TWMS J. App. and Eng. Math. V.10, Special Issue, 2020, pp. 9-14

A THEOREM ON SUMMABILITY FACTORS FOR THE NÖRLUND METHOD FOR DOUBLE SERIES IN ULTRAMETRIC FIELDS

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ABSTRACT. Throughout this paper, K denotes a complete, non-trivially valued, ultrametric (or non-archimedean) field. 4-dimensional infinite matrices, double sequences and double series have entries in K. In this paper, we prove a theorem on summability factors for the Nörlund method for double series in K.

Keywords: ultrametric (or non-archimedean) field, summability factor, double sequence, double series, 4-dimensional infinite matrix, regular method, Nörlund method.

AMS Subject Classification: 40C05, 40G05, 46S10.

1. Introduction and Preliminaries

Throughout this paper, K denotes a complete, non-trivially valued, ultrametric (or non-archimedean) field. 4-dimensional infinite matrices, double sequences and double series have entries in K. We recall the following definitions and results briefly (see [2]) for the sake of completeness.

Definition 1.1. *Given a double sequence $\{x_{m,n}\}$ in K and $x \in K$, we write*

$$\lim_{m+n\to\infty} x_{m,n} = x,$$

if for every $\epsilon > 0$, the set $\{(m,n) \in \mathbb{N}^2 : |x_{m,n} - x| \geq \epsilon\}$ is finite, $\mathbb{N}$ being the set of all non-negative integers. In such a case, x is unique and x is called the limit of the double sequence $\{x_{m,n}\}$. We also say that $\{x_{m,n}\}$ converges to x.

Definition 1.2. *Let $\{x_{m,n}\}$ be a double sequence in K and $s \in K$. We write*

$$\sum_{m,n=0}^{\infty,\infty} x_{m,n} = s$$

if

$$\lim_{m+n\to\infty} s_{m,n} = s,$$

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§ Manuscript received: April 16, 2019; accepted: October 10, 2019.
TWMS Journal of Applied and Engineering Mathematics, Vol.10, Special Issue;

where

$$s_{m,n} = \sum_{k,\ell=0}^{m,n} x_{k,\ell}, \ m,n = 0,1,2,\ldots.$$

In such a case, we say that the double series $\sum_{m,n=0}^{\infty,\infty} x_{m,n}$ *converges to* s.

Remark 1.1. *If* $\{x_{m,n}\}$ *converges, then* $\{x_{m,n}\}$ *is bounded.*

Theorem 1.1. [2, Lemma 1] $\lim_{m+n\to\infty} x_{m,n} = x$ *if and only if*

(i) $\lim_{m\to\infty} x_{m,n} = x, \ n = 0,1,2,\ldots;$

(ii) $\lim_{n\to\infty} x_{m,n} = x, \ m = 0,1,2,\ldots;$ *and*

(iii) for every $\epsilon > 0$, *there exists* $N \in \mathbb{N}$ *such that* $|x_{m,n} - x| < \epsilon$, $m,n \geq N$, *which is written as*

$$\lim_{m,n\to\infty} x_{m,n} = x,$$

noting that this is Pringsheim's definition of convergence of a double sequence.

Theorem 1.2. [2, Lemma 2] $\sum_{m,n=0}^{\infty,\infty} x_{m,n}$ *converges if and only if*

$$\lim_{m+n\to\infty} x_{m,n} = 0.$$

Remark 1.2. *In the case of simple series, it is well-known that* $\sum_{n=0}^{\infty} x_n$ *converges if and only if*

$$\lim_{n\to\infty} x_n = 0$$

(see [1, p. 25, Theorem 1.1]*). Theorem 1.2 shows that Definition 1.1 is more suited in the ultrametric case than Pringsheim's definition of convergence of a double sequence.*

