TWMS J. App. and Eng. Math. V.10, Special Issue, 2020, pp. 15-27

ON THE CONVERGENCE OF (p,q)-BERNSTEIN OPERATORS OF THE RATIONAL FUNCTIONS WITH POLES IN $[0,1]$

FAISAL KHAN[1], MOHD SAIF[2], M. MURSALEEN[3], ABDUL HAKIM KHAN[4], §

Abstract. In the present paper, we obtain the approximation results of (p,q)-Bernstein operators $B^n_{p,q}(h;x)$ to a rational function for $q > p > 1$ and investigate convergence properties of $B^n_{p,q}(h;x)$ for the function $h(x) = (x - p^m q^{-m})^{-\eta}$ with $\eta > 2$. Here, we observe that the approximation properties for the (p,q) -Bernstein operators are more precise in nature than the previously obtained results given in [23, 25].

Keywords: (p,q)-integer, (p,q)-Bernstein operators, convergence, poles.

AMS Subject Classification: 41A10, 41A25, 41A36.

1. Introduction and preliminaries

The development of q-calculus and (p,q)-calculus [4, 11, 12, 26] plays an important role in the field of approximation theory, number theory, quantum physics and other branches of physical sciences. Mursaleen et.al. were the first to apply the concept of (p,q)-calculus in approximation theory [15, 18, 19]. After that (p,q)-analogue of well known operators were studied by many authors (see [1, 2, 3, 5, 6, 7, 9, 10, 21]). We recall certain definitions and well known notations of (p,q)-calculus:
The (p,q)-integers $[n]_{p,q}$ is defined as

$$[n]_{p,q} := \frac{p^n - q^n}{p - q}, \qquad (n \in \mathbb{N} \cup \{0\},\ p > q \geq 1).$$

[1] Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India.
e-mail: faisalamu2011@gmail.com; ORCID: http://orcid.org/0000-0001-5241-3729.
[2] Department of Applied Mathematics, Aligarh Muslim University, Aligarh 202002, India.
e-mail: usmanisaif153@gmail.com; ORCID: http://orcid.org/0000-0002-2719-6002.
[3] (a) Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India.
(b) Department of Medical Research, China Medical University Hospital, China Medical University (Taiwan),Taichung, Taiwan.
(c) Department of Computer Science and Information Engineering, Asia University, Taichung, Tai wan.
e-mail: mursaleenm@gmail.com; ORCID: http://orcid.org/0000-0003-4128-0427.
[4] Department of Applied Mathematics, Aligarh Muslim University, Aligarh 202002, India.
e-mail: ahkhan.amu@gmail.com; ORCID: http://orcid.org/0000-0002-1204-0064.
§ Manuscript received: May 3, 2019; accepted: January 11, 2020.
TWMS Journal of Applied and Engineering Mathematics, Vol.10, Special Issue;

The (p,q)-binomial expansion is given as

$$(x+y)^n_{p,q} := \prod_{s=0}^{n-1}(p^s x + q^s y) \quad and \quad (x,p;q)_k := \prod_{s=0}^{n-1}(p^s - q^s x).$$

It can be easily verified by induction that

$$\prod_{s=0}^{n-1}(p^s + q^s x) := (1+x)(p+qx)(p^2+q^2x)\cdots(p^{n-1}+q^{n-1}x)$$

$$= \sum_{r=0}^{k} p^{\frac{(n-r)(n-r-1)}{2}} q^{\frac{r(r-1)}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_{p,q} x^r,$$

and the (p,q)-binomial coefficients are defined by

$$\begin{bmatrix} n \\ r \end{bmatrix}_{p,q} := \frac{[n]_{p,q}!}{[r]_{p,q}![n-r]_{p,q}!}.$$

Let $h \in \mathbb{C}[0,1]$ be such that $h : [0,1] \longrightarrow \mathbb{R}$ and $q > p > 1$. Then the (p,q)-Bernstein operators [18] of h are defined as:

$$B^n_{p,q}(h;x) := \sum_{k=0}^{n} h\left(p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}}\right) p_{n,k}(p,q;x) \quad n \in \mathbb{N},$$

where, polynomial $p_{n,k}(p,q;x)$ is given by

$$p_{n,k}(p,q;x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{s=0}^{n-k-1}(p^s - q^s x), x \in [0,1], 0 < q < p < 1. \quad (1)$$

If we set $p = 1$, $B^n_{p,q}(h;x)$ reduces to q-Bernstein operators [24] and note that they are used only for the case $q \neq 1$.
The end point interpolation property of (p,q)-Bernstein operators is given by (see [23]).

$$B^n_{p,q}(h;0) = h(0), \quad B^n_{p,q}(h;1) = h(1). \quad (2)$$

The (p,q)-divided difference of Bernstein operators $B^n_{p,q}(h;x)$ (see [20]) is defined as:

$$B^n_{p,q}(h;x) := \sum_{r=0}^{n} \lambda^n_{p,q}\, h\left[0, \frac{p^{n-1}[1]_{p,q}}{[n]_{p,q}}, \cdots, \frac{p^{n-r}[r]_{p,q}}{[n]_{p,q}}\right] x^r, \quad (3)$$

where, the coefficients $\lambda^n_{p,q}$ are given by

$$\lambda^n_{p,q} = \begin{bmatrix} n \\ r \end{bmatrix}_{p,q} \frac{[r]_{p,q}\,!}{[n]^r_{p,q}}\, p^{\frac{(n-r)(n-r-1)}{2}}\, q^{\frac{r(r-1)}{2}},$$

and the k-th order divided-difference of the function h with pairwise distinct nodes are given by

$$h[x_0] = h(x_0),\ h[x_0,x_1] = \frac{h(x_1)-h(x_0)}{x_1 - x_0}, \ldots$$

$$\ldots h[x_0,x_1,\ldots,x_k] = \frac{h[x_1,\ldots,x_k] - h[x_0,\ldots,x_{k-1}]}{[x_k - x_0]},$$

$$= \left(1 - \frac{p^{n-1}[1]_{p,q}}{[n]_{p,q}}\right)\left(1 - \frac{p^{n-2}[2]_{p,q}}{[n]_{p,q}}\right)\cdots\left(1 - \frac{p^{n-r+1}[r-1]_{p,q}}{[n]_{p,q}}\right) \quad (4)$$

and $\lambda^0_{p,q} = \lambda^1_{p,q} = 1, \quad 0 \leq \lambda^r_{p,q} \leq 1, \quad r = 0,1,\ldots,n.$

Let $\mathbb{T}_{p,q}$ denote the time scale defined as

$$\mathbb{T}_{p,q} = \{0\} \cup \{p^k q^{-k}\}_{k=0}^{\infty}.$$

In our present study we mainly focus on the (p, q)-Bernstein operators with $q > p > 1$. We consider the (p, q)-Bernstein operators of the rational function $\frac{M(x)}{N(x)}$ and it can be seen that the approximation properties for the (p, q)-Bernstein operators are more precise in nature than the previously obtained results [8, 13, 14, 16, 17, 22]. Some known results lead to the following conclusion:

