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ON GENERALIZATION OF SOME INTEGRAL INEQUALITIES FOR MULTIPLICATIVELY P-FUNCTIONS

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Abstract. In this paper, by using Hölder-İşcan, Hölder and power-mean integral inequality and an general identity for differentiable functions we can obtain new estimates on generalization of Hadamard, Ostrowski and Simpson type inequalities for functions whose derivatives in absolute value at certain power are multiplicatively P-functions. In addition, It is proved that the result obtained Hölder-İşcan integral inequality is better than the result obtained Hölder integral inequality. Some applications to special means of real numbers are also given.
Keywords: Hölder-İşcan inequality, Hermite-Hadamard inequality, Simpson type inequality, Ostrowski type inequality, trapezoid type inequality, midpoint type inequality, convex function, multiplicatively P-functions.

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1. Introduction

Integral inequalities have played an important role in the development of all branches of Mathematics and the other sciences. The inequalities discovered by Hermite and Hadamard for convex functions are very important in the literature. The classical Hermite-Hadamard integral inequality provides estimates of the mean value of a continuous convex function $f : [a, b] \to \mathbb{R}$. Firstly, let's recall the Hermite-Hadamard integral inequality.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integ-ral inequality for convex functions [1]. Note that some of the classical inequalities for means can be derived from (1) for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if the function f is concave.

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Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a mapping differentiable in I°, the interior of I, and let $a, b \in I^{\circ}$ with $a < b$. If $|f'(x)| \leq M$, $x \in [a, b]$, then we the following inequality holds

$$\left| f(x) - \frac{1}{b-a} \int\limits_{a}^{b} f(t)dt \right| \leq \frac{M}{b-a} \left[\frac{(x-a)^2 + (b-x)^2}{2} \right]$$

for all $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller one. This result is known in the literature as the Ostrowski inequality [3].

The following inequality is well known in the literature as Simpson's inequality .

Let $f : [a, b] \to \mathbb{R}$ be a four times continuously differentiable mapping on (a, b) and $\left\| f^{(4)} \right\|_{\infty} = \sup\limits_{x \in (a,b)} \left| f^{(4)}(x) \right| < \infty$. Then the following inequality holds:

$$\left| \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2} \right) \right] - \frac{1}{b-a} \int\limits_{a}^{b} f(x)dx \right| \leq \frac{1}{2880} \left\| f^{(4)} \right\|_{\infty} (b-a)^4 .$$

In recent years many authors have studied error estimations for Simpson's inequality; for refinements, counterparts, generalizations and new Simpson's type inequalities, see [12, 13] and therein.

Definition 1.1. *A nonnegative function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be P-function if the inequality*

$$f\left(tx + (1-t)\,y\right) \leq f\left(x\right) + f\left(y\right)$$

holds for all $x, y \in I$ and $t \in (0, 1)$.

We will denote by $P(I)$ the set of P-function on the interval I. Note that $P(I)$ contain all nonnegative convex and quasi-convex functions.

In [2], Dragomir et al. proved the following inequality of Hadamard type for class of P-functions.

Theorem 1.1. *Let $f \in P\left(I\right)$, $a, b \in I$ with $a < b$ and $f \in L\left[a, b\right]$. Then*

$$f\left(\frac{a+b}{2} \right) \leq \frac{2}{b-a} \int_{a}^{b} f\left(x\right) dx \leq 2\left[f\left(a\right) + f\left(b\right) \right].$$

It should be noted that the concept of log-P-convex, which we consider in our study and given below, was first defined by Noor et al in 2013. Then, the algebraic properties of this definition with the name of multiplicatively P-function are examined in detail by us.

Definition 1.2 ([6],[10])**.** *Let $I \neq \emptyset$ be an interval in $\mathbb{R}$. The function $f : I \to [0, \infty)$ is said to be multiplicatively P-function (or* log-*P-function), if the inequality*

$$f\left(tx + (1-t)y\right) \leq f(x)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

We will denote by $MP\left(I\right)$ the class of all multiplicatively P-convex functions on interval I. Clearly, $f : I \to [0, \infty)$ is multiplicatively P-function if and only if $\log f$ is P-function. We state that the range of the multiplicatively P-functions is greater than or equal to 1. In recent years many authors have studied P-functions and multiplicatively P-function, see [2, 5, 8, 9, 11] and therein.

