TWMS J. App. and Eng. Math. V.10, Special Issue, 2020, pp. 73-84

SUBLINEAR OPERATORS WITH ROUGH KERNEL GENERATED BY FRACTIONAL INTEGRALS AND COMMUTATORS ON GENERALIZED VANISHING LOCAL MORREY SPACES

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ABSTRACT. In this work, we consider the norm inequalities for sublinear operators with rough kernel generated by fractional integrals and commutators on generalized vanishing local Morrey spaces including their weak versions under generic size conditions which are satisfied by most of the operators in harmonic analysis, respectively.

Keywords: Sublinear operator, Fractional integral operator, Rough kernel, Generalized vanishing local Morrey space, Commutator, Local Campanato space.

AMS Subject Classification: 42B20, 42B25, 42B35.

1. INTRODUCTION

The classical fractional integral which is also known as Riesz potential was introduced by Riesz in 1949 [13], defined by

$$\begin{aligned} I_\alpha f(x) &= (-\Delta)^{-\frac{\alpha}{2}} f(x) \qquad 0<\alpha<n, \\ &= \frac{1}{\gamma(\alpha)} \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \end{aligned}$$

with

$$\gamma(\alpha) = \frac{\pi^{\frac{n}{2}} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n}{2}-\frac{\alpha}{2}\right)},$$

where $\Gamma(\cdot)$ is the standard gamma function, that is, $\Gamma(n) = \int\limits_0^\infty t^{n-1} e^{-t} dt$ $(n>0)$ and I_α plays an important role in partial diferential equation as the inverse of power of Laplace operator. Especially, its most significant feature is that I_α maps $L_p(\mathbb{R}^n)$ continuously into $L_q(\mathbb{R}^n)$, with $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$ and $1<p<\frac{n}{\alpha}$, through the well known Hardy-Littlewood-Sobolev imbedding theorem (see pp. 119-121, Theorem 1 and its proof in [17]) for I_α. Throughout the paper we assume that $x\in\mathbb{R}^n$ and $r>0$ and also let $B(x,r)$ denotes the open ball centered at x of radius r, $B^C(x,r)$ denotes its complement and $|B(x,r)|$ is the Lebesgue

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§ Manuscript received: August 26, 2019; accepted: January 15, 2020.

measure of the ball $B(x,r)$ and $|B(x,r)| = v_n r^n$, where $v_n = |B(0,1)|$. On the other hand, let $f \in L_1^{loc}(\mathbb{R}^n)$. The fractional maximal operator M_α is defined by

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int\limits_{B(x,t)} |f(y)|dy, \qquad 0 \le \alpha < n.$$

It is well known that M_α and I_α play an important role in harmonic analysis (see [18]).

Suppose that S^{n-1} is the unit sphere on $\mathbb{R}^n$ $(n \ge 2)$ equipped with the normalized Lebesgue measure $d\sigma$. Let $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$, $\Omega(\mu x) = \Omega(x)$ for any $\mu > 0$, $x \in \mathbb{R}^n \setminus \{0\}$ and satisfy the cancellation condition

$$\int\limits_{S^{n-1}} \Omega(x')d\sigma(x') = 0,$$

where $x' = \frac{x}{|x|}$ for any $x \neq 0$.

Let $\Omega \in L_s(S^{n-1})$ with $1 < s \le \infty$ be homogeneous of degree zero. We define $s' = \frac{s}{s-1}$ for any $s > 1$. Suppose that $T_{\Omega,\alpha}$, $\alpha \in (0,n)$ represents a linear or a sublinear operator, which satisfies that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin supp f$

$$|T_{\Omega,\alpha} f(x)| \le c_0 \int\limits_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)|\, dy, \tag{1}$$

where c_0 is independent of f and x.

We point out that the condition (1) in the case of $\Omega \equiv 1$, $\alpha = 0$ has been introduced by Soria and Weiss in [16]. The condition (1) is satisfied by many interesting operators in harmonic analysis, such as fractional maximal operator, fractional integral operator(Riesz potential), fractional Marcinkiewicz operator and so on (see [16] for details).

We first recall the definitions of rough fractional integral operator $I_{\Omega,\alpha}$ and a related rough fractional maximal operator $M_{\Omega,\alpha}$ and their commutators as follows:

Definition 1.1. *Define*

$$I_{\Omega,\alpha} f(x) = \int\limits_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)dy \qquad 0 < \alpha < n,$$

$$M_{\Omega,\alpha} f(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int\limits_{|x-y|<r} |\Omega(x-y)|\, |f(y)|\, dy \qquad 0 < \alpha < n,$$

$$\begin{aligned}[b, I_{\Omega,\alpha}] f(x) &= b(x)\, I_{\Omega,\alpha} f(x) - I_{\Omega,\alpha}(bf)(x) \\ &= \int\limits_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} (b(x) - b(y))\, f(y)dy \qquad 0 < \alpha < n, b \in L_1^{loc}(\mathbb{R}^n)\end{aligned}$$

and

$$\begin{aligned}[b, M_{\Omega,\alpha}] f(x) &= b(x)\, M_{\Omega,\alpha} f(x) - M_{\Omega,\alpha}(bf)(x) \\ &= \sup_{r>0} \frac{1}{r^{n-\alpha}} \int\limits_{|x-y|<r} |\Omega(x-y)|\, |b(x) - b(y)|\, |f(y)|\, dy \qquad 0 < \alpha < n, b \in L_1^{loc}(\mathbb{R}^n).\end{aligned}$$