- If $\alpha = 0$, that is $h(x) = \frac{1}{x^\eta}, x \neq 0$ and $h(0) = b$ $(b \in \mathbb{R})$, then, for $q \geq 2$,

$$\lim_{n\to\infty} B^n_{p,q}(h; x) = \begin{cases} \text{h(x)} & x \in \mathbb{T}_{p,q} \\ \infty & x \notin \mathbb{T}_{p,q}. \end{cases}$$

- If $\alpha \in [0, 1] \setminus \mathbb{T}_{p,q}$, that is $h(x) = \frac{1}{(x-\alpha)^\eta}$ $(x \neq \alpha)$ and $f(\alpha) = b$ $(b \in \mathbb{R})$, then

$$\lim_{n\to\infty} B^n_{p,q}(h; x) = h(x), \quad x \in \mathbb{T}_{p,q}.$$

2. Statement of main results

Let $m \in \mathbb{N} \cup \{0\}$ with $\eta \in \mathbb{N}$, $b \in \mathbb{R}$ and $h_m : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$h_m(x) = \begin{cases} \frac{1}{(x-p^m q^{-m})^\eta} & x \in \mathbb{R} \setminus \{p^m q^{-m}\} \\ b & x = p^m q^{-m}. \end{cases} \tag{5}$$

The first result shows that for $m \in \mathbb{N}$, the function in (5) is uniformly approximated by its (p, q)-Bernstein operators on any compact set in $(-p^{(m+\eta)}q^{-(m+\eta)}, p^{(m+\eta)}q^{-(m+\eta)})$. The sharpness of this result is demonstrated in part (ii) of Theorem 2.1, which claims that outside of the interval, the sequence of operators $\{B^n_{p,q}(h; x)\}$ diverges everywhere, except for a finite number of indicated points.

Theorem 2.1. *(i). For* $m \in \mathbb{N}$, $\lim_{n\to\infty} B^n_{p,q}(h_m; x) = h_m(x)$ *uniformly on any compact subset of* $\left(-p^{(m+\eta)}q^{-(m+\eta)}, p^{(m+\eta)}q^{-(m+\eta)}\right)$.

(ii). For $m \in \mathbb{N}$, $\lim_{n\to\infty} B^n_{p,q}(h_m; x) = \infty$ *with* $|x| > p^{(m+1)}q^{-(m+1)}$, $x \neq p^{(m+1)}q^{-(m+1)}$, $x \neq p^{(m-1)}q^{-(m-1)}$, $x \neq p^{(m-2)}q^{-(m-2)}, \ldots, 1$.

Since, the function h_m given by (5) is continuous at all the nodes $x_0, x_1, ..., x_n$, when n is large enough. At $m = 0$ the function h_0 has infinite discontinuity at the nodes $x_n = \frac{[n]_{p,q}}{[n]_{p,q}} = 1$, $n \in \mathbb{N}$.

Theorem 2.2. *Let* h_0 *be given by (5) with* $m = 0$*. Then*
(i) For all $x \in (-1, 1]$,

$$\lim_{n\to\infty} B^n_{p,q}(h_0; x) = h_0(x)$$

uniformly on any compact subset of $(-1, 1)$.
(ii) For all $x \in (-\infty, -1) \cup [1, \infty)$,

$$\lim_{n\to\infty} B^n_{p,q}(h_0; x) \to \infty.$$

Using the above result, the following phenomenon can be established:
Let $h(x) = \frac{1}{(x-\alpha)^{\eta}}, \alpha \in \mathbb{R} \setminus \{0\}$, and

$$\mathcal{T}(\alpha) = \begin{cases} |\alpha|, \ \alpha \notin \{p\,q^{-1}, p^2 q^{-2}, \ldots\} \\ \frac{p^{\eta}\,\alpha}{q^{\eta}}, \ \alpha \in \{p\,q^{-1}, p^2 q^{-2}, \ldots\}. \end{cases} \tag{6}$$

Then, $B^n_{p,q}(h_m; x) \to h_m(x)$ as $n \to \infty$ uniformly on any compact set in $\{x : |x| < \mathcal{T}(\alpha)\}$. Since, sequence of operators $\{B^n_{p,q}(h; x)\}$ does not converge uniformly on any interval in $\{x : |x| > \mathcal{T}(\alpha)\}$. It is noticed that the set of convergence for the (p, q)-Bernstein operators $B^n_{p,q}(h_m; x)$ depends not only on the distance of pole from the origin but also on whether or not the pole α belongs to the sequence $\{p^l\, q^{-l}\}_{l=1}^{\infty}$. If $\alpha \in \{p^l\, q^{-l}\}_{l=1}^{\infty}$ then the set of convergence of $B^n_{p,q}(h_m; x)$ also depends on the multiplicity of the pole.
Finally, let $h(x)$ be the rational function with poles $\alpha_1, \alpha_2, \ldots, \alpha_s \in \mathbb{R} \setminus \{0\}$ having multiplicities $\eta_1, \eta_2, \ldots, \eta_s$ respectively. Set $\mathcal{T} = min_{1 \le k \le s}\{\mathcal{T}(\alpha_k)\}$. Then, $h(x)$ is uniformly approximated by $\{B^n_{p,q}(h; x)\}$ on any compact set in $\{x : |x| < \mathcal{T}\}$.

3. Some auxiliary results

To prove the main theorems, we present here some technical lemmas. The first of which describe the behavior of $B^n_{p,q}(h_m; .)$ on the time scale $\mathbb{T}_{p,q}$.

Lemma 3.1. *Let h_m be a function defined by (5)*
(a) If $m \in \mathbb{N}$, then

$$\lim_{n\to\infty} B^n_{p,q}(h_m; p^l q^{-l}) = \begin{cases} h_m(p^l q^{-l}) \ \eta \in \mathbb{N} \cup \{0\} \setminus \{m, m+1, \ldots, m+\eta\} \\ \infty \ l \in \{m, m+1, \ldots, m+\eta-1\} \end{cases}$$

and

$$\lim_{n\to\infty} B^n_{p,q}(h_m; p^{(m+\eta)} q^{-(m+\eta)}) = h_m(p^{m+\eta} q^{-(m+\eta)}) + \mathcal{D}, \quad \mathcal{D} \neq 0, \eta \in \mathbb{N} \cup \{0\}.$$

(b) If $m = 0$, then the following holds

$$\lim_{n\to\infty} B^n_{p,q}(h_0; p^l q^{-l}) = h_0\,(p^l q^{-l}), \ l \in \mathbb{N} \cup \{0\},$$

that is, $B^n_{p,q}(h_0; .)$ approximates h_0 on $\mathbb{T}_{p,q}$.