[6], Kadakal proved the following inequalities of Hermite-Hadamard type integral inequalities for class of multiplicatively P-functions.

Theorem 1.2. *Let the function $f : I \to [1,\infty)$ be a multiplicatively P-function. If $f \in L[a,b]$, then the following inequalities hold:*

$$\begin{aligned} i)\quad f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}\int_a^b f(x)f(a+b-x)\,dx \leq [f(a)f(b)]^2 \\ ii)\quad f\left(\frac{a+b}{2}\right) &\leq f(a)f(b)\frac{1}{b-a}\int_a^b f(x)dx \leq [f(a)f(b)]^2 \end{aligned}$$

Lemma 1.1 ([4])**.** *Let the function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$ and $\theta, \lambda \in [0,1]$. Then the following equality holds:*

$$\begin{aligned} &(1-\theta)(\lambda f(a) + (1-\lambda) f(b)) + \theta f((1-\lambda)a + \lambda b) - \frac{1}{b-a}\int_a^b f(x)dx \\ = \ &(b-a)\left[-\lambda^2 \int_0^1 (t-\theta) f'(ta + (1-t)[(1-\lambda)a + \lambda b])\,dt \right. \\ &\left. + (1-\lambda)^2 \int_0^1 (t-\theta) f'(tb + (1-t)[(1-\lambda)a + \lambda b])\,dt \right]. \end{aligned}$$

An refinement of Hölder integral inequality better approach than Hölder integral inequality can be given as follows:

Theorem 1.3 (Hölder-İşcan Integral Inequality [7])**.** *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on interval $[a,b]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[a,b]$ then*

$$\begin{aligned} \int_a^b |f(x)g(x)|\,dx \leq \ & \frac{1}{b-a}\left\{ \left(\int_a^b (b-x)|f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_a^b (b-x)|g(x)|^q dx\right)^{\frac{1}{q}} \right. \\ & \left. + \left(\int_a^b (x-a)|f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_a^b (x-a)|g(x)|^q dx\right)^{\frac{1}{q}} \right\} \end{aligned}$$

Our aim is to obtain the general integral inequalities giving the Hermite-Hadamard, Ostrowsky and Simpson type inequalities for the multiplicatively P-function in the special case using the Hölder, Hölder-İşcan, power mean integral inequalities and above lemma.

2. Main results

Theorem 2.1. *Let the function $f : I \subseteq [1,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$ and $\theta, \lambda \in [0,1]$. If $|f'|^q$ is multiplicatively P-function on $[a,b]$, $q \geq 1$, then the following inequality holds:*

$$\begin{aligned} &\left|(1-\theta)(\lambda f(a) + (1-\lambda) f(b)) + \theta f((1-\lambda)a + \lambda b) - \frac{1}{b-a}\int_a^b f(x)dx\right| \qquad (2) \\ \leq \ &(b-a)A_1(\theta)\left|f'(A_\lambda)\right| \left(\lambda^2 \left|f'(a)\right| + (1-\lambda)^2 \left|f'(b)\right|\right) \end{aligned}$$

where $A_1(\theta) = \theta^2 - \theta + \frac{1}{2}$ and $A_\lambda = (1-\lambda)a + \lambda b$.

Proof. Suppose that $q \geq 1$ and $A_\lambda = (1-\lambda)a + \lambda b$. From Lemma 1.1 and using the well known power mean integral inequality, we have

$$\left|(1-\theta)(\lambda f(a) + (1-\lambda) f(b)) + \theta f(A_\lambda) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq (b-a)$$

$$\left[\lambda^2 \int_0^1 |t-\theta| \left|f'(ta + (1-t)A_\lambda)\right| dt + (1-\lambda)^2 \int_0^1 |t-\theta| \left|f'(tb+(1-t)A_\lambda)\right| dt\right]$$