In 1938, Morrey [11] considered regularity of the solution of elliptic partial differential equations(PDEs) in terms of the solutions themselves and their derivatives. Later, there are many applications of Morrey space to the Navier-Stokes equations (see [10]), the

Schrödinger equations (see [14]) and the elliptic problems with discontinuous coefficients (see [2]).

Its definition is defined as follows:

Definition 1.2. *For $0 < q \le p < \infty$, the Morrey space $M_q^p(\mathbb{R}^n)$ is the collection of all measurable functions f whose Morrey space norm is*

$$\|f\|_{M_q^p(\mathbb{R}^n)} = \sup_{\substack{Q\subset\mathbb{R}^n\\ Q:Cubes}} |Q|^{\frac{1}{p}-\frac{1}{q}} \|f\chi_Q\|_{L_q(\mathbb{R}^n)} < \infty.$$

In order to illustrate the relationship of the Lebesgue space, the classical Morrey space, we give the following remark.

Remark 1.1. *Obviously, the Morrey space is the generalization of the Lebesgue space that can be seen from the special case $M_q^q(\mathbb{R}^n) = L_q(\mathbb{R}^n)$ with $1 \le q < \infty$.*

Here, we would like to mention that in many research papers, such as in [8, 9], the Morrey space is defined in another way.

Definition 1.3. *Let $0 \le \lambda \le n$ and $1 \le q < \infty$. Then for $f \in L_q^{loc}(\mathbb{R}^n)$ and any cube $B = B(x,r)$, the Morrey space $L_{q,\lambda}(\mathbb{R}^n)$ is defined by*

$$\|f\|_{L_{q,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n}\sup_{r>0} r^{-\frac{\lambda}{q}} \|f\|_{L_q(B)} \equiv \sup_B r^{-\frac{\lambda}{q}} \|f\|_{L_q(B)} < \infty.$$

Corollary 1.1. [9] *Recall that $0 < q \le p < \infty$ and $0 \le \lambda \le n$. By checking the definitions of $M_q^p(\mathbb{R}^n)$ and $L_{q,\lambda}(\mathbb{R}^n)$, it is easy to see that if we take $\lambda = \left(1-\frac{q}{p}\right)n \in [0,n]$, then $L_{q,\left(1-\frac{q}{p}\right)n}(\mathbb{R}^n) = M_q^p(\mathbb{R}^n)$. Moreover, if we choose $p = \frac{qn}{n-\lambda} \le q$, $M_q^{\frac{qn}{n-\lambda}}(\mathbb{R}^n) = L_{q,\lambda}(\mathbb{R}^n)$. Thus, we conclude that $M_q^p(\mathbb{R}^n)$ is equivalent to $L_{q,\lambda}(\mathbb{R}^n)$.*

The study of the operators of harmonic analysis in vanishing Morrey space, in fact has been almost not touched. A version of the classical Morrey space $L_{p,\lambda}(\mathbb{R}^n)$ where it is possible to approximate by "nice" functions is the so called vanishing Morrey space $VL_{p,\lambda}(\mathbb{R}^n)$ has been introduced by Vitanza in [19] and has been applied there to obtain a regularity result for elliptic PDEs. This is a subspace of functions in $L_{p,\lambda}(\mathbb{R}^n)$, which satisfies the condition

$$\lim_{r\to 0^+} \sup_{\substack{x\in\mathbb{R}^n\\ 0<t<r}} t^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,t))} = 0.$$

For the properties and applications of vanishing Morrey spaces, see also [3]. Then, Gürbüz [8] has considered the boundedness of a class of sublinear operators and their commutators by with rough kernels associated with Calderón-Zygmund operator, Hard-Littlewood maximal operator, fractional integral operator, fractional maximal operator by with rough kernels both on vanishing generalized Morrey spaces and vanishing Morrey spaces, respectively.

In this work, extending the definition of vanishing Morrey spaces [19] and vanishing generalized Morrey spaces [8], the author introduces the generalized vanishing local Morrey spaces $VLM_{p,\varphi}^{\{x_0\}}$, including their weak versions and studies the boundedness of the sublinear operators with rough kernel generated by fractional integrals and commutators in these spaces. These conditions are satisfied by most of the operators in harmonic analysis, such as fractional maximal operator, fractional integral operator(Riesz potential), fractional Marcinkiewicz operator and so on. In all the cases the conditions for the boundedness of $T_{\Omega,\alpha}$ and $T_{\Omega,b,\alpha}$ are given in terms of Zygmund-type integral inequalities on (φ_1, φ_2), where there is no assumption on monotonicity of φ_1, φ_2 in r.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Generalized vanishing local Morrey spaces

Recall that the generalized local Morrey space $LM_{p,\varphi}^{\{x_0\}}$ has been defined by Gürbüz in [1, 7, 9].