Proof. From (1), we see that $p_{n,n-k}(p, q; p^l q^{-l}) = 0$, for $k > l$ whence

$$B^n_{p,q}(h; p^{-l} q^l) = \sum_{k=0}^{min\{k,l\}} h\left(\frac{[n-k]_{p,q}}{[n]_{p,q}}\right) p_{n,n-k}(p, q; p^l q^{-l}). \tag{7}$$

Besides

$$\lim_{n\to\infty} p_{n,n-k}(p, q; p^{\eta} q^{-l}) = \delta_{k,l} \ and \ \lim_{n\to\infty} \frac{[n-k]_{p,q}}{[n]_{p,q}} = p^k q^{-k}, \ k \in \mathbb{N} \cup \{0\}. \tag{8}$$

Therefore

$$\lim_{n\to\infty} h\left(\frac{[n-k]_{p,q}}{[n]_{p,q}}\right) p_{n,n-k}(p, q; p^l q^{-l}) = h_m(p^k q^{-k})\,\delta_{l,k}, \ for\, all\, k \neq m$$

which implies that

$$\lim_{n\to\infty} B^n_{p,q}(h_m; p^l q^{-l}) = h_m(p^l q^{-l}) \ for\, l < m.$$

Now consider,

$$h_m\left(\frac{[n-m]_{p,q}}{[n]_{p,q}}\right)=\left(\frac{[n-m]_{p,q}}{[n]_{p,q}}-\frac{1}{[q]^m}\right)^{-l}$$
$$=\frac{p^{-nl}q^{ml}(p^{-n}q^{n}-1)^{l}}{(1-p^{-m}q^{m})^{l}}\sim\frac{p^{-nl}q^{nl}}{(p^{m}q^{-m}-1)^{l}},\ n\to\infty$$
$$h_m\left(\frac{[n-m]_{p,q}}{[n]_{p,q}}\right)p_{n,n-m}(q;q^{-l})\sim\mathcal{D}\,p^{-(m+l-l)n}q^{(m+l-l)n},\ n\to\infty,\tag{9}$$

where $\mathcal{D}=\mathcal{D}(l)$. It follows that,

$$\lim_{n\to\infty}h\left(\frac{[n-k]_{p,q}}{[n]_{p,q}}\right)p_{n,n-k}(p,q;p^{l}q^{-l})=\begin{cases}\infty\ m\leq l\leq m+\eta-1,\\ \mathcal{D}\neq0\ l=m+\eta,\\ 0\ l\geq m+\eta+1.\end{cases}$$

As a result, for $l>m$

$$\lim_{n\to\infty}B^{n}_{p,q}(h_m;p^{l}q^{-l})=\lim_{n\to\infty}\left\{h_m\left(\frac{[n-l]_{p,q}}{[n]_{p,q}}\right)p_{n,n-l}(p,q;p^{l}q^{-l})\right\}$$
$$+\lim_{n\to\infty}\left\{h_m\left(\frac{[n-k]_{p,q}}{[n]_{p,q}}\right)p_{n,n-l}(p,q,p^{l}q^{-l})\right\}$$
$$=\begin{cases}\infty,\ m<l\leq m+\eta-1,\\ h_m(p^{(m+\eta)}q^{-(m+\eta)})+\mathcal{D}\ l=m+\eta\\ h_m(p^{l}q^{-l})\ l\geq m+\eta+1\end{cases}$$

with the observation

$$\lim_{n\to\infty}B^{n}_{p,q}(h_m;p^{m}q^{-m})=\lim_{n\to\infty}h_m\left(\frac{[n-m]_{p,q}}{[n]_{p,q}}\right)p_{n,n-m}(p,q;p^{m}q^{-m})=\infty.$$

This completes the proof.

(b) As we know that h_0 is continuous at all points $p^{l}q^{-l},l\in\mathbb{N}$ and with the help of end-point interpolation property (2) and formula (8), the statement follows.

Lemma 3.2. *For $n\in\mathbb{N}$, denote $x_k=p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}},k=0,1,2,\dots,n$ with $0\leq i\leq k\leq n$, then following relation holds:*

$$\prod_{0\leq s\leq k,s\neq i}\left(\frac{1}{x_i-x_s}\right)=\frac{(-1)^{k-i}\,p^{-nk}\,q^{nk}\,(1-p^{n}q^{-n})^{k}}{p^{\frac{-k(k+1)}{2}-\frac{i(i-1)}{2}}q^{\frac{k(k+1)}{2}+\frac{i(i-1)}{2}}\,(\frac{p}{q},\frac{p}{q})_i\,(\frac{p}{q},\frac{p}{q})_{k-i}}.\tag{10}$$

Proof. It can be easily seen that

$$\prod_{0\leq s\leq k,s\neq i}\left(\frac{1}{x_i-x_s}\right)=\frac{[n]^{k}_{p,q}}{p^{n-i}[i]_{p,q}([i]_{p,q}-p^{i-1}[1]_{p,q})\cdots([i]_{p,q}-p^{i-2}[2]_{p,q})\cdots}$$
$$\times\frac{1}{([i]_{p,q}-p[i-1]_{p,q})\cdots([i]_{p,q}-p^{i-k}[k]_{p,q})}$$
$$=\frac{p^{-k(n-i)}[n]_{p,q}}{p^{k(i-1)}(p^{-i}q^{i}-1)\cdots(p^{-i}q^{i}-p^{-(i-1)}q^{i-1})\cdots(p^{-i}q^{i}-p^{-k}q^{k})}$$
$$=\frac{(-1)^{k-i}p^{-nk}q^{nk}\,(1-p^{n}q^{-n})^{k}}{p^{\frac{-k(k+1)}{2}-\frac{i(i-1)}{2}}q^{\frac{k(k+1)}{2}+\frac{i(i-1)}{2}}\,(\frac{p}{q},\frac{p}{q})_i\,(\frac{p}{q},\frac{p}{q})_{k-i}}.$$

Corollary 3.1. *For integers $0 \leq u \leq v \leq m$, the following estimate holds:*

$$\prod_{\substack{o \leq s \leq n-m+v, \\ s \neq n-m+u}} \left(\frac{1}{x_{n-m+u} - x_s} \right) \sim \mathcal{D}\, p^{-n(m-u)} q^{n(m-u)}, \ n \to \infty,$$

where

$$\mathcal{D} = \mathcal{D}(u,v) = \frac{(-1)^{v-u} p^{-n(m-u)} q^{n(m-u)} q^{-m^2+m(u+v)-\frac{u^2+v^2}{2}-\frac{(v-u)}{2}} (1-p^n q^{-n})^{n-m+u}}{(\frac{p}{q};\frac{p}{q})_{v-u} (\frac{p}{q};\frac{p}{q})_\infty}.$$