$$\begin{aligned} \leq \quad & (b-a)\left\{\lambda^2 \left(\int_0^1 |t-\theta|\,dt\right)^{1-\frac{1}{q}} \left(\int_0^1 |t-\theta| \left|f'(ta+(1-t)A_\lambda)\right|^q dt\right)^{\frac{1}{q}}\right. \\ & \left. + (1-\lambda)^2 \left(\int_0^1 |t-\theta|\,dt\right)^{1-\frac{1}{q}} \left(\int_0^1 |t-\theta| \left|f'(tb+(1-t)A_\lambda)\right|^q dt\right)^{\frac{1}{q}}\right\}. \end{aligned} \tag{3}$$

Since $|f'|^q$ is multiplicatively P-function on $[a,b]$, we know that for $t \in [0,1]$

$$\left|f'(ta + A_\lambda(1-t))\right|^q \leq \left|f'(a)\right|^q \left|f'(A_\lambda)\right|^q \tag{4}$$

$$\left|f'(tb + A_\lambda(1-t))\right|^q \leq \left|f'(b)\right|^q \left|f'(A_\lambda)\right|^q. \tag{5}$$

Hence, by simple computation

$$\int_0^1 |t-\theta| \left|f'(ta+(1-t)A_\lambda)\right|^q dt \leq \left|f'(a)\right|^q \left|f'(A_\lambda)\right|^q \int_0^1 |t-\theta|\,dt \tag{6}$$

$$\int_0^1 |t-\theta| \left|f'(tb+(1-t)A_\lambda)\right|^q dt \leq \left|f'(a)\right|^q \left|f'(A_\lambda)\right|^q \left[\theta^2 - \theta + \frac{1}{2}\right] \tag{7}$$

$$\int_0^1 |t-\theta|\,dt = \theta^2 - \theta + \frac{1}{2}. \tag{8}$$

Thus, using (6), (7) and (8) in (3), we obtain the inequality (2). This completes the proof. □

Corollary 2.1. *Under the assumptions of Theorem 2.1 with $\theta = 1$, then we have following generalized midpoint type inequality*

$$\left|f((1-\lambda)a + \lambda b) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2}\left|f'(A_\lambda)\right|\left(\lambda^2 \left|f'(a)\right| + (1-\lambda)^2 \left|f'(b)\right|\right).$$

Corollary 2.2. *Under the assumptions of Theorem 2.1 with $\theta = 1$, if $|f'(x)| \leq M$, $x \in [a,b]$, then we have the following Ostrowski type inequality*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)du\right| \leq M^2 \left[\frac{(x-a)^2 + (b-x)^2}{2(b-a)}\right] \tag{9}$$

for each $x \in [a,b]$.

Proof. For each $x \in [a,b]$, there exist $\lambda_x \in [0,1]$ such that $x = (1-\lambda_x)\,a + \lambda_x b$. Hence, we have $\lambda_x = \frac{x-a}{b-a}$ and $1-\lambda_x = \frac{b-x}{b-a}$. Therefore for each $x \in [a,b]$, from the inequality (2) we obtain the inequality (9). □

Corollary 2.3. *Under the assumptions of Theorem 2.1 with $\theta = 0$, then we have following generalized trapezoid type inequality*

$$\begin{aligned} &\left|\lambda f(a) + (1-\lambda)\, f(b) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq\ & \frac{b-a}{2}\left|f'(A_\lambda)\right|\left(\lambda^2\left|f'(a)\right| + (1-\lambda)^2\left|f'(b)\right|\right). \end{aligned}$$

Corollary 2.4. *Under the assumptions of Theorem 2.1 with $\lambda = \frac{1}{2}$ and $\theta = \frac{2}{3}$, then we have the following Simpson type inequality*

$$\begin{aligned} &\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq\ & \frac{5}{36}(b-a)\left|f'\left(\frac{a+b}{2}\right)\right| A\left(\left|f'(a)\right|, \left|f'(b)\right|\right), \end{aligned}$$

where A is arithmetic mean.

