'(x)P(u)\varepsilon(u)f(u)du \\
&= \frac{P'(x)\varepsilon(x)}{(P(x))^2}\int_0^x P(u)f(u)du - \frac{P'(x)}{(P(x))^2}\int_0^x \varepsilon'(u)\int_0^u P(t)f(t)dtdu \\
&= \frac{P'(x)\varepsilon(x)\nu(x)}{P(x)} - \frac{P'(x)}{(P(x))^2}\int_0^x P(u)\varepsilon'(u)\left(\frac{1}{P(u)}\int_0^u P(t)f(t)dt\right)du \\
&= \frac{P'(x)\varepsilon(x)\nu(x)}{P(x)} - \frac{P'(x)}{(P(x))^2}\int_0^x P(u)\,\varepsilon'(u)\,\nu(u)du \\
&= T_1(x) + T_2(x).
\end{aligned}$$

Applying Minkowski's inequality, we have

$$|T'(x)|^k = |T_1 + T_2|^k < 2^k\left(|T_1|^k + |T_2|^k\right). \tag{23}$$

Furthermore, by Hölder's inequality, we have

$$\begin{aligned}
\int_0^x \left(\frac{P(t)}{p(t)}\right)^{\delta k+k-1}|T_1(t)|^k dt &= \int_0^x \left(\frac{P(t)}{p(t)}\right)^{\delta k+k-1}\left|\frac{P'(x)\varepsilon(x)\nu(x)}{P(x)}\right|^k dt \\
&= \int_0^x \left(\frac{P(t)}{p(t)}\right)^{\delta k+k-1)}\frac{|P'(t)|^k|\nu(t)|^k|\varepsilon(t)|^k}{|P(t)|^k}dt \\
&= \int_0^x \left(\frac{P(t)}{p(t)}\right)^{\delta k-1}\left(\frac{|P'(t)|}{p(t)}\right)^k|\nu(t)|^k|\varepsilon(t)|^{k-1}|\varepsilon(t)|dt \\
&\le \int_0^x \left(\frac{P(t)}{p(t)}\right)^{\delta k-1}\left(\frac{|P'(t)|}{p(t)}\right)^k|\nu(t)|^k|\varepsilon(t)|dt \\
&= |\varepsilon(x)|\int_0^x \left(\frac{P(t)}{p(t)}\right)^{\delta k-1}\left(\frac{|P'(t)|}{p(t)}\right)^k|\nu(t)|^k \\
&- \int_0^x \left|\varepsilon'(t)\left(\int_0^t \left(\frac{P(y)}{p(y)}\right)^{\delta k-1}\left(\frac{|P'(t)|}{p(y)}\right)^k|\nu(y)|^k|dy\right)\right|dt \\
&= O(1)|\varepsilon(x)|\chi(x) - \int_0^x \beta(t)\chi(t)dt \\
&= O(1) - \beta(x)\int_0^x \chi(u)du + \int_0^x |\beta'(t)|\left(\int_0^t \chi(u)du\right)dx \\
&\le O(1) - \beta(x)\int_0^x \chi(u)du + \int_0^x t|\beta'(t)|\chi(t)dt \\
&= O(1), \text{as } x \to \infty.
\end{aligned}$$

By virtue of the hypothesis of Theorem 3.1,

$$\begin{aligned}
\int_0^x \left(\frac{P(t)}{p(t)}\right)^{\delta k+k-1} |T_2(t)|^k dt &= \int_0^x \left(\frac{P(t)}{p(t)}\right)^{\delta k+k-1} \left|\frac{P'(t)}{(P(t))^2}\int_0^t P(u)\varepsilon'(u)\nu(u)du\right|^k dt \\
&\leq \int_0^x \left(\frac{P(t)}{p(t)}\right)^{\delta k} \frac{P'(t)}{(p(t))^{k-1}(P(t))^2}\left(\int_0^t (P(u))^k|\varepsilon'(u)|^k|\nu(u)|^k du\right) \\
&\quad \cdot \left(\frac{1}{P(t)}\int_0^t P''(u)du\right)^{k-1} dt \\
&= \int_0^x |P(u)\varepsilon'(u)|^{k-1}|P(u)\varepsilon'(u)||\nu(u)|^k du \int_u^x \left(\frac{P(t)}{p(t)}\right)^{\delta k} \frac{P'(t)}{(p(t))^{k-1}(P(t))^2} dt \\
&= O\left(\left(\frac{P(t)}{p(t)}\right)^{\delta k}\frac{1}{(p(x))^{k-1}}\right)\int_0^x |P(u)\varepsilon'(u)||\nu(u)|^k\left(\frac{1}{P(u)}-\frac{1}{P(x)}\right)du \\
&\leq O\left(\left(\frac{P(t)}{p(t)}\right)^{\delta k}\frac{1}{(p(x))^{k-1}}\right)\int_0^x |P(u)\varepsilon'(u)||\nu(u)|^k\left(\frac{1}{P(u)}\right)du \\
&= O\left(\left(\frac{P(t)}{p(t)}\right)^{\delta k}\frac{1}{(p(x))^{k-1}}\right)P(x)|\varepsilon'(x)|\int_0^x |\nu(u)|^k\left(\frac{1}{P(u)}\right)du \\
&\quad - O\left(\left(\frac{P(t)}{p(t)}\right)^{\delta k}\frac{1}{(p(x))^{k-1}}\right)\int_0^x \left(P(u)\ |\varepsilon'(u)|\right)'\int_0^u |\nu(t)|^k\left(\frac{1}{P(t)}\right)dt\ du \\
&= O\left(\left(\frac{P(t)}{p(t)}\right)^{\delta k}\frac{1}{(p(x))^{k-1}}\right)\left\{P(x)|\beta(x)|\chi(x) - \int_0^x (P(u)|\beta(u)|)'\,\chi(u)du\right\} \\
&= O\left(\left(\frac{P(t)}{p(t)}\right)^{\delta k}\frac{1}{(p(x))^{k-1}}\right) \\
&\quad \cdot \left\{P(x)|\beta(x)|\chi(x) - \int_0^x P'(u)|\beta(u)|\chi(u)du - \int_0^x \left(P(u)|\beta'(u)|\right)\chi(u)du\right\} \\
&= O(1).
\end{aligned}$$

Thus,

$$\int_0^x \left(\frac{P(t)}{p(t)}\right)^{\delta k+k-1} |T_2(t)|^k dt = O(1),\ (x\to\ \infty). \tag{24}$$

On collecting (22) – (24), we have

$$\int_0^x \left(\frac{P(t)}{p(t)}\right)^{\delta k+k-1} |T'(t)|^k dt = O(1).$$

□

5. Conclusion

Through the investigation, we have obtained minimal sufficient conditions for the improper integrability via $|N,p_n,\delta|_k$-mean. Also, the result established here is more general in the sense that putting $\delta=0$, $|N,p_n,\delta|_k$-summability of improper integrals reduces to $|N,p_n|_k$- integrability demonstrated by Sonker and Munjal [7]. Moreover, for $\delta=0$ and $p_n=1$, our proposed method reduces to $|C,1|_k$ $(k\geq 1)$ integrability which was introduced by Özgen [5].

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