Proof. On using $i = n-m+u$ and $k = n-m+v$ into (10), we get

$$\prod_{\substack{o \leq s \leq n-m+v, \\ s \neq n-m+u}} \left(\frac{1}{x_{n-m+u} - x_s} \right) = \frac{p^{-n(m-u)} q^{n(m-u)} (-1)^{v-u} q^{-m^2+m(u+v)-\frac{(u^2+v^2)}{2}-\frac{(v-u)}{2}}}{(\frac{p}{q};\frac{p}{q})_{n-m+u}}$$

$$\times \frac{(1-q^{-n})^{n-m+v}}{(\frac{p}{q};\frac{p}{q})_{v-u}}$$

$$\sim p^{-n(m-u)} q^{n(m-u)} \frac{(-1)^{v-u} q^{-m^2+m(u+v)} - \frac{(u(u-1))}{2} - \frac{v(v+1)}{2}}{(\frac{p}{q};\frac{p}{q})_\infty (\frac{p}{q};\frac{p}{q})_{v-u}}$$

$$=: \mathcal{D}(u,v) p^{-n(m-u)} q^{n(m-u)}.$$

Lemma 3.3. *For $n \in \mathbb{N}$, denote $x_k = p^{n-k} \frac{[k]_{p,q}}{[n]_{p,q}}, k = 0, 1, \ldots, n$. Then, for $0 \leq v \leq m$, the following estimates holds:*

$$\sum_{u=0}^{v} \frac{h_m(x_{n-m+u})}{\prod_{o \leq s \leq n-m+v, s \neq n-m+u} (x_{n-m+u} - x_s)} \sim \mathcal{D}\, p^{-n(m+\eta)} q^{n(m+\eta)}, \ n \to \infty$$

where,

$$\mathcal{D} = \mathcal{D}(v) = \frac{(-1)^v p^{\frac{v(v+1)}{2}+mv-m^2} q^{-\frac{v(v+1)}{2}-mv+m^2}}{(q^{-m}-1)^\eta (p^m-1)^\eta (\frac{p}{q};\frac{p}{q})_v (\frac{p}{q};\frac{p}{q})_\infty} \tag{11}$$

Proof. As we know that for $n \to \infty$, $h_m(x_{n-m}) \to \infty$ while $h_m(x_{n-m+u}) \to h_m(q^{-m+u} p^{m-u}) \in \mathbb{R}$. For $u > 0$, Corollary 2.5 implies that,

$$\frac{h_m(x_{n-m+u})}{\prod_{o \leq s \leq n-m+v, s \neq n-m+u} (x_{n-m+u} - x_s)} \sim h_m(x_{n-m+u})\, \mathcal{D}(u,v)\, p^{-n(m-u)} q^{n(m-u)}$$

$$= o\left(h_m(x_{n-m}) p^{-nm} q^{nm}\right), \ n \to \infty.$$

Therefore,

$$\sum_{u=0}^{v} \frac{h_m(x_{n-m+u})}{\prod_{o \leq s \leq n-m+v, s \neq n-m+u} (x_{n-m+u} - x_s)} \sim \frac{h_m(x_{n-m})}{\prod_{o \leq s \leq n-m+v, s \neq n-m+u} (x_{n-m} - x_s)}. \tag{12}$$

On substituting $u = 0$ in Corollary 2.5, we get

$$\frac{1}{\prod_{o \leq s \leq n-m+v, s \neq n-m+u} (x_{n-m} - x_s)} \sim \mathcal{D}(0,v)\, p^{-nm} q^{nm}. \tag{13}$$

Finally, by using (9) and (13) into (12), we get our desired result.

Lemma 3.4. *For $m \in \mathbb{N} \cup \{0\}, k \leq n-m-1$ with $\varepsilon > 0$ there exists a positive constant $\mathcal{D} = \mathcal{D}(\varepsilon)$, then the following estimate holds:*

$$\left| D_{k,n}^{p,q} \right| \leq \mathcal{D} p^{-k(m+\epsilon)} q^{k(m+\epsilon)}. \tag{14}$$

Proof. It is assumed that the function h is analytic in and on the contour $\mathcal{L}$ encircling the k distinct nodes $x_0, \ldots, x_k$, then the k-th order divided difference of the function h can be seen as:

$$h[x_0, x_1, \ldots, x_k] = \frac{1}{2\pi i} \oint_{|\tau|=\rho} \frac{h(\tau) d\tau}{(\tau - x_0)(\tau - x_1) \cdots (\tau - x_k)}.$$

For $k \leq n-m-1$, the nodes $x_s = p^{n-s} \frac{[s]_{p,q}}{[n]_{p,q}}$ and the function $h(z) = h_m(z) = \frac{1}{z - p^m q^{-m}}$. Then

$$\begin{aligned} h_m[x_0, x_1, \ldots, x_k] &= \frac{1}{2\pi i} \oint_{|\tau|=\rho} \frac{h_m(\tau)\, d\tau}{\tau(\tau - x_1) \cdots (\tau - x_k)} \\ &= \frac{1}{2\pi i} \oint_{|\tau|=\rho} \frac{h_m(\tau)\, d\tau}{\tau^{k+1}(1 - \frac{x_1}{\tau}) \cdots (1 - \frac{x_k}{\tau})}, \end{aligned} \tag{15}$$

where $\rho = p^{(m+\varepsilon)} q^{-(m+\varepsilon)}, \varepsilon \in (0, 1)$. Here we choose ε in such a way that all the poles $\alpha = p^m q^{-m}$ are outside of $\{z : |z| = \rho\}$ and nodes $x_0, x_1, \ldots, x_k$ are inside the circle $\{z : |z| = \rho\}$. To estimate (14), let us consider

$$\begin{aligned} \left| \left(1 - \frac{x_1}{\tau}\right) \cdots \left(1 - \frac{x_k}{\tau}\right) \right| &\geq \left(1 - \frac{x_1}{\rho}\right) \cdots \left(1 - \frac{x_k}{\rho}\right) \\ &\geq \left(1 - \frac{x_1}{\rho}\right) \cdots \left(1 - \frac{x_{n-(m+1)}}{\rho}\right) \\ &\geq \left(1 - \frac{1}{\rho p^{-(n-1)} q^{n-1}}\right) \cdots \left(1 - \frac{1}{\rho p^{-(m+1)} q^{m+1}}\right) \\ &\geq \left(\frac{p^{m+1}}{\rho q^{m+1}}; \frac{p}{q} \right) = \mathcal{D} \geq 0. \end{aligned} \tag{16}$$

From equations (15) and (16), we get

$$|h_m[x_0, x_1, \ldots, x_k]| \leq \frac{M(h_m; \rho)}{\mathcal{D} \rho^k} =: \mathcal{D}\ (\varepsilon)\, p^{-(m+\varepsilon)k} q^{(m+\varepsilon)k}.$$

Since $h_m(z) = h_m(x)$ for $z = x \in [0, 1]$, the statement follows from the divided difference representation (3).