Corollary 2.5. *Under the assumptions of Theorem 2.1 with $\lambda = \frac{1}{2}$ and $\theta = 1$, then we have following midpoint type inequality*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right| A\left(\left|f'(a)\right|, \left|f'(b)\right|\right),$$

where A is arithmetic mean.

Corollary 2.6. *Under the assumptions of Theorem 2.1 with $\lambda = \frac{1}{2}$, and $\theta = 0$, then we get the following trapezoid type inequality*

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right| A\left(\left|f'(a)\right|, \left|f'(b)\right|\right),$$

where A is arithmetic mean.

Using Lemma 1.1 we shall give another result for multiplicatively P-functions as follows.

Theorem 2.2. *Let $f : I \subseteq [1,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a,b \in I^\circ$ with $a < b$ and $\theta, \lambda \in [0,1]$. If $|f'|^q$ is multiplicatively P-function on $[a,b]$, $q > 1$, then the following inequality holds:*

$$\begin{aligned} &\left|(1-\theta)\left(\lambda f(a) + (1-\lambda)\, f(b)\right) + \theta f((1-\lambda)\, a + \lambda b) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\leq (b-a)\left(\frac{\theta^{p+1} + (1-\theta)^{p+1}}{p+1}\right)^{\frac{1}{p}}\left|f'(A_\lambda)\right|\left[\lambda^2\left|f'(a)\right| + (1-\lambda)^2\left|f'(b)\right|\right]. \end{aligned} \tag{10}$$

where $A_\lambda = (1-\lambda)\, a + \lambda b$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Suppose that $A_\lambda = (1-\lambda)\, a + \lambda b$. From Lemma 1.1 and by well-known Hölder's integral inequality, we have

$$\begin{aligned}
&\left|(1-\theta)\left(\lambda f(a) + (1-\lambda) f(b)\right) + \theta f(A_\lambda) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq (b-a) \\
&\times \left[\lambda^2 \int_0^1 |t-\theta| \left|f'\left(ta + (1-t) A_\lambda\right)\right| dt \right. \\
&\left. + (1-\lambda)^2 \int_0^1 |t-\theta| \left|f'\left(tb + (1-t) A_\lambda\right)\right| dt \right] \\
\leq\ & (b-a) \left\{ \lambda^2 \left(\int_0^1 |t-\theta|^p dt\right)^{\frac{1}{p}} \left(\int_0^1 \left|f'\left(ta + (1-t) A_\lambda\right)\right|^q dt\right)^{\frac{1}{q}} \right. \\
&\left. + (1-\lambda)^2 \left(\int_0^1 |t-\theta|^p dt\right)^{\frac{1}{p}} \left(\int_0^1 \left|f'\left(tb + (1-t) A_\lambda\right)\right|^q dt\right)^{\frac{1}{q}} \right\}. \qquad (11)
\end{aligned}$$

Since $|f'|^q$ is multiplicatively P-function on $[a,b]$, the inequalities (4) and (5) holds. Hence, by simple computation

$$\int_0^1 \left|f'\left(ta + (1-t) A_\lambda\right)\right|^q dt \leq \left|f'(a)\right|^q \left|f'(A_\lambda)\right|^q \qquad (12)$$

$$\int_0^1 \left|f'\left(tb + (1-t) A_\lambda\right)\right|^q dt \leq \left|f'(b)\right|^q \left|f'(A_\lambda)\right|^q \qquad (13)$$

$$\int_0^1 |t-\theta|^p dt = \frac{\theta^{p+1} + (1-\theta)^{p+1}}{p+1} \qquad (14)$$

thus, using (12)-(14) in (11), we obtain the inequality (10). This completes the proof. □