Definition 2.1. ***(generalized local Morrey space)*** *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \leq p < \infty$. For any fixed $x_0 \in \mathbb{R}^n$ we denote by $LM_{p,\varphi}^{\{x_0\}} \equiv LM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ the generalized local Morrey space, the space of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{LM_{p,\varphi}^{\{x_0\}}} = \sup_{r>0} \varphi(x_0,r)^{-1} |B(x_0,r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x_0,r))} < \infty.$$

Also by $WLM_{p,\varphi}^{\{x_0\}} \equiv WLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ we denote the weak generalized local Morrey space of all functions $f \in WL_p^{loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WLM_{p,\varphi}^{\{x_0\}}} = \sup_{r>0} \varphi(x_0,r)^{-1} |B(x_0,r)|^{-\frac{1}{p}} \|f\|_{WL_p(B(x_0,r))} < \infty.$$

According to this definition, we recover the local Morrey space $LL_{p,\lambda}^{\{x_0\}}$ and the weak local Morrey space $WLL_{p,\lambda}^{\{x_0\}}$ under the choice $\varphi(x_0,r) = r^{\frac{\lambda-n}{p}}$:

$$LL_{p,\lambda}^{\{x_0\}} = LM_{p,\varphi}^{\{x_0\}} \Big|_{\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}}, \quad WLL_{p,\lambda}^{\{x_0\}} = WLM_{p,\varphi}^{\{x_0\}} \Big|_{\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}}.$$

The main goal of [1, 7, 9] is to give some sufficient conditions for the boundedness of a large class of rough sublinear operators and their commutators on the generalized local Morrey space $LM_{p,\varphi}^{\{x_0\}}$. For the properties and applications of generalized local Morrey spaces $LM_{p,\varphi}^{\{x_0\}}$, see also [1, 7, 9].

For brevity, in the sequel we use the notations

$$\mathfrak{M}_{p,\varphi}(f;x_0,r) := \frac{|B(x_0,r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x_0,r))}}{\varphi(x_0,r)}$$

and

$$\mathfrak{M}_{p,\varphi}^{W}(f;x_0,r) := \frac{|B(x_0,r)|^{-\frac{1}{p}} \|f\|_{WL_p(B(x_0,r))}}{\varphi(x_0,r)}.$$

Extending the definition of vanishing generalized Morrey spaces [8] to the case of generalized local Morrey spaces, we introduce the following definitions.

Definition 2.2. ***(generalized vanishing local Morrey space)*** *The generalized vanishing local Morrey space $VLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in LM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ such that*

$$\lim_{r\to 0^+} \sup_{r>0} \mathfrak{M}_{p,\varphi}(f;x_0,r) = 0. \quad (2)$$

Definition 2.3. ***(weak generalized vanishing local Morrey space)*** *The weak generalized vanishing local Morrey space $WVLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in WLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ such that*

$$\lim_{r\to 0^+} \sup_{r>0} \mathfrak{M}_{p,\varphi}^{W}(f;x_0,r) = 0. \tag{3}$$

Everywhere in the sequel we assume that

$$\lim_{r\to 0^+} \frac{r^{\frac{n}{p}}}{\inf\limits_{r>0} \varphi(x_0,r)} = 0, \tag{4}$$

and

$$\sup_{0<r<\infty} \frac{r^{\frac{n}{p}}}{\inf\limits_{r>0} \varphi(x_0,r)} < \infty, \tag{5}$$

which make the spaces $VLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ non-trivial, because bounded functions with compact support belong to this space. The spaces $VLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ and $WVLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ are Banach spaces with respect to the norm

$$\|f\|_{VLM_{p,\varphi}^{\{x_0\}}} \equiv \|f\|_{LM_{p,\varphi}^{\{x_0\}}} = \sup_{r>0} \mathfrak{M}_{p,\varphi}(f;x_0,r), \tag{6}$$

$$\|f\|_{WVLM_{p,\varphi}^{\{x_0\}}} = \|f\|_{WLM_{p,\varphi}^{\{x_0\}}} = \sup_{r>0} \mathfrak{M}_{p,\varphi}^{W}(f;x_0,r), \tag{7}$$

respectively. The spaces $VLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ and $WVLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ are closed subspaces of the Banach spaces $LM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ and $WLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$, respectively, which may be shown by standard means.

3. Sublinear operators with rough kernel $T_{\Omega,\alpha}$ on the spaces $VLM_{p,\varphi}^{\{x_0\}}$

In this section, we will prove the boundedness of the operator $T_{\Omega,\alpha}$ satisfying (1) on the generalized vanishing local Morrey spaces $VLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$, including their weak versions.

Now using results in [1], we get the boundedness of the operator $T_{\Omega,\alpha}$ on the generalized vanishing local Morrey spaces $VLM_{p,\varphi}^{\{x_0\}}$.