Lemma 3.5. *For $m \in \mathbb{N}$, $v = 0, 1, \ldots, m$ and $x_k = p^{n-k} \frac{[k]_{p,q}}{[n]_{p,q}}, k = 0, 1, 2, \ldots, n-m+v$, the following asymptomatic relation hold good:*

$$h_m[x_0, x_1, \ldots, x_{n-m+v}] \sim \mathcal{D}\, p^{-(m+\eta)n} q^{(m+\eta)n}, \tag{17}$$

where $\mathcal{D} = \mathcal{D}(v)$ is given by (11).

Proof. By using well-known representation for the divided difference analogue, we have

$$h_m[x_0, x_1, \ldots, x_{n-m+\nu}] = \sum_{r=0}^{n-m+\nu} \frac{h_m(x_r)}{(x_r - x_0) \cdots (x_r - x_{n-m+v})} =: \sum_{r=0}^{n-m-1} + \sum_{r=n-m}^{n-m+v}.$$

For very large n, it can be easily seen that,

$$\sum_{r=0}^{n-m-1} = \frac{1}{2\pi i} \oint_{|\tau|=\rho} \frac{h_m(\tau) d\tau}{(\tau - x_0)(\tau - x_1) \cdots (\tau - x_{n-m+v})}$$

$$= \frac{1}{2\pi i}\oint_{|\tau|=\rho}\frac{h(\tau)d\tau}{\tau^{n-m+v+1}(1-\frac{x_1}{\tau})\cdots(1-\frac{x_{n-m-1}}{\tau})(1-\frac{x_{n-m}}{\tau})\cdots(1-\frac{x_{n-m+v}}{\tau})}$$

where $\rho = p^{(m+\varepsilon)}q^{-(m+\varepsilon)}, \varepsilon \in (0,1)$. By using (15) and taking modulus of denominator, we get

$$\left|\left(1-\frac{x_1}{\tau}\right)\cdots\left(1-\frac{x_{n-m-1}}{\tau}\right)\left(1-\frac{x_{n-m}}{\tau}\right)\cdots\left(1-\frac{x_{n-m+v}}{\tau}\right)\right|$$

$$\geq \left(1-\frac{x_1}{\rho}\right)\cdots\left(1-\frac{x_{n-m-1}}{\rho}\right)\left(1-\frac{x_{n-m}}{\rho}\right)\cdots\left(1-\frac{x_{n-m+v}}{\rho}\right)$$

$$\geq \left(\frac{p^{(m+1)}}{\rho q^{(m+1)}};\frac{p}{q}\right)_\infty \cdot \left(\frac{x_{n-m}}{\rho}-1\right)\cdots\left(\frac{x_{n-m+v}}{\rho}-1\right).$$

Since $x_{n-m} \to p^m q^{-m}$ as $n \to \infty$, it can be seen that $x_{n-m} > p^{(m+\varepsilon)/2}q^{-(m+\varepsilon)/2}$ for n large enough, whence for these values of n, $\left(\frac{x_{n-m}}{\rho}-1\right) \geq q^{\varepsilon/2}-1$.
Now, for remaining factor, it can be seen tat

$$\left(\frac{x_{n-m+r}}{\rho}-1\right) > \left(\frac{x_{n-m}}{\rho}-1\right) \geq q^{\frac{\varepsilon}{2}}-1 > 0.$$

Therefore,

$$\left(\frac{x_{n-m}}{\rho}-1\right)\cdots\left(\frac{x_{n-m+v}}{\rho}-1\right) \geq (q^{\frac{\varepsilon}{2}}-1)^{v+1} =: \mathcal{D}(\varepsilon, v) > 0.$$

Aggregating all the estimates above, we see that

$$\left|\sum_{r=0}^{n-m-1}\right| \leq \frac{M(\rho;h_m)}{\rho^{n-m}\left(\frac{p^{m+1}}{\rho q^{m+1}};\frac{p}{q}\right)_\infty \mathcal{D}(\varepsilon,v)} \tag{18}$$

$$=: \mathcal{D}\, p^{-(m+\varepsilon)n}q^{(m+\varepsilon)n} = o\left(p^{-(m+\varepsilon)n}q^{(m+\varepsilon)n}\right),\ n\to\infty.$$

Also, with the help of Lemma 3.4

$$\left|\sum_{r=n-m}^{n-m+v}\right| \sim \mathcal{D}\, p^{-(m+\eta)n}q^{(m+\eta)n}.$$

where $\mathcal{D} = \mathcal{D}(v)$ is expressed by (11), This completes the proof.

Corollary 3.2. *For $m \in \mathbb{N}$, there exists $\mathcal{D} > 0$ independent of k and n such that the estimation*

$$\left|D_{k,n}^{p,q}\right| \leq \mathcal{D}\, p^{-(m+\eta)n}q^{(m+\eta)n}$$

is valid for all $n \in \mathbb{N}, k = 0,1,2\ldots,n$.

Lemma 3.6. *For $v = 1,2,\ldots,m$, the coefficients of $B_{p,q}^n(h_m;x)$ satisfy the following relation*

$$\lim_{n\to\infty}\frac{D_{n-m+v,n}^{p,q}}{D_{n-m,n}^{p,q}} = (-1)^v \begin{bmatrix} m \\ v \end{bmatrix}_{p,q} p^{\frac{(n-v)(n-v-1)}{2}}q^{\frac{v(v-1)}{2}}.$$

Proof. By using formula (11) and Lemma 2.10, we get

$$\frac{D^{p,q}_{n-m+v,n}}{D^{p,q}_{n-m,n}} = \frac{(-1)^v q^{mv-\frac{v(v+1)}{2}} p^{-mv+\frac{v(v+1)}{2}} \lambda^{p,q}_{n-m+v,n}}{(\frac{p}{q};\frac{p}{q})_v \lambda^{p,q}_{n-m,n}},$$

$$= \frac{vq^{mv}p^{-mv}\left(1-p^{n-m}\frac{[n-m]_{p,q}}{[n]_{p,q}}\right)\cdots\left(1-p^{n-m+v-1}\frac{[n-m]_{p,q}}{[n]_{p,q}}\right)}{(\frac{p}{q};\frac{p}{q})_v},$$

$$\sim \frac{p^{-mv}q^{mv}(-1)^{m-v}(p^{-(m-v)}q^{m-v};p^{-1}q)_v}{(\frac{p}{q};\frac{p}{q})_v q^{m+\cdots+m-v+1}p^{-(m+\cdots+m-v+1)}},$$

$$\to (-1)^v \begin{bmatrix} m \\ v \end{bmatrix}_{p,q} p^{\frac{(n-v)(n-v-1)}{2}} q^{\frac{v(v-1)}{2}} \quad (n\to\infty).$$