Corollary 2.7. *Under the assumptions of Theorem 2.2 with $\theta = 1$, then we have the following generalized midpoint type inequality*

$$\begin{aligned}
&\left|f((1-\lambda)\, a + \lambda b) - \frac{1}{b-a}\int_a^b f(x)dx\right| \qquad (15) \\
\leq\ & (b-a)\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left|f'(A_\lambda)\right| \left[\lambda^2 \left|f'(a)\right| + (1-\lambda)^2 \left|f'(b)\right|\right].
\end{aligned}$$

where $A_\lambda = (1-\lambda)\, a + \lambda b$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Corollary 2.8. *Under the assumptions of Theorem 2.2 with $\theta = 0$, then we have the following generalized trapezoid type inequality*

$$\begin{aligned} & \left|\lambda f(a) + (1-\lambda) f(b) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq \; & \frac{(b-a)}{(p+1)^{\frac{1}{p}}} \left|f'(A_\lambda)\right| \left[\lambda^2 \left|f'(a)\right| + (1-\lambda)^2 \left|f'(b)\right|\right], \end{aligned}$$

where $A_\lambda = (1-\lambda)a + \lambda b$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Corollary 2.9. *Under the assumptions of Theorem 2.2 with $\theta = 1$, if $|f'(x)| \leq M$, $x \in [a,b]$, then we have the following Ostrowski type inequality*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{M^2}{(p+1)^{\frac{1}{p}}}\left[\frac{(x-a)^2 + (b-x)^2}{b-a}\right] \tag{16}$$

for each $x \in [a,b]$.

Proof. For each $x \in [a,b]$, there exist $\lambda_x \in [0,1]$ such that $x = (1-\lambda_x)a + \lambda_x b$. Hence we have $\lambda_x = \frac{x-a}{b-a}$ and $1 - \lambda_x = \frac{b-x}{b-a}$. Therefore, for each $x \in [a,b]$, from the inequality (10) we obtain the inequality (16). □

Corollary 2.10. *Under the assumptions of Theorem 2.2 with $\lambda = \frac{1}{2}$ and $\theta = \frac{2}{3}$, then we have the following Simpson type inequality*

$$\begin{aligned} & \left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq \; & \frac{b-a}{6}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}} \left|f'\left(\frac{a+b}{2}\right)\right| A\left(\left|f'(a)\right|, \left|f'(b)\right|\right), \end{aligned}$$

where A is the arithmetic mean.

Corollary 2.11. *Under the assumptions of Theorem 2.2 with $\lambda = \frac{1}{2}$ and $\theta = 1$, then we have the following midpoint type inequality*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left|f'\left(\frac{a+b}{2}\right)\right| A\left(\left|f'(a)\right|, \left|f'(b)\right|\right),$$

where A is the arithmetic mean.

Corollary 2.12. *Under the assumptions of Theorem 2.2 with $\lambda = \frac{1}{2}$ and $\theta = 0$, then we have the following trapezoid type inequality*

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left|f'\left(\frac{a+b}{2}\right)\right| A\left(\left|f'(a)\right|, \left|f'(b)\right|\right),$$

where A is the arithmetic mean.

Using Lemma 1.1 we shall give another result for multiplicatively P-functions as follows using the Hölder-İşcan integral inequality:

Theorem 2.3. *Let $f : I \subseteq [1,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I^{\circ}$ with $a < b$ and $\theta, \lambda \in [0,1]$. If $|f'|^q$ is multiplicatively P-function on $[a,b]$, $q > 1$, then the following inequality holds:*

$$\begin{aligned} &\left| (1-\theta)\left(\lambda f(a) + (1-\lambda) f(b)\right) + \theta f((1-\lambda) a + \lambda b) - \frac{1}{b-a}\int_a^b f(x)dx \right| \qquad (17) \\ \leq\ & (b-a)\left|f'(A_\lambda)\right| \left(\frac{1}{2}\right)^{\frac{1}{q}} \left\{ C^{\frac{1}{p}}(\theta,p) + D^{\frac{1}{p}}(\theta,p) \right\} \left[\lambda^2 \left|f'(a)\right| + (1-\lambda)^2 \left|f'(b)\right| \right] \end{aligned}$$

where $A_\lambda = (1-\lambda) a + \lambda b$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Suppose that $A_\lambda = (1-\lambda) a + \lambda b$. From Lemma 1.1 and by Hölder-İşcan integral inequality, we have