Definition 1.3. *Given a 4-dimensional infinite matrix* $A = (a_{m,n,k,\ell})$, $a_{m,n,k,\ell} \in K$, $m,n,k,\ell = 0,1,2,\ldots$ *and a double sequence* $x = \{x_{k,\ell}\}$, $x_{k,\ell} \in K$, $k,\ell = 0,1,2,\ldots$, *by the* A*-transform of* $x = \{x_{k,\ell}\}$, *we mean the double sequence* $A(x) = \{(Ax)_{m,n}\}$,

$$(Ax)_{m,n} = \sum_{k,\ell=0}^{\infty,\infty} a_{m,n,k,\ell} x_{k,\ell}, \ m,n = 0,1,2,\ldots,$$

where we suppose that the double series on the right converge. If $\lim_{m+n\to\infty} (Ax)_{m,n} = s$, *we say that the double sequence* $x = \{x_{k,\ell}\}$ *is* A*-summable or summable* A *to* s, *written as*

$$x_{k,\ell} \to s(A).$$

If $\lim_{m+n\to\infty} (Ax)_{m,n} = s$, *whenever* $\lim_{k+\ell\to\infty} x_{k,\ell} = s$, *we say that* A *is regular. A double series* $\sum_{m,n=0}^{\infty,\infty} x_{m,n}$ *is said to be* A*-summable to* s, *if* $\{s_{m,n}\}$ *is* A*-summable to* s, *where* $s_{m,n} = \sum_{k,\ell=0}^{m,n} x_{k,\ell}, \ m,n = 0,1,2,\ldots.$

The following important result, due to Natarajan and Srinivasan [2], gives a criterion for a 4-dimensional infinite matrix to be regular in terms of its entries.

Theorem 1.3 (Silverman-Toeplitz). *The 4-dimensional infinite matrix $A = (a_{m,n,k,\ell})$ is regular if and only if*

$$\sup_{m,n,k,\ell} |a_{m,n,k,\ell}| < \infty; \tag{1}$$

$$\lim_{m+n\to\infty} a_{m,n,k,\ell} = 0, \ \ k,\ell = 0,1,2,\ldots; \tag{2}$$

$$\lim_{m+n\to\infty} \sum_{k,\ell=0}^{\infty,\infty} a_{m,n,k,\ell} = 1; \tag{3}$$

$$\lim_{m+n\to\infty} \sup_{k\geq 0} |a_{m,n,k,\ell}| = 0, \ \ \ell = 0,1,2,\ldots; \tag{4}$$

and

$$\lim_{m+n\to\infty} \sup_{\ell\geq 0} |a_{m,n,k,\ell}| = 0, \ \ k = 0,1,2,\ldots. \tag{5}$$

The Nörlund method $(N, p_{m,n})$ for double sequence and double series in K was introduced earlier by Natarajan and Srinivasan in [2].

Definition 1.4. *Given $p_{m,n} \in K$, $m,n = 0,1,2,\ldots$, the Nörlund method (mean), denoted by $(N, p_{m,n})$, is defined by the 4-dimensional infinite matrix $(a_{m,n,k,\ell})$, $m,n,k,\ell = 0,1,2,\ldots$, where*

$$a_{m,n,k,\ell} = \begin{cases} \frac{p_{m-k,n-\ell}}{P_{m,n}}, & \text{if } k \leq m \text{ and } \ell \leq n; \\ 0, & \text{otherwise,} \end{cases}$$

$P_{m,n} = \sum_{k,\ell=0}^{m,n} p_{k,\ell}$, $m,n = 0,1,2,\ldots$, $|p_{k,\ell}| < |p_{0,0}|$, $(k,\ell) \neq (0,0)$, $k,\ell = 0,1,2,\ldots$.

Remark 1.3. *From the above definition, it follows that $p_{0,0} \neq 0$.*

It is easy to prove the following result, which is very useful in the sequel.

Theorem 1.4. *If $\sum_{m,n=0}^{\infty,\infty} a_{m,n}$ is $(N, p_{m,n})$ summable, then $\{a_{m,n}\}$ is bounded.*

Some properties of the Nölund method for double sequences in K were studied in [2].

For the definition of summability factors for simple series in the classical case, see ([4, pp. 38–39]). We retain the same definition for double series in the ultrametric set up too with suitable changes.

2. Main Result

We now prove the main result of the paper, which deals with summability factors for the Nörlund method for double series in K.

