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AN INTEGRAL EQUATION INVOLVING SAIGO-MAEDA OPERATOR

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ABSTRACT. The aim of this paper is to obtain a solution of integral equation of the Saigo- Maeda operator which contain Appell-hypergeometric function as a kernel. The integral equation and its solution gives new form of generalised fractional integral and generalised fractional derivative. Further various consequences also investigated.

Keywords: Saigo-Maeda fractional integral operator and derivatives, Appell hypergeometric function.
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1. INTRODUCTION

The Appell hypergeometric function of the third type $F_3(-)$ as [8]

$$F_3(a,a',b,b';c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_m(a')_n(b)_m(b')_n}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!},\ (|x|<1,|y|<1). \tag{1}$$

Which is also written as

$$F_3(a,a',b,b';c;x,y)=\sum_{n=0}^{\infty}\frac{(a')_n(b')_n}{(c)_n}\ {}_2F_1(a,b;c+n;x)\frac{y^n}{n!}, \tag{2}$$

where ${}_2F_1$(-) is the Gauss hypergeometric function defined as

$${}_2F_1(a,b;c;x)=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(c)_m}\frac{x^m}{m!}, \tag{3}$$

where $(|x|<1)$ and $c\neq 0$ or negative integer and $(a)_m$ is the pochhammer symbol such as

$$(a)_m=a(a+1)...(a+m-1),\ (a)_0=1,\ where\ a\in C\ m\in N. \tag{4}$$

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The function F_3(-) in (1) is reduces to the Gauss hypergeometric function

$$\begin{aligned} F_3(a,a',b,b';c;x,y) &= F_3(a,a',b,b';c;x,0) = F_3(a,a',b,0;c;x,y) \\ &= F_3(a,0,b,b';c;x,y) = {}_2F_1(a,b;c;x) = F(a,b;c;x) \end{aligned} \tag{5}$$

and

$$\begin{bmatrix} F_3(a,a',b,b';c;x,y) = (1-x)^{a'}F(b,a+a';c;x), \\ F(a,b;c;x) = (1-x)^{-a}F(a,c-b;c;\dfrac{x}{(1-x)}) \end{bmatrix} \tag{6}$$

Definition 1.1. *Let* $\alpha,\ \alpha',\ \beta,\ \beta' \in \mathbb{C}$ *and* $\gamma \in \mathbb{R}_+$ *and* ($0 < \gamma < 1$), *Here* $\mathbb{C}$ *is the class of analytic function* $f(z)$*in a simply-connected region containing the origin and if the multiplicity of* $(t-x)^{(\gamma-1)}$ *to be real* $x < t$.
Then consider the following integral equation

$$\left(I^{\alpha,\alpha',\beta,\beta',\gamma}\right)f(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)}\int_x^\infty (t-x)^{\gamma-1}t^{-\alpha}F_3\left(\alpha,\alpha',\beta,\beta',\gamma;1-\frac{t-x}{t},\frac{x-t}{x}\right)f(t)dt. \tag{7}$$

2. Main Result

In this paper, we obtain a formal solution of integral equation (7) involving the Appell hypergeometric function in the kernel. The integral equations with the F_3 kernel used by Higgins [2] and Maricev [4] and applied the method for obtaining the solution follows similar works of studying analogous. These references are similar as well as the book written by Srivastava and Buschman [7] and these describe in a comprehensive manner, which are useful in various application such as theory of convolution type integral equations.
To obtain the solution of integral equation (7) formally, let

$$\left(I^{\alpha,\alpha',\beta,\beta',\gamma}\right)f(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)}\int_x^\infty (t-x)^{\gamma-1}t^{-\alpha}F_3\left(\alpha,\alpha',\beta,\beta',\gamma;1-\frac{t-x}{t},1-\frac{x-t}{x}\right)f(t)dt = g(x). \tag{8}$$

Using equation (2) in (8), we have

$$g(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)}\int_x^\infty (t-x)^{\gamma+r-1}t^{-\alpha-r}\sum_{r=0}^\infty \frac{(\alpha)_r(\beta)_r}{(\gamma)_r r!}F\left(\alpha,\alpha',\beta,\beta',\gamma+r;\frac{x-t}{x}\right)f(t)dt \tag{9}$$