$$\begin{aligned} &\left| (1-\theta)\left(\lambda f(a) + (1-\lambda) f(b)\right) + \theta f(A_\lambda) - \frac{1}{b-a}\int_a^b f(x)dx \right| \qquad (18) \\ \leq\ & (b-a)\lambda^2 \left[\int_0^1 |t-\theta| \left|f'(ta + (1-t) A_\lambda)\right| dt \right. \\ & \left. + (1-\lambda)^2 \int_0^1 |t-\theta| \left|f'(tb + (1-t) A_\lambda)\right| dt \right] \\ \leq\ & (b-a)\lambda^2 \left\{ \left(\int_0^1 (1-t)|t-\theta|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 (1-t)\left|f'(ta + (1-t) A_\lambda)\right|^q dt \right)^{\frac{1}{q}} \right. \\ & \left. + \left(\int_0^1 t|t-\theta|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 t\left|f'(ta + (1-t) A_\lambda)\right|^q dt \right)^{\frac{1}{q}} \right\} \\ & + (b-a)(1-\lambda)^2 \left\{ \left(\int_0^1 (1-t)|t-\theta|^p dt \right)^{\frac{1}{p}} \right. \\ & \times \left(\int_0^1 (1-t)\left|f'(tb + (1-t) A_\lambda)\right|^q dt \right)^{\frac{1}{q}} \\ & \left. + \left(\int_0^1 t|t-\theta|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 t\left|f'(tb + (1-t) A_\lambda)\right|^q dt \right)^{\frac{1}{q}} \right\} \\ \leq\ & (b-a)\left|f'(A_\lambda)\right| \left(\frac{1}{2}\right)^{\frac{1}{q}} \left\{ C^{\frac{1}{p}}(\theta,p) + D^{\frac{1}{p}}(\theta,p) \right\} \left[\lambda^2 \left|f'(a)\right| + (1-\lambda)^2 \left|f'(b)\right| \right]. \end{aligned}$$

Since $|f'|^q$ is multiplicatively P-function on interval $[a,b]$, the following inequalities holds.

$$\int_0^1 \left|f'\left(ta+(1-t)A_\lambda\right)\right|^q dt \leq \left|f'(a)\right|^q \left|f'(A_\lambda)\right|^q \tag{19}$$

$$\int_0^1 \left|f'\left(tb+(1-t)A_\lambda\right)\right|^q dt \leq \left|f'(b)\right|^q \left|f'(A_\lambda)\right|^q \tag{20}$$

Here, by simple computation we obtain

$$\int_0^1 (1-t)dt = \int_0^1 tdt = \frac{1}{2},$$

$$\begin{aligned} C(\theta,p) &= \int_0^1 (1-t)\left|t-\theta\right|^p dt \qquad (21)\\ &= (1-\theta)\left[\frac{\theta^{p+1}+(1-\theta)^{p+1}}{p+1}\right]+\left[\frac{\theta^{p+2}-(1-\theta)^{p+2}}{p+2}\right] \\ D(\theta,p) &= \int_0^1 t\left|t-\theta\right|^p dt \qquad (22)\\ &= \theta\left[\frac{\theta^{p+1}+(1-\theta)^{p+1}}{p+1}\right]-\left[\frac{\theta^{p+2}-(1-\theta)^{p+2}}{p+2}\right]. \end{aligned}$$

Thus, using (19)-(22) in (18), we obtain the inequality (17). This completes the proof. □

Corollary 2.13. *Under the assumptions of Theorem 2.3 with $\theta=1$, then we have the following generalized midpoint type inequality*

$$\begin{aligned} &\left|f((1-\lambda)a+\lambda b)-\frac{1}{b-a}\int_a^b f(x)dx\right| \qquad (23)\\ \leq\ & (b-a)\left|f'(A_\lambda)\right|\left(\frac{1}{2}\right)^{\frac{1}{q}}\left\{C^{\frac{1}{p}}(1,p)+D^{\frac{1}{p}}(1,p)\right\}\left[\lambda^2\left|f'(a)\right|+(1-\lambda)^2\left|f'(b)\right|\right]. \end{aligned}$$

where $A_\lambda=(1-\lambda)a+\lambda b$ and $\frac{1}{p}+\frac{1}{q}=1$.