Theorem 3.1. *(Our main result) Let $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$, be homogeneous of degree zero. Let $0 < \alpha < n$, $1 \le p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Let $T_{\Omega,\alpha}$ be a sublinear operator satisfying condition (1), bounded on $L_p(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$. Let for $s' \le p$, $p \ne 1$, the pair (φ_1, φ_2) satisfies conditions (4)-(5) and*

$$C_\psi := \int_\psi^\infty \sup_{t>0} \varphi_1(x_0,t) \frac{t^{\frac{n}{p}}}{t^{\frac{n}{q}+1}} dt < \infty \tag{8}$$

for every $\psi > 0$, and

$$\int_r^\infty \varphi_1(x_0,t) \frac{t^{\frac{n}{p}}}{t^{\frac{n}{q}+1}} dt \le C_0 \varphi_2(x_0,r), \tag{9}$$

and for $q < s$ the pair (φ_1, φ_2) satisfies conditions (4)-(5) and also

$$C_{\psi'} := \int_{\psi'}^\infty \sup_{t>0} \varphi_1(x_0,t) \frac{t^{\frac{n}{p}}}{t^{\frac{n}{q}-\frac{n}{s}+1}} dt < \infty \tag{10}$$

for every $\psi' > 0$*, and*

$$\int_r^{\infty} \varphi_1(x_0,t) \frac{t^{\frac{n}{p}}}{t^{\frac{n}{q}-\frac{n}{s}+1}} dt \leq C_0 \varphi_2(x_0,r) r^{\frac{n}{s}}, \tag{11}$$

where C_0 *does not depend on* $r > 0$.

Then the operator $T_{\Omega,\alpha}$ *is bounded from* $VLM^{\{x_0\}}_{p,\varphi_1}$ *to* $VLM^{\{x_0\}}_{q,\varphi_2}$ *for* $p > 1$ *and from* $VLM^{\{x_0\}}_{1,\varphi_1}$ *to* $WVLM^{\{x_0\}}_{q,\varphi_2}$ *for* $p = 1$*. Moreover, we have for* $p > 1$

$$\|T_{\Omega,\alpha} f\|_{VLM^{\{x_0\}}_{q,\varphi_2}} \lesssim \|f\|_{VLM^{\{x_0\}}_{p,\varphi_1}}, \tag{12}$$

and for $p = 1$

$$\|T_{\Omega,\alpha} f\|_{WVLM^{\{x_0\}}_{q,\varphi_2}} \lesssim \|f\|_{VLM^{\{x_0\}}_{1,\varphi_1}}. \tag{13}$$

Proof. The estimation of (12) follows from Lemma 2 in [1] and (9). That is,

$$\begin{aligned}
\|T_{\Omega,\alpha} f\|_{VLM^{\{x_0\}}_{q,\varphi_2}} &= \sup_{r>0} \frac{|B(x_0,r)|^{-\frac{1}{q}} \|T_{\Omega,\alpha} f\|_{L_q(B(x_0,r))}}{\varphi_2(x_0,r)} \\
&\lesssim \sup_{r>0} \frac{1}{\varphi_2(x_0,r)} \int_r^{\infty} \varphi_1(x_0,t) t^{\frac{n}{p}} \left[\frac{\|f\|_{L_p(B(x_0,t))}}{\varphi_1(x_0,t) t^{\frac{n}{p}}} \right] \frac{dt}{t^{\frac{n}{q}+1}} \\
&\lesssim \|f\|_{VLM^{\{x_0\}}_{p,\varphi_1}} \sup_{r>0} \frac{1}{\varphi_2(x_0,r)} \int_r^{\infty} \varphi_1(x_0,t) \frac{t^{\frac{n}{p}}}{t^{\frac{n}{q}+1}} dt \\
&\lesssim \|f\|_{VLM^{\{x_0\}}_{p,\varphi_1}}.
\end{aligned}$$

Now, let us show that

$$\lim_{r\to 0^+} \sup_{r>0} \mathfrak{M}_{p,\varphi_1}(f;x_0,r) = 0 \text{ implies } \lim_{r\to 0^+} \sup_{r>0} \mathfrak{M}_{q,\varphi_2}(T_{\Omega,\alpha} f;x_0,r) = 0. \tag{14}$$

Indeed, for $0 < r < \psi_0 < \infty$, by Lemma 2 in [1], we have

$$\frac{r^{-\frac{n}{q}} \|T_{\Omega,\alpha} f\|_{L_q(B(x_0,r))}}{\varphi_2(x_0,r)} \leq C \left[\mathcal{F}_{\psi_0}(x_0,r) + \mathcal{G}_{\psi_0}(x_0,r) \right], \tag{15}$$

where

$$\mathcal{F}_{\psi_0}(x_0,r) := \frac{r^{\frac{n}{q}}}{\varphi_2(x_0,r)} \int_r^{\psi_0} \frac{\varphi_1(x_0,t) t^{\frac{n}{p}}}{t^{\frac{n}{q}+1}} \left[\sup_{0<r<t} \frac{\|f\|_{L_p(B(x_0,r))}}{\varphi_1(x_0,r) r^{\frac{n}{p}}} \right] dt$$

and

$$\mathcal{G}_{\psi_0}(x_0,r) := \frac{r^{\frac{n}{q}}}{\varphi_2(x_0,r)} \int_{\psi_0}^{\infty} \frac{\varphi_1(x_0,t) t^{\frac{n}{p}}}{t^{\frac{n}{q}+1}} \left[\sup_{0<r<t} \frac{\|f\|_{L_p(B(x_0,r))}}{\varphi_1(x_0,r) r^{\frac{n}{p}}} \right] dt.$$