Corollary 3.3. *The following relation holds good.*

$$If\ g'_n(x) =: \frac{D_{n-m,n}+\cdots+D_{n-m+\eta,n}x^\eta+\cdots+D_{n,n}x^n}{D_{n-m,n}}$$

$$then \quad \lim_{n\to\infty} g'_n(x) = (x;p,q)_m. \tag{19}$$

Proof. The statement follows from Rothe's Identity

$$(x;p,q)_m = \sum_{v=0}^{m}(-1)^v \begin{bmatrix} m \\ v \end{bmatrix}_{p,q} p^{\frac{(n-v)(n-v-1)}{2}} q^{\frac{v(v-1)}{2}}.$$

4. Proof of the theorems

In this section we prove some results related to approximation of analytic function of sequence of (p, q)-Bernstein operators by using (p, q)-analogue of divided differences on compact disk $\{z : |z| \le \rho\}$ in the complex plane. Concerning the simultaneous approximation, we prove the following:

Proof of Theorem 2.1. (i) Let us consider the complex (p, q)-Bernstein operators as:

$$B^n_{p,q}(h;z) = \sum_{k=0}^{n} h\left(p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}}\right) p_{n,k}(p,q;z),\ n\in\mathbb{N}, z\in\mathbb{D} \tag{20}$$

and the function $h_m(z) = \frac{1}{(z-p^mq^{-m})^\eta}$ analytic in $z \in \mathbb{D}\backslash\{p^mq^{-m}\}$. As $\rho \in (0, p^{(m+\eta)}q^{-(m+\eta)})$ with $|z| \le \rho$, and by using Corollary 2.9, we seen that

$$\left|B^n_{p,q}(h_m;z)\right| \le \sum_{k=0}^{n}\left|D^{p,q}_{k,n}\rho^k\right| \le \mathcal{D}_{p,q,m}\sum_{k=0}^{n}\left(p^{-(m+1)}q^{(m+1)}\rho\right)^k \le \mathcal{D}_{p,q,m}\frac{1}{(1-p^{-(m+1)}q^{(m+1)}\rho)}.$$

Therefore, the (p, q)-Bernstein operators $\{B^n_{p,q}(h_m, z)\}$ are uniformly bounded in $\{z : |z| \le \rho\}$. Also by Lemma 2.3, they converge to the sequence $\{p^lq^{-l}\}^\infty_{l=\eta+1}$ which has a limit point 0 to the function $h_m(z)$ analytic in this disk. Now, by using Vitali's Convergence Theorem, $B^n_{p,q}(h_m;z) \to h_m(z)$ as $n \to \infty$ uniformly on any compact set in $\{z : |z| \le \rho\}$. This completes the proof.

(ii) Here we discuss for 'particular' points $p^{(m-1)}q^{-(m-1)}, p^{(m-2)}q^{-(m-2)}, \ldots, 1$ which can be analyzed in Lemma 2.3 (i). Let $|x| > p^{(m-1)}q^{-(m-1)}$ be different from these values. By Lemma 2.7 one obtains:

$$\left|\sum_{k=0}^{n-m-1}\mathcal{D}^{p,q}_{k,n}x^k\right| \le D_{m,p,q}\sum_{k=0}^{n-m-1}p^{-m(k+\varepsilon)}q^{m(k+\varepsilon)}x^k$$

$$= \mathcal{D}_{m,p,q} \frac{(p^{-(m+\varepsilon)}q^{(m+\varepsilon)}x)^{n-m} - 1}{p^{-(m+\varepsilon)}q^{(m+\varepsilon)}x - 1}$$
$$= o\left((p^{-(m+1)}q^{(m+1)}x)^n\right).$$

Therefore

$$B^n_{p,q}(h_m; x) = \sum_{k=n-m}^{n} D^{p,q}_{k,n} x^k + o\left((p^{-(m+1)}q^{(m+1)}x)^n\right)$$
$$= D^{p,q}_{n-m} x^{n-m} g_n(x) + o\left((p^{-(m+1)}q^{(m+1)}x)^n\right).$$

To calculate the coefficients $D^{p,q}_{n-m,n}$, we see that

$$\lambda^{p,q}_{n-m,n} = (1-x_0)(1-x_1)\cdots(1-x_{n-m-1})$$
$$\geq (1-p^{(m+1)q^{-(m+1)}})\cdots(1-p^{(n-1)q^{-(n-1)}})$$
$$\geq \left(\frac{p^{(m+1)}}{q^{(m+1)}}; \frac{p}{q}\right)_\infty > 0. \tag{21}$$

Therefore $\left|D^{p,q}_{n-m}\right| \to \infty$ as $n \to \infty$ whenever $|x| > p^{(m+1)}q^{-(m+1)}$. Since by (15) $\lim_{n\to\infty} g_n(x) = (x; p, q)_m \neq 0$, when $x \notin \{p^{(m+1)}q^{-(m+1)}, \ldots, 1\}$. This completes the proof.
The next lemma asserts about estimation of coefficient of $B^n_{p,q}$, which is used in the proof of Theorem 2.2.

Lemma 4.1. *If $B^n_{p,q}(h; x) = \sum_{k=0}^{n} D^{p,q}_{k,n} x^k$, and h_0 is defined by (5) with $m = 0$, then we have the following results:*

(i) $0 < \mathcal{D}_1(k+1)^{\eta-1} \leq \left|D^{p,q}_{k,n}\right| \leq \mathcal{D}_2(k+1)^{\eta-1}$ for $k = 0, 1, 2 \ldots, n-1$.
(ii) $0 < \mathcal{D}_1 n^\eta \leq |D^{p,q}_{n,n}| \leq \mathcal{D}_2 n^\eta$ for n large enough.
(iii) $\left|D^{p,q}_{k,n}\right| \leq \left|D^{p,q}_{k+1,n}\right|$ for $k = 0, 1, 2 \ldots, n-2$.