Remark 2.1. *The inequality (23) is better than the inequality (15). For this, we need to show that*

$$M(p):=\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left\{\left(\frac{1}{p+1}\right)^{\frac{1}{p}}+1\right\}\leq N(p):=\left(\frac{1}{p+1}\right)^{\frac{1}{p}}.$$

If we write as $M(p) = N(p)\,X(p)$*, then* $X(p) = \left(\frac{1}{2}\right)^{\frac{1}{q}}\left[\left(\frac{1}{p+2}\right)^{\frac{1}{p}} + \left(\frac{p+1}{p+2}\right)^{\frac{1}{p}}\right]$*. Therefore, by using concavity of the function* $\psi : [0,\infty) \to \mathbb{R}, \psi(x) = x^s, 0 < s \leq 1,$ *we have*

$$\begin{aligned} X(p) &= 2^{\frac{1}{p}}\left[\frac{1}{2}\left(\frac{1}{p+2}\right)^{\frac{1}{p}} + \frac{1}{2}\left(\frac{p+1}{p+2}\right)^{\frac{1}{p}}\right] \\ &\leq 2^{\frac{1}{p}}\left(\frac{\frac{1}{p+2} + \frac{p+1}{p+2}}{2}\right)^{\frac{1}{p}} \\ &= 1. \end{aligned}$$

Hence, $M(p) \leq N(p)$*.*

Corollary 2.14. *Under the assumptions of Theorem 2.3 with* $\theta = 0$*, then we have the following generalized trapezoid type inequality*

$$\begin{aligned} &\left|\lambda f(a) + (1-\lambda) f(b) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq\ & (b-a)\left|f'(A_\lambda)\right|\left(\frac{1}{2}\right)^{\frac{1}{q}}\left\{\left(\frac{1}{p+1}\right)^{\frac{1}{p}} + \left(\frac{1}{p+2}\right)^{\frac{1}{p}}\right\}\left[\lambda^2\left|f'(a)\right| + (1-\lambda)^2\left|f'(b)\right|\right], \end{aligned}$$

where $A_\lambda = (1-\lambda)a + \lambda b$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*.*

Corollary 2.15. *Under the assumptions of Theorem 2.3 with* $\theta = 1$*, if* $|f'(x)| \leq M$*,* $x \in [a,b]$*, then we have the following Ostrowski type inequality*

$$\begin{aligned} &\left|f(x) - \frac{1}{b-a}\int_a^b f(u)du\right| \\ \leq\ & M^2\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left\{\left(\frac{1}{p+1}\right)^{\frac{1}{p}} + 1\right\}\left[\frac{(x-a)^2 + (b-x)^2}{b-a}\right] \end{aligned}$$

for each $x \in [a,b]$*.*

Proof. For each $x \in [a,b]$, there exist $\lambda_x \in [0,1]$ such that $x = (1-\lambda_x)a + \lambda_x b$. Hence we have $\lambda_x = \frac{x-a}{b-a}$ and $1 - \lambda_x = \frac{b-x}{b-a}$. Therefore, for each $x \in [a,b]$, from the inequality (17) we obtain the desired inequality. □

Corollary 2.16. *Under the assumptions of Theorem 2.3 with* $\lambda = \frac{1}{2}$ *and* $\theta = \frac{2}{3}$*, then we have the following Simpson type inequality*

$$\begin{aligned} &\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq\ & (b-a)\left(\frac{1}{2}\right)^{1+\frac{1}{q}}\left\{C^{\frac{1}{p}}\left(\frac{2}{3},p\right) + D^{\frac{1}{p}}\left(\frac{2}{3},p\right)\right\}\left|f'\left(\frac{a+b}{2}\right)\right| A\left(\left|f'(a)\right|, \left|f'(b)\right|\right) \end{aligned}$$

where A *is the arithmetic mean.*

Corollary 2.17. *Under the assumptions of Theorem 2.3 with $\lambda = \frac{1}{2}$ and $\theta = 1$, then we have the following midpoint type inequality*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \\ \leq\ & (b-a) \left| f'\left(\frac{a+b}{2}\right) \right| \left(\frac{1}{2}\right)^{1+\frac{1}{q}} \left\{ C^{\frac{1}{p}}(1,p) + D^{\frac{1}{p}}(1,p) \right\} A\left(\left|f'(a)\right|, \left|f'(b)\right| \right), \end{aligned}$$

where A is the arithmetic mean.