For any $\epsilon > 0$, now we can select any fixed $\psi_0 > 0$ such that whenever $r \in (0,\psi_0)$,

$$\sup_{r>0} \sup_{0<r<\psi_0} \frac{\|f\|_{L_p(B(x_0,r))}}{\varphi_1(x_0,r) r^{\frac{n}{p}}} < \frac{\epsilon}{2CC_0},$$

where C_0 and C are constants from (11) and (15), which is possible since $f \in VLM^{\{x_0\}}_{p,\varphi_1}$.

This allows to guess the first term properly from the type $r \in (0,\psi_0)$ such that

$$\sup_{r>0} C\mathcal{F}_{\psi_0}(x_0,r) < \frac{\epsilon}{2}.$$

For the second term, in view of (8) and by choosing r small enough, we obtain

$$\mathcal{G}_{\psi_0}(x_0,r) \leq C_{\psi_0} \frac{r^{\frac{n}{q}}}{\varphi_2(x_0,r)} \|f\|_{VLM^{\{x_0\}}_{p,\varphi_1}},$$

where C_{ψ_0} is the constant from (8). Since φ_2 satisfies conditions (4)-(5), it is sufficient to choose r small enough such that

$$\sup_{r>0} \frac{r^n}{\varphi_2^q(x_0,r)} \leq \left(\frac{\epsilon}{2CC_{\psi_0} \|f\|_{VLM^{\{x_0\}}_{p,\varphi_1}}} \right)^q.$$

Hence,

$$\sup_{r>0} C\mathcal{G}_{\psi_0}(x_0,r) < \frac{\epsilon}{2}.$$

As a result,

$$\frac{r^{-\frac{n}{q}}}{\varphi_2(x_0,r)} \|T_{\Omega,\alpha} f\|_{L_q(B(x_0,r))} < \epsilon,$$

which means that

$$\lim_{r\to 0^+} \sup_{r>0} \frac{r^{-\frac{n}{q}}}{\varphi_2(x_0,r)} \|T_{\Omega,\alpha} f\|_{L_q(B(x_0,r))} = 0,$$

which completes the proof of (12).

The proof of (13) is similar to the proof of (12). For the case of $q < s$, we can also use the same method, so we omit the details. □

Corollary 3.1. *Let $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, be homogeneous of degree zero. Let $0 < \alpha < n$, $1 \leq p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Let also for $s' \leq p$, $p \neq 1$, the pair (φ_1, φ_2) satisfies conditions (4)-(5) and (8)-(9) and for $q < s$ the pair (φ_1, φ_2) satisfies conditions (4)-(5) and (10)-(11). Then the operators $M_{\Omega,\alpha}$ and $I_{\Omega,\alpha}$ are bounded from $VLM^{\{x_0\}}_{p,\varphi_1}$ to $VLM^{\{x_0\}}_{q,\varphi_2}$ for $p > 1$ and from $VLM^{\{x_0\}}_{1,\varphi_1}$ to $WVLM^{\{x_0\}}_{q,\varphi_2}$ for $p = 1$, respectively.*

In the case of $q = \infty$ by Theorem 3.1, we get

Corollary 3.2. *Let $x_0 \in \mathbb{R}^n$,$1 \leq p < \infty$ and the pair (φ_1, φ_2) satisfies conditions (4)-(5) and (8)-(9). Then the operators M_α and I_α are bounded from $VLM^{\{x_0\}}_{p,\varphi_1}$ to $VLM^{\{x_0\}}_{q,\varphi_2}$ for $p > 1$ and from $VLM^{\{x_0\}}_{1,\varphi_1}$ to $WVLM^{\{x_0\}}_{q,\varphi_2}$ for $p = 1$, respectively.*

4. Commutators of the linear operators with rough kernel $T_{\Omega,\alpha}$ on the spaces $VLM^{\{x_0\}}_{p,\varphi}$

In this section, we will prove the boundedness of the operator $T_{\Omega,b,\alpha}$ with $b \in LC^{\{x_0\}}_{p_2,\lambda}$ on the generalized vanishing local Morrey spaces $VLM^{\{x_0\}}_{p,\varphi}$.

Let T be a linear operator. For a locally integrable function b on $\mathbb{R}^n$, we define the commutator $[b,T]$ by

$$[b,T]f(x) = b(x)\,Tf(x) - T(bf)(x)$$

for any suitable function f.