Proof. (i) For $0 \leq k \leq n-1$ and $x_s = p^{n-s}\frac{[s]_{p,q}}{[n]_{p,q}}$, $s = 0, 1, \ldots, k$, $\lambda^{p,q}_{k,n} = \prod_{l=0}^{k-1}(1-x_l)$ and by using (3), and also with the help of residue theorem with $\rho \in (\frac{p}{q}; 1)$

$$D^{p,q}_{k,n} = \frac{\lambda^{p,q}_{k,n}}{2\pi i} \oint_{|\tau\ |=\rho} \frac{h_0(\tau)d\tau}{(\tau - x_0)(\tau - x_1)\cdots(\tau - x_k)}$$
$$= -\lambda^{p,q}_{k,n}\ Res_{|z=1} \left[\frac{h_0(z)dz}{z(z-x_1)\cdots(z-x_k)}\right]$$
$$= \frac{-\lambda^{p,q}_{k,n}}{(\eta-1)!} \lim_{z\to 1} \left[\frac{1}{z(z-x_1)\cdots(z-x_k)}\right]^{(\eta-1)}$$
$$= \frac{(-1)^\eta}{(\eta-1)!} \sum_{s_0+\cdots+s_k=\eta-1} \binom{\eta-1}{s_0,\ldots,s_k} \frac{1}{(1-x_0)^{s_0}\cdots(1-x_{k-1})^{s_{k-1}}(1-x_k)^{1+s_k}} \tag{22}$$

Therefore, we have

$$\left|D^{p,q}_{k,n}\right| \geq \frac{1}{(\eta-1)!}(k+1)^\eta.$$

Now,

$$\left|D^{p,q}_{k,n}\right| \leq \frac{1}{(\eta-1)!}. \sum_{s_0+\cdots+s_k=\eta-1} \binom{\eta-1}{s_0,\ldots,s_k} \frac{1}{(1-x_{x_{n-1}})^\eta} \leq \frac{1}{(\eta-1)!}\left(\frac{q}{q-p}\right)^\eta (k+1)^\eta.$$

(ii). To calculate the coefficients $D_{n,n}^{p,q}$, we use end-point interpolation property (2), such that

$$B_{p,q}^{n}(h_0;1)=\sum_{k=0}^{n} D_{k,n}^{p,q}=h_0(1)=b$$

whence $D_{n,n}^{p,q}=b-\sum_{k=0}^{n-1}D_{n,n}^{p,q}$ and, since all $D_{n,n}^{p,q}, k=0,1\cdots n-1$ are of the same sign, we get

$$\begin{aligned} D_{n,n}^{p,q} &\geq \sum_{k=0}^{n-1}\left|D_{n,n}^{p,q}\right|-|b| \geq \frac{1}{(\eta-1)!}\sum_{k=0}^{n-1}(k+1)^{\eta-1}-|b| \\ &\geq \frac{1}{(\eta-1)!}\int_0^n x^{\eta-1}dx=\frac{n^{\eta}}{\eta!}-|b|\geq \mathcal{D}_1 n^{\eta}>0, \end{aligned}$$

for vary large value of n. Now we have,

$$\left|D_{n,n}^{p,q}\right| \leq |b|+\sum_{k=0}^{n-1}\left|D_{n,n}^{p,q}\right| \leq |b|\,\mathcal{D}\sum_{k=0}^{n-1}(k+1)^{\eta-1}\leq |b|+n.(n)^{\eta-1}\leq \mathcal{D}_2 n^{\eta}.$$

(iii) By using (22), it can be easily seen that

$$\begin{aligned} (\eta-1)!\left|D_{k+1,n}^{p,q}\right| &= \sum_{s_0+\cdots+s_k=\eta-1}\binom{\eta-1}{s_0,...,s_k}\frac{1}{(1-x_0)^{s_0}\cdots(1-x_{k-1})^{s_{k-1}}(1-x_k)^{1+s_k}} \\ &= \sum_{s_{k+1}=0}+\sum_{s_{k+1}\neq 0}\geq \sum_{s_{k+1}=0} \\ &= \sum_{s_0+\cdots+s_k=\eta-1}\binom{\eta-1}{s_0,...,s_k}\frac{1}{(1-x_0)^{s_0}\cdots(1-x_{k-1})^{s_{k-1}}(1-x_k)^{s_k}(1-x_{k+1})} \\ &= \sum_{s_0+\cdots+s_k=\eta-1}\binom{\eta-1}{s_0,...,+s_k}\frac{1}{(1-x_0)^{s_0}\cdots(1-x_{k-1})^{s_{k-1}}(1-x_k)^{1+s_k}}.\frac{1-x_k}{1-x_{k+1}} \\ &\geq \sum_{s_0+\cdots+s_k=\eta-1}\binom{\eta-1}{s_0,...,s_k}\frac{1}{(1-x_0)^{s_0}\cdots(1-x_{k-1})^{s_{k-1}}(1-x_k)^{1+s_k}} \\ &= (\eta-1)!\left|D_{k,n}^{p,q}\right|,\quad as\ \frac{1-x_k}{1-x_{k+1}}>1. \end{aligned}$$

Proof of Theorem 2.2. (i) We know that $B_{p,q}^{n}(h_0;1)=h_0(1)$. Let us consider the complex (p,q)-Bernstein operators (20). As for any $\rho\in(0,1)$ and $|z|\leq\rho$, using Lemma 3.1 (i), we obtain

$$\left|B_{p,q}^{n}(h_0;z)\right|=\left|\sum_{k=0}^{n}D_{k,n}^{p,q}z^k\right|\leq \mathcal{D}\sum_{k=0}^{n}(k+1)^{\eta}\rho^k=:\mathcal{D}<\infty.$$

Therefore, the sequence of operators $\{B_{p,q}^{n}(h;z)\}$ is uniformly bounded in any disk $\{z:|z|\leq\rho\}$. Again using Lemma 3.1 (ii) and with the help of Vitali's Convergence Theorem, we get the required result.

(ii). For $|x|\geq 1$ and using Abel's inequality, we have

$$\left|\sum_{k=0}^{n-1}D_{k,n}^{p,q}x^k\right|\leq \frac{|x|^n-1}{|x|-1}(D_{0,n}^{p,q}+2D_{n-1,n}^{p,q})\leq \mathcal{D}n^{\eta-1}|x|^n,\ \mathcal{D}=\mathcal{D}(x).$$

Also, $|D_{n,n}^{p,q}|\geq \mathcal{D}n^{\eta}|x|^n$ by Lemma 3.1 (ii). Therefore

$$\left|B_{p,q}^{n}(h_0;x)\right|\geq \mathcal{D}\,(n^{\eta}-n^{\eta-1})\,|x|^n\to\infty\ as\ n\to\infty.$$

For $x=-1$, we have

$$B^n_{p,q}(h_0;-1)=\sum_{k=0}^{n-1} D^{p,q}_{k,n}(-1)^k + D^{p,q}_{n,n}(-1)^n.$$

Again applying Abel's inequality, we obtain

$$\left|\sum_{k=0}^{n-1} D^{p,q}_{k,n}(-1)^k\right| \leq \left|D^{p,q}_{0,n}\right| + 2\left|D^{p,q}_{n-1,n}\right| \leq \mathcal{D}n^{\eta-1}.$$

Moreover, $|D^{p,q}_{n,n}| \geq \mathcal{D}n^j$ lead to $\left|B^n_{p,q}(h_0;-1)\right| \geq \mathcal{D}(n^\eta - n^{\eta-1}) \to \infty$ as $n\to\infty$.