Theorem 2.1. *If $\sum_{m,n=0}^{\infty,\infty} a_{m,n}$ is $(N, p_{m,n})$ summable, $(N, p_{m,n})$ being regular and if $\{b_{m,n}\}$ converges, then $\sum_{m,n=0}^{\infty,\infty} a_{m,n} b_{m,n}$ is $(N, p_{m,n})$ summable too.*

Proof. Let $s_{m,n} = \sum_{k,\ell=0}^{m,n} a_{k,\ell}$, $t_{m,n} = \sum_{k,\ell=0}^{m,n} a_{k,\ell} b_{k,\ell}$, $m, n = 0, 1, 2, \dots$. Let $\{\alpha_{m,n}\}$, $\{\beta_{m,n}\}$ be the $(N, p_{m,n})$-transforms of $\{s_{m,n}\}$, $\{t_{m,n}\}$ respectively so that

$$\alpha_{m,n} = \frac{1}{P_{m,n}} \sum_{k,\ell=0}^{m,n} p_{m-k,n-\ell} s_{k,\ell},$$

$$\beta_{m,n} = \frac{1}{P_{m,n}} \sum_{k,\ell=0}^{m,n} p_{m-k,n-\ell} t_{k,\ell},$$

$m, n = 0, 1, 2, \dots$.

Let $\lim_{m+n\to\infty} \alpha_{m,n} = s$ and $\lim_{m+n\to\infty} b_{m,n} = t$. Let

$$b_{m,n} = t + \epsilon_{m,n}, \; m, n = 0, 1, 2, \dots$$

so that

$$\lim_{m+n\to\infty} \epsilon_{m,n} = 0.$$

Now,

$$\begin{aligned}
\alpha_{m,n} &= \frac{1}{P_{m,n}} \sum_{k,\ell=0}^{m,n} p_{m-k,n-\ell} s_{k,\ell} \\
&= \frac{1}{P_{m,n}} \sum_{k,\ell=0}^{m,n} p_{m-k,n-\ell} \left(\sum_{i,j=0}^{k,\ell} a_{i,j} \right) \\
&= \frac{1}{P_{m,n}} \sum_{k,\ell=0}^{m,n} a_{k,\ell} \left(\sum_{i,j=0}^{m-k,n-\ell} p_{i,j} \right) \\
&= \frac{1}{P_{m,n}} \sum_{k,\ell=0}^{m,n} a_{k,\ell} P_{m-k,n-\ell},
\end{aligned}$$

where, we recall that $P_{m,n} = \sum_{k,\ell=0}^{m,n} p_{k,\ell}$, $m, n = 0, 1, 2, \dots$. Since $(N, p_{m,n})$ is regular, using (2), we have,

$$\begin{aligned}
&\lim_{m+n\to\infty} a_{m,n,0,0} = 0, \\
i.e., &\lim_{m+n\to\infty} \frac{p_{m,n}}{P_{m,n}} = 0, \\
i.e., &\lim_{m+n\to\infty} \left| \frac{p_{m,n}}{P_{m,n}} \right| = 0, \\
i.e., &\lim_{m+n\to\infty} |p_{m,n}| = 0, \quad \text{since } |P_{m,n}| = |p_{0,0}|, \\
&\qquad \text{using the fact that the} \\
&\qquad \text{valuation is non-archimedean} \\
i.e., &\lim_{m+n\to\infty} p_{m,n} = 0.
\end{aligned}$$

In view of Theorem 1.2, $\sum_{m,n=0}^{\infty,\infty} p_{m,n}$ converges (say) to P. Since $|P_{m,n}| = |p_{0,0}|$, $m, n = 0, 1, 2, \dots$, $|P| = |p_{0,0}|$. Since $p_{0,0} \neq 0$, it follows that $P \neq 0$. Also $\lim_{m+n\to\infty} P_{m,n} = P$.

We now have,

$$\begin{aligned}
\beta_{m,n} &= \frac{1}{P_{m,n}} \sum_{k,\ell=0}^{m,n} a_{k,\ell} b_{k,\ell} P_{m-k,n-\ell} \\
&= \frac{1}{P_{m,n}} \sum_{k,\ell=0}^{m,n} a_{k,\ell}(t + \epsilon_{k,\ell}) P_{m-k,n-\ell} \\
&= \frac{1}{P_{m,n}} \left[t \sum_{k,\ell=0}^{m,n} a_{k,\ell} P_{m-k,n-\ell} \right. \\
&\qquad \left. + \sum_{k,\ell=0}^{m,n} a_{k,\ell}\epsilon_{k,\ell} P_{m-k,n-\ell} \right] \\
&= t\alpha_{m,n} + \frac{1}{P_{m,n}} \sum_{k,\ell=0}^{m,n} a_{k,\ell}\epsilon_{k,\ell} P_{m-k,n-\ell}. \qquad (6)
\end{aligned}$$