For $b \in L_1^{loc}(\mathbb{R}^n)$, the commutator $[b, I_\alpha]$ of fractional integral operator (also known as the Riesz potential) is defined by

$$[b,I_\alpha]f(x) = b(x)I_\alpha f(x) - I_\alpha(bf)(x) = \int_{\mathbb{R}^n} \frac{b(x)-b(y)}{|x-y|^{n-\alpha}} f(y)dy \qquad 0<\alpha<n$$

for any suitable function f. Also, the sublinear commutator of the fractional maximal operator is defined as follows

$$M_{b,\alpha}(f)(x)=\sup_{r>0}\frac{1}{r^{n-\alpha}}\int\limits_{|x-y|<r}|b(x)-b(y)||f(y)|dy.$$

The function b is also called the symbol function of $[b,I_\alpha]$. The characterization of (L_p,L_q)-boundedness of the commutator $[b,I_\alpha]$ of fractional integral operator has been given by Chanillo [4]. A well known result of Chanillo [4] states that the commutator $[b,I_\alpha]$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$, $1<p<q<\infty$, $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n}$ if and only if $b\in BMO(\mathbb{R}^n)$ (Bounded Mean Oscillation).

There are two major reasons for considering the problem of commutators. The first one is that the boundedness of commutators can produce some characterizations of function spaces (see [1, 4, 7, 8, 9, 12]). The other one is that the theory of commutators plays an important role in the study of the regularity of solutions to elliptic and parabolic PDEs of the second order (see [5, 6, 15]).

The definition of local Campanato space $LC_{p,\lambda}^{\{x_0\}}$ is as follows.

Definition 4.1. [1, 7, 9] *Let $1\le p<\infty$ and $0\le\lambda<\frac{1}{n}$. A function $f\in L_p^{loc}(\mathbb{R}^n)$ is said to belong to the $LC_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n)$ (local Campanato space), if*

$$\|f\|_{LC_{p,\lambda}^{\{x_0\}}}=\sup_{r>0}\left(\frac{1}{|B(x_0,r)|^{1+\lambda p}}\int\limits_{B(x_0,r)}\left|f(y)-f_{B(x_0,r)}\right|^p dy\right)^{\frac{1}{p}}<\infty, \tag{16}$$

where

$$f_{B(x_0,r)}=\frac{1}{|B(x_0,r)|}\int\limits_{B(x_0,r)}f(y)\,dy.$$

Define

$$LC_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n)=\left\{f\in L_p^{loc}(\mathbb{R}^n):\|f\|_{LC_{p,\lambda}^{\{x_0\}}}<\infty\right\}.$$

Remark 4.1. *If two functions which differ by a constant are regarded as a function in the space $LC_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n)$, then $LC_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n)$ becomes a Banach space. The space $LC_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n)$ when $\lambda=0$ is just the $LC_p^{\{x_0\}}(\mathbb{R}^n)$. That is,*

$$\|f\|_{LC_p^{\{x_0\}}(\mathbb{R}^n)}\approx\|f\|_{BMO(\mathbb{R}^n)}$$

so that $LC_p^{\{x_0\}}(\mathbb{R}^n)$ is the well known $BMO(\mathbb{R}^n)$ space. Apparently, (16) is equivalent to the following condition:

$$\sup_{r>0}\inf_{c\in\mathbb{C}}\left(\frac{1}{|B(x_0,r)|^{1+\lambda p}}\int\limits_{B(x_0,r)}|f(y)-c|^p dy\right)^{\frac{1}{p}}<\infty.$$

Now using results in [1], we also obtain the boundedness of the operator $T_{\Omega,b,\alpha}$ on the generalized vanishing local Morrey spaces $VLM_{p,\varphi}^{\{x_0\}}$.

Theorem 4.1. *(Our main result) Let $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, be homogeneous of degree zero. Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $b \in LC_{p_2,\lambda}^{\{x_0\}}(\mathbb{R}^n)$, $0 \leq \lambda < \frac{1}{n}$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $\frac{1}{q_1} = \frac{1}{p_1} - \frac{\alpha}{n}$, and $T_{\Omega,\alpha}$ is a linear operator satisfying condition (1) and bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$. Let for $s' \leq p$ the pair (φ_1, φ_2) satisfies conditions (4)-(5) and*

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right) \varphi_1(x_0, t) \frac{t^{\frac{n}{p_1}}}{t^{\frac{n}{q_1}+1-n\lambda}} dt \leq C_0 \varphi_2(x_0, r), \tag{17}$$

where C_0 does not depend on $r > 0$,

$$\lim_{r \to 0} \frac{\ln \frac{1}{r}}{\inf\limits_{r>0} \varphi_2(x_0, r)} = 0 \tag{18}$$

and

$$C_\psi := \int_\psi^\infty (1 + |\ln t|) \sup_{t>0} \varphi_1(x_0, t) \frac{t^{\frac{n}{p_1}}}{t^{\frac{n}{q_1}+1-n\lambda}} dt < \infty \tag{19}$$

for every $\psi > 0$, and for $q_1 < s$ the pair (φ_1, φ_2) satisfies conditions (4)-(5) and also