5. Conclusions

In this paper, we have studied the approximation results of (p,q)-Bernstein operators $B^n_{p,q}(h;x)$ to a rational function for $q>p>1$ and investigated convergence properties of $B^n_{p,q}(h;x)$ for the function $h(x)=(x-p^mq^{-m})^{-\eta}$ with $\eta>2$. We observed that the approximation properties for the (p,q)-Bernstein operators are more precise in nature than the previously obtained results for q-Bernstein operators.

References

[1] Acar, T., Aral, A. and Mohiuddine, S. A. (2018), Approximation by bivariate (p, q)-Bernstein-Kantorovich operators, Iran. J. Sci. Technol., Trans. A, Sci., 42, pp. 655-662.
[2] Acar, T., Aral, A. and Mohiuddine, S.A., (2016), On Kantorovich modification of (p,q)-Baskakov operators, J. Inequal. Appl., 98, pp. 1-14.
[3] Acar, T., Mohiuddine, S.A. and Mursaleen, M. (2018), Approximation by (p,q)-Baskakov-Durrmeyer-Stancu operators, Complex. Anal. Oper. Th., 12, pp. 1453-1468.
[4] Acar, T., Aral, A. and Mohiuddine, S.A., (2018), On Kantorovich modification of (p,q)-Bernstein operators, Iran. J. Sci. Technol., Trans. A, Sci., 42, pp. 1459-1464.
[5] Anderson, J.D., Campbell, J.K., Ekelund, J.E., Ellis, J. and Jordan, J.F., (2008), Phys. Rev. Lett., 100, 091102.
[6] Bohman, H., (1952), On approximation of continuous and analytic functions, Ark. Math., 2, pp. 43–56.
[7] Burban, I., (1995), Two-parameter deformation of the oscillator algebra and (p,q)-analogue of two dimensional conformal field theory, Nonlinear Math. Phys., 2 (3-4), pp. 384-391.
[8] Erdelyi, A., (1956), Asymptotic expansions, Dover publications, New York.
[9] Gal, S.G.,(2012), Approximation by complex q-Lorentz polynomial, $q>1$, Mathematica (Cluj), 54, pp. 53-63.
[10] Korovkin, P.P., (1953), Convergence of linear positive operator in the space of continuous function, Dokl. Akad. Nauk. Russian. SSSR (N.S.), 90, 961–964.
[11] Lorentz, G.G., (1986) Bernstein Polynomials, 2nd edn. Chelsea Publication, New York.
[12] Lupas, A., (1987), A q-analogue of the Bernstein operators, University of Cluj-Napoca seminar on numerical and statistical calculus, Univ Babeş-Bolyai Cluj-Napoca, pp. 85-92.
[13] Mahmudov, N.I., (2010), Convergence properties and iterations for $q-$Stancu polynomials in compact disks, Comput. Math. Appl., 59, pp. 3763-3769.
[14] Mahmudov, N.I. and Kara, M., (2012), Approximation theorems for generalized complex Kantorovich-type operators, J. Appl. Math., Article ID 454579, pp. 1-14.
[15] Mohiuddine, S. A., Acar, T. and Alghamdi, M.A., (2018), Genuine modified Bernstein-Durrmeyer operators, J. Inequal. App., 104, pp. 1-13
[16] Mohiuddine, S. A., Acar, T. and Alotaibi, A. (2017), Construction of a new family of Bernstein-Kantorovich operators, Math. Method. Appl. Sci., 40, pp. 7749-7759.
[17] Mohiuddine, S. A., Acar, T. and Alotaibi, A., (2018), Durrmeyer type (p,q)-Baskakov operators preserving linear functions, J. Math. Inequal., 12, pp. 961-973.
[18] Mursaleen, M., Ansari, K.J. and Khan, A., (2015), On (p,q)-analogue of Bernstein operators, Appl. Math. Comput., 266, pp. 874-882. [Erratum: Appl. Math. Comput., 278 (2016) 70–71.
[19] Mursaleen, M., Nasiruzzaman, Md., Khan, A. and Ansari, K.J., (2016), Some Approximation result on Bleimann-Butzar-Hahn operator defined by (p,q)-integer, Filomat, 30, pp. 639-648.

[20] Mursaleen, M., Nasiruzzaman, Md., Khan, F. and Khan, A., (2017), On (p,q)-analogue of divided difference and Bernstein operators, J. Nonlinear. Funct. Anal., 2017, Art. Id. 25, pp. 1-13.
[21] Mursaleen, M., Nasiruzzaman, Md. and Nurgali, A., (2015), Some Approximation results on Bernstein-Schurer operator defined by (p,q)-integers, J. Inequal. Appl., 249, pp. 1-12.
[22] Mursaleen, M., Khan, F. and Khan, A., (2016), Approximation by (p,q)-Lorentz polynomial on a compact disk, Complex Anal. Oper. Theory, 10, pp. 1725-1740.
[23] Ostrovska, S., (2008), q-Bernstein polynomial of the Cauchy kernel, Appl. Math. Comput. 198 (1), pp. 261-270.
[24] Ostrovska, S., (2003), q-Bernstein polynomial and their iterates, Jour. Approx. Theory, 123, pp. 232-255.
[25] Ostrovaska, S. and Özban, A.Y., (2013), On the q-Bernstein polynomial of unbounded function with $q > 1$, Abstr. Appl. Anal. Art. Id. 349156, pp. 1-7.
[26] Phillips, G.M., (1997), Bernstein polynomials based on the q-integers, Ann. Numer. Math. 4, pp. 511-518.
[27] Sadjang, P.N., (2018), On the fundamental theorem of (p,q)-calculus and some (p,q)-Taylor formulas, Results Math., 73:39, pp. 1-21.

Faisal Khan is an assistant professor in the Department of Mathematics, Aligarh Muslim University, Aligarh, India. He has completed his MSc and PhD degree from Department of Mathematics, Aligarh Muslim University, Aligarh. He has published more than 35 research articles in various reputed journals. His research interest include Mathematical Analysis, Functional Analysis, Approximation theory, Operator theory, Fourier Analysis.

Mohd Saif is currently enrolled in PhD in the faculty of Engineering and Technology, Department of Applied Mathematics, Aligarh Muslim University, Aligarh, India. He persuaded his Master in Mathematics from Department of Mathematics, Aligarh Muslim University, Aligarh. He has 5 research article in his name published in reputed journals. His research interest are in the area of Approximation theory, Special Function, Integral Transforms.

M. Mursaleen is a presently Principal Investigator, SERB Core Research Grant . His research interests are in the areas of pure and applied mathematics including Approximation Theory, Summability Theory, Operator Theory, Fixed Point Theory, Differential and Integral Equations. He has published about 340 research papers in reputed international journals and eight books. He is member of several scientific committees, advisory boards as well as member of editorial board of a number of scientific journals. He has visited a number of foreign universities and institutions as a visiting scientist/ visiting professor. Recently he has been recognized as Highly Cited Researcher 2019 by Web of Science.