We note the following. Since $\sum_{m,n=0}^{\infty,\infty} a_{m,n}$ is $(N, p_{m,n})$ summable, $\{a_{m,n}\}$ is bounded in view of Theorem 1.4. Also $\lim_{m+n\to\infty} \epsilon_{m,n} = 0$. So $\lim_{m+n\to\infty} a_{m,n}\epsilon_{m,n} = 0$. Consequently, using Theorem 1.2 again, $\sum_{m,n=0}^{\infty,\infty} a_{m,n}\epsilon_{m,n}$ converges. Also,

$$\begin{aligned}
&\sum_{k,\ell=0}^{m,n} a_{k,\ell}\epsilon_{k,\ell} P_{m-k,n-\ell} \\
&\qquad = \sum_{k,\ell=0}^{m,n} a_{k,\ell}\epsilon_{k,\ell}(P_{m-k,n-\ell} - P) \\
&\qquad\qquad + P \sum_{k,\ell=0}^{m,n} a_{k,\ell}\epsilon_{k,\ell}.
\end{aligned}$$

In view of the fact that $\lim_{m+n\to\infty} a_{m,n}\epsilon_{m,n} = 0$ and $\lim_{m+n\to\infty}(P_{m,n} - P) = 0$, using Theorem 2 of [3], it follows that

$$\lim_{m+n\to\infty} \sum_{k,\ell=0}^{m,n} a_{k,\ell}\epsilon_{k,\ell}(P_{m-k,n-\ell} - P) = 0.$$

Thus

$$\lim_{m+n\to\infty} \sum_{k,\ell=0}^{m,n} a_{k,\ell}\epsilon_{k,\ell} P_{m-k,n-\ell} = P \sum_{m,n=0}^{\infty,\infty} a_{m,n}\epsilon_{m,n}.$$

So, taking limit as $m+n \to \infty$ in (6), we have,

$$\lim_{m+n\to\infty} \beta_{m,n} = ts + \sum_{m,n=0}^{\infty,\infty} a_{m,n}\epsilon_{m,n}.$$

In other words, $\sum_{m,n=0}^{\infty,\infty} a_{m,n}b_{m,n}$ is $(N, p_{m,n})$ summable to $ts + \sum_{m,n=0}^{\infty,\infty} a_{m,n}\epsilon_{m,n}$. This completes the proof of the theorem. □

3. Conclusion

We conclude the paper with an important observation in the context of Theorem 1.2.

$$\lim_{m,n\to\infty} x_{m,n} = 0$$

does not ensure the convergence of $\sum_{m,n=0}^{\infty,\infty} x_{m,n}$ in the sense of Definition 1.1, as illustrated by the following example. Let $K = \mathbb{Q}_2$, the 2-adic field. Consider the series $\sum_{m,n=0}^{\infty,\infty} x_{m,n}$, where,

$$x_{m,n} = 3^m 2^n,\ m,n = 0,1,2,\ldots.$$

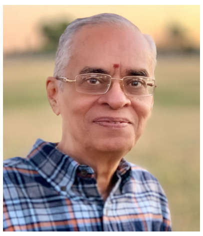
Note that

$$\lim_{m,n\to\infty} x_{m,n} = 0,$$

while, a simple computation shows that $\sum_{m,n=0}^{\infty,\infty} x_{m,n}$ does not converge in the sense of Definition 1.1.

References

[1] Bachman, G., (1964), Introduction to p-adic numbers and valuation theory, Academic Press.
[2] Natarajan, P.N. and Srinivasan, V., (2002), Silverman-Toeplitz theorem for double sequences and series and its application to Nörlund means in non-archimedean fields, Ann. Math. Blaise Pascal, 9, pp. 85-100.
[3] Natarajan, P.N. and Sakthivel, S., (2008), Multiplication of double series and convolution of double infinite matrices in non-archimedean fields, Indian J. Math., 50, pp. 115-123.
[4] Peyerimhoff, A., (1969), Lectures on summability, Lecture Notes in Mathematics, 107, Springer.



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