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right) \varphi_1(x_0, t) \frac{t^{\frac{n}{p_1}}}{t^{\frac{n}{q_1}-\frac{n}{s}+1-n\lambda}} dt \leq C_0 \varphi_2(x_0, r) r^{\frac{n}{s}}, \tag{20}$$

where C_0 does not depend on $r > 0$,

$$\lim_{r \to 0} \frac{\ln \frac{1}{r}}{\inf\limits_{r>0} \varphi_2(x_0, r)} = 0$$

and

$$C_{\psi'} := \int_{\psi'}^\infty (1 + |\ln t|) \sup_{t>0} \varphi_1(x_0, t) \frac{t^{\frac{n}{p_1}}}{t^{\frac{n}{q_1}-\frac{n}{s}+1-n\lambda}} dt < \infty \tag{21}$$

for every $\psi' > 0$.

Then the operator $T_{\Omega,b,\alpha}$ is bounded from $VLM_{p_1,\varphi_1}^{\{x_0\}}$ to $VLM_{q,\varphi_2}^{\{x_0\}}$. Moreover,

$$\|T_{\Omega,b,\alpha} f\|_{VLM_{q,\varphi_2}^{\{x_0\}}} \lesssim \|b\|_{LC_{p_2,\lambda}^{\{x_0\}}} \|f\|_{VLM_{p_1,\varphi_1}^{\{x_0\}}}. \tag{22}$$

Proof. The estimation of (22) follows from Lemma 4 in [1] and (17). That is,

$$\begin{aligned}
\|T_{\Omega,b,\alpha} f\|_{VLM_{q,\varphi_2}^{\{x_0\}}} &= \sup_{r>0} \frac{|B(x_0,r)|^{-\frac{1}{q}} \|T_{\Omega,b,\alpha} f\|_{L_q(B(x_0,r))}}{\varphi_2(x_0,r)} \\
&\lesssim \|b\|_{LC_{p_2,\lambda}^{\{x_0\}}} \sup_{r>0} \frac{1}{\varphi_2(x_0,r)} \int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{t^{\frac{n}{p_1}}}{t^{\frac{n}{q_1}+1-n\lambda}} \varphi_1(x_0,t) \left[\frac{\|f\|_{L_{p_1}(B(x_0,t))}}{\varphi_1(x_0,t) t^{\frac{n}{p_1}}}\right] dt \\
&\lesssim \|b\|_{LC_{p_2,\lambda}^{\{x_0\}}} \|f\|_{VLM_{p_1,\varphi_1}^{\{x_0\}}} \sup_{r>0} \frac{1}{\varphi_2(x_0,r)} \int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{t^{\frac{n}{p_1}}}{t^{\frac{n}{q_1}+1-n\lambda}} \varphi_1(x_0,t) dt \\
&\lesssim \|b\|_{LC_{p_2,\lambda}^{\{x_0\}}} \|f\|_{VLM_{p_1,\varphi_1}^{\{x_0\}}}.
\end{aligned}$$

Now, let us show that

$$\lim_{r\to 0^+}\sup_{r>0}\frac{r^{-\frac{n}{q}}\left\|T_{\Omega,b,\alpha}f\right\|_{L_q(B(x_0,r))}}{\varphi_2(x_0,r)}=0.$$

For $0<r<\psi_0<\infty$, by Lemma 4 in [1], we have

$$\frac{r^{-\frac{n}{q}}\left\|T_{\Omega,b,\alpha}f\right\|_{L_q(B(x_0,r))}}{\varphi_2(x_0,r)}\le C\left[\mathcal{F}_{\psi_0}(x_0,r)+\mathcal{G}_{\psi_0}(x_0,r)\right], \tag{23}$$

where

$$\mathcal{F}_{\psi_0}(x_0,r):=\|b\|_{LC_{p_2,\lambda}^{\{x_0\}}}\frac{r^{\frac{n}{q}}}{\varphi_2(x_0,r)}\int_r^{\psi_0}\left(1+\ln\frac{t}{r}\right)\varphi_1(x_0,t)\,\frac{t^{\frac{n}{p_1}}}{t^{\frac{n}{q_1}+1-n\lambda}}\left[\sup_{0<r<t}\frac{\|f\|_{L_{p_1}(B(x_0,r))}}{\varphi_1(x_0,r)r^{\frac{n}{p_1}}}\right]dt$$

and

$$\mathcal{G}_{\psi_0}(x_0,r):=\|b\|_{LC_{p_2,\lambda}^{\{x_0\}}}\frac{r^{\frac{n}{q}}}{\varphi_2(x_0,r)}\int_{\psi_0}^{\infty}\left(1+\ln\frac{t}{r}\right)\varphi_1(x_0,t)\,\frac{t^{\frac{n}{p_1}}}{t^{\frac{n}{q_1}+1-n\lambda}}\left[\sup_{0<r<t}\frac{\|f\|_{L_{p_1}(B(x_0,r))}}{\varphi_1(x_0,r)r^{\frac{n}{p_1}}}\right]dt.$$

For any $\epsilon>0$, now we can choose any fixed $\psi_0>0$ such that whenever $r\in(0,\psi_0)$,

$$\sup_{r>0}\sup_{0<r<\psi_0}\frac{\|f\|_{L_{p_1}(B(x_0,r))}}{\varphi_1(x_0,r)r^{\frac{n}{p_1}}}<\frac{\epsilon}{2CC_0\|b\|_{LC_{p_2,\lambda}^{\{x_0\}}}},$$

where C_0 and C are constants from (17) and (23), which is possible since $f\in VLM_{p_1,\varphi_1}^{\{x_0\}}$.