Replacing x by t and t by p in equation (9), we have

$$g(t) = \frac{t^{-\alpha'}}{\Gamma(\gamma)}\int_t^\infty (p-t)^{\gamma+r-1}p^{-\alpha-r}\sum_{r=0}^\infty \frac{(\alpha)_r(\beta)_r}{(\gamma)_r r!}F\left(\alpha,\alpha',\beta,\beta',\gamma+r;\frac{t-p}{x}\right)f(p)dp \tag{10}$$

Now multiplying both the sides by

$(t-x)^{m-\gamma-1}t^{-\alpha'}F_3\left(-\alpha',-\alpha,-\beta',m-\beta;m-\gamma;\frac{t-x}{t},\frac{x-t}{x}\right)$

where $m \in N$, then the above expression is equivalent to

$(t-x)^{m-\gamma-1}t^{\alpha'}\sum_{s=0}^\infty \frac{(\alpha)_s(m-\beta)_s}{(m-\gamma)_s s!}\left(\frac{x-t}{x}^q\right)F\left(-\alpha',-\beta',m-\gamma+s;\frac{t-x}{t}\right)$

Using equation (6) in above expression, we have

$$= \sum_{s=0}^{\infty}(t-x)^{m-\gamma+s-1}t^{\alpha'}\frac{x^{-s}(-1)^s(-\alpha)_s(m-\beta)_s}{(m-\gamma)_s s!}$$

$$\times\left[1-\frac{(t-x)}{t}\right]^{\alpha'} F\left(-\alpha', m-\gamma+s+\beta'; m-\gamma+s; \frac{t-x}{x}\right) \tag{11}$$

using equation (11) in equation (10) and integrate both side from x to ∞.

$$\int_x^{\infty}\sum_{s=0}^{\infty}(t-x)^{m-\gamma+s-1}\frac{x^{\alpha'-s}(-1)^s(-\alpha)_s(m-\beta)_s}{(m-\gamma)_s s!}$$

$$\times F\left(-\alpha', m-\gamma+s+\beta'; m-\gamma+s; \frac{(t-x)}{x}\right) g\left(t\right) dt$$

$$= \int_x^{\infty}\sum_{s=0}^{\infty}(t-x)^{m-\gamma+s-1}\frac{x^{\alpha'-s}(-1)^s(-\alpha)_s(m-\beta)_s}{(m-\gamma)_s s!}$$

$$F\left(-\alpha', m-\gamma+s+\beta'; m-\gamma+s; \frac{(t-x)}{x}\right)$$

$$\times\frac{t^{-\alpha'}}{\Gamma\left(\gamma\right)}\int_t^{\infty}(p-t)^{\gamma+r-1}p^{-\alpha-r}\sum_{r=0}^{\infty}\frac{(\alpha)_r(\beta)_r}{(\gamma)_r r!}F\left(\alpha', \beta', \gamma+r; \frac{t-p}{t}\right) f(p)dpdt \tag{12}$$

Let us consider the right hand side of equation (12), and changing order of integration, we have

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\alpha)_r(\beta)_r}{(\gamma)_r r!}\frac{x^{\alpha'-s}(-1)^s(-\alpha)_s(m-\beta)_s}{\Gamma\left(\gamma\right)(m-\gamma)_s s!}$$

$$\times\int_x^{\infty}\int_x^{p}(p-t)^{\gamma+r-1}p^{-\alpha-r}t^{-\alpha'}(t-x)^{m-\gamma+s-1}F\left(\alpha', \beta', \gamma+r; \frac{t-p}{t}\right)$$

$$\times F\left(-\alpha', m-\gamma+s+\beta'; m-\gamma+s; \frac{(t-x)}{x}\right) f\left(p\right)\ dt\ dp$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\alpha)_r(\beta)_r}{(\gamma)_r r!}\frac{x^{\alpha'-s}(-1)^s(-\alpha)_s(m-\beta)_s}{\Gamma\left(\gamma\right)(m-\gamma)_s s!}\int_x^{\infty} p^{-\alpha-r}f\left(p\right)\ dp$$

$$\times\int_x^{p}(p-t)^{\gamma+r-1}t^{-\alpha'}(t-x)^{m-\gamma+s-1}F\left(\alpha', \beta', \gamma+r; \frac{t-p}{t}\right)$$

$$\times F\left(-\alpha', m-\gamma+s+\beta'; m-\gamma+s; \frac{(t-x)}{x}\right) dt \tag{13}$$