Corollary 2.18. *Under the assumptions of Theorem 2.3 with $\lambda = \frac{1}{2}$ and $\theta = 0$, then we have the following trapezoid type inequality*

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \\ \leq\ & (b-a) \left(\frac{1}{2}\right)^{1+\frac{1}{q}} \left(\frac{1}{p+2}\right)^{\frac{1}{p}} \left\{ \left(\frac{1}{p+1}\right)^{\frac{1}{p}} + 1 \right\} \left| f'\left(\frac{a+b}{2}\right) \right| A\left(\left|f'(a)\right|, \left|f'(b)\right| \right), \end{aligned}$$

where A is the arithmetic mean.

3. Some applications for special means

Let us recall the following special means of arbitrary real numbers a, b with $a \neq b$ and $\alpha \in [0,1]$:

(1) The weighted arithmetic mean

$$A_\alpha(a,b) := \alpha a + (1-\alpha)b,\ a, b \in \mathbb{R}.$$

(2) The weighted geometric mean

$$G_\alpha(a,b) := a^\alpha b^{1-\alpha}, \quad a, b > 0.$$

(3) The Logarithmic mean

$$L(a,b) := \frac{b-a}{\ln b - \ln a}, \quad a \neq b, \quad a, b > 0.$$

Now, using the results of Section 2, some new inequalities are derived for the above means.

Proposition 3.1. *Let $a, b \in \mathbb{R}$ with $0 < a < b$ and $\lambda, \theta \in [0,1]$ we have the following inequality:*

$$\begin{aligned} &\left| (1-\theta) A_\lambda\left(e^a, e^b\right) + \theta G_\lambda\left(e^a, e^b\right) - L\left(e^a, e^b\right) \right| \\ \leq\ & (b-a) A_1(\theta) e^{A_\lambda(a,b)} \left(\lambda^2 e^a + (1-\lambda)^2 e^b \right) \end{aligned}$$

where $A_1(\theta)$ is defined as in Theorem 2.1.

Proof. The assertion follows from Theorem 2.1 for the function $f(t) = e^t$, $t \in [0, \infty)$. □

Proposition 3.2. *Let $a,b\in\mathbb{R}$ with $0<a<b,\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1$ and $\lambda,\theta\in[0,1]$ we have the following inequality:*

$$\begin{aligned}&\left|(1-\theta)\,A_\lambda\left(e^a,e^b\right)+\theta G_\lambda\left(e^a,e^b\right)-L\left(e^a,e^b\right)\right|\\ \leq\ & (b-a)\left(\frac{\theta^{p+1}+(1-\theta)^{p+1}}{p+1}\right)^{\frac{1}{p}}e^{A_\lambda(a,b)}\left(\lambda^2e^a+(1-\lambda)^2e^b\right).\end{aligned}$$

Proof. The assertion follows from Theorem 2.2 for the function $f(t)=e^t,\ t\in[0,\infty)$. □

Proposition 3.3. *Let $a,b\in\mathbb{R}$ with $0<a<b,\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1$ and $\lambda,\theta\in[0,1]$ we have the following inequality:*

$$\begin{aligned}&\left|(1-\theta)\,A_\lambda\left(e^a,e^b\right)+\theta G_\lambda\left(e^a,e^b\right)-L\left(e^a,e^b\right)\right|\\ \leq\ & (b-a)\,e^{A_\lambda(a,b)}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left\{C^{\frac{1}{p}}(\theta,p)+D^{\frac{1}{p}}(\theta,p)\right\}\left[\lambda^2e^a+(1-\lambda)^2e^b\right].\end{aligned}$$

Proof. The assertion follows from Theorem 2.3 for the function $f(t)=e^t,\ t\in[0,\infty)$. □

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