This allows to estimate the first term uniformly, for $0<r<\psi_0$,

$$\sup_{r>0}C\mathcal{F}_{\psi_0}(x_0,r)<\frac{\epsilon}{2}.$$

For the second term, writing $1+\ln\frac{t}{r}\le 1+|\ln t|+\ln\frac{1}{r}$, from (19), it follows that

$$\mathcal{G}_{\psi_0}(x_0,r)\le\|b\|_{LC_{p_2,\lambda}^{\{x_0\}}}\left(C_{\psi_0}+\widetilde{C_{\psi_0}}\ln\frac{1}{r}\right)\|f\|_{VLM_{p_1,\varphi_1}^{\{x_0\}}}\frac{r^{\frac{n}{q}}}{\varphi_2(x_0,r)},$$

where C_{ψ_0} is the constant from (19) with $\psi=\psi_0$ and $\widetilde{C_{\psi_0}}$ is a similar constant with omitted logarithmic factor in the integrand. Since φ_2 satisfies conditions (4)-(5) and by (18), it is sufficient to choose r small enough such that

$$\sup_{r>0}\frac{r^n}{\varphi_2^q(x_0,r)}\le\left(\frac{\epsilon}{2\|b\|_{LC_{p_2,\lambda}^{\{x_0\}}}C\left(C_{\psi_0}+\widetilde{C_{\psi_0}}\ln\frac{1}{r}\right)\|f\|_{VLM_{p_1,\varphi_1}^{\{x_0\}}}}\right)^q.$$

Thus,

$$\sup_{r>0}C\mathcal{G}_{\psi_0}(x_0,r)<\frac{\epsilon}{2}.$$

As a result,

$$\frac{r^{-\frac{n}{q}}}{\varphi_2(x_0,r)}\left\|T_{\Omega,b,\alpha}f\right\|_{L_q(B(x_0,r))}<\epsilon,$$

which means that

$$\lim_{r\to 0^+}\sup_{r>0}\frac{r^{-\frac{n}{q}}}{\varphi_2(x_0,r)}\left\|T_{\Omega,b,\alpha}f\right\|_{L_q(B(x_0,r))}=0,$$

which completes the proof of (22).

For the case of $q_1 < s$, we can also use the same method, so we omit the details. □

Corollary 4.1. *Suppose that* $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, *is homogeneous of degree zero. Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $b \in LC_{p_2,\lambda}^{\{x_0\}}(\mathbb{R}^n)$, $0 \leq \lambda < \frac{1}{n}$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $\frac{1}{q_1} = \frac{1}{p_1} - \frac{\alpha}{n}$. *If for* $s' \leq p$ *the pair* (φ_1, φ_2) *satisfies conditions (4)-(5)-(18) and (17)-(19) and for* $q_1 < s$ *the pair* (φ_1, φ_2) *satisfies conditions (4)-(5)-(18) and (20)-(21). Then, the operators* $M_{\Omega,b,\alpha}$ *and* $[b, I_{\Omega,\alpha}]$ *are bounded from* $VLM_{p_1,\varphi_1}^{\{x_0\}}$ *to* $VLM_{q,\varphi_2}^{\{x_0\}}$, *respectively.*

In the case of $q = \infty$ by Theorem 4.1, we get

Corollary 4.2. *Let* $x_0 \in \mathbb{R}^n$, $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $b \in LC_{p_2,\lambda}^{\{x_0\}}(\mathbb{R}^n)$, $0 \leq \lambda < \frac{1}{n}$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $\frac{1}{q_1} = \frac{1}{p_1} - \frac{\alpha}{n}$ *and the pair* (φ_1, φ_2) *satisfies conditions (4)-(5)-(18) and (17)-(19). Then the operators* $M_{b,\alpha}$ *and* $[b, I_\alpha]$ *are bounded from* $VLM_{p_1,\varphi_1}^{\{x_0\}}$ *to* $VLM_{q,\varphi_2}^{\{x_0\}}$, *respectively.*

5. Conclusions

In this article, the author has established some norm inequalities for sublinear operators with rough kernel generated by fractional integrals and commutators on generalized vanishing local Morrey spaces. These type of inequalities also include their weak versions under generic size conditions which are satisfied by most of the operators in harmonic analysis. On the other hand, the results presented here are sure to be new and potentially useful. Since the research subject here and its related ones are so popular, the content of this paper may attract interested readers who have been interested in this and related research subjects. Therefore, the results in this paper are worthwhile to record.

Acknowledgement. The author cordially thanks the anonymous referees for their valuable comments which led to the improvement of this paper.

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