Put $t = p + (1-y)(x-p)$ in equation (13) in right hand side, we have

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(\alpha)_r(\beta)_r}{(\gamma)_r r!} \frac{x^{\alpha'-s}(-1)^s(-\alpha)_s(m-\beta)_s}{\Gamma(\gamma)(m-\gamma)_s s!} \int_x^{\infty} p^{-\alpha-r} f(p)\ dp$$

$$\times \int_0^1 (1-y)^{\gamma+r-1}(s-x)^{m+r+s-1}(x)^{-\alpha'}(y)^{m+\gamma+s-1}\left(1-\frac{y(x-s)}{x}\right)^{-\alpha'}$$

$$\times F\left(\alpha',\beta',\gamma+r;\frac{\frac{(1-y)(x-p)}{x}}{1-\frac{y(x-p)}{x}}\right) F\left(-\alpha', m-\gamma+s+\beta'; m-\gamma+s; \frac{y(p-x)}{x}\right)\ dy \quad (14)$$

Using the following know formula [1].

$$F(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(\lambda)\Gamma(c-\lambda)} \int_0^1 s^{\lambda-1}(1-s)^{c-\lambda-1}(1-sx)^{a'} F(a-a',b;\lambda;sx)$$
$$\times F\left(a', b-\lambda; c-\lambda; \frac{x(1-s)}{1-sx}\right)\ dx$$

in (14), we obtain

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(\alpha)_r(\beta)_r}{(\gamma)_r r!} \frac{x^{-s}(-1)^s(-\alpha)_s(m-\beta)_s}{\Gamma(\gamma)(m-\gamma)_s s!} \frac{\Gamma(m-\gamma+s)\ \Gamma(\gamma+r)}{\Gamma(m+r+s)}$$

$$\int_x^{\infty} p^{-\alpha-r}(p-x)^{m+r+s-1} f(p) F\left(-0, m-\gamma+q+\beta'; m+q+p; \frac{(x-s)}{x}\right) dp$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(\alpha)_r(\beta)_r}{r!} \frac{x^{-s}(-1)^s(-\alpha)_s(m-\beta)_s}{(m)_{r+s}\ s!} \frac{\Gamma(m-\gamma)}{\Gamma(m)} \int_x^{\infty} p^{-\alpha-r}(p-x)^{m+r+s-1} f(p)\, dp$$

$$= \frac{\Gamma(m-\gamma)}{\Gamma(m)} \int_x^{\infty} p^{-\alpha}(p-x)^{m+r+s-1} F_3\left(\alpha,-\alpha,\beta,m-\beta;m;\frac{p-x}{p},\frac{x-p}{x}\right) f(p)\, dp \quad (15)$$

Using equation (6) in equation (15), we obtain

$$= \frac{\Gamma(m-\gamma)}{\Gamma(m)} x^{-\alpha} \int_x^{\infty} (p-x)^{m-1} f(p)\ dp \quad (16)$$

Using equation (16) in equation (12), we have

$$\int_x^{\infty} \sum_{s=0}^{\infty} (t-x)^{m-\gamma+s-1} \frac{x^{\alpha'-s}(-1)^s(-\alpha)_s(m-\beta)_s}{(m-\gamma)_s s!}$$
$$\times F\left(-\alpha', m-\gamma+s+\beta'; m-\gamma+s; \frac{(t-x)}{x}\right) g(t)\, dt$$

$$= \frac{\Gamma(m-\gamma)}{\Gamma(m)}\ x^{-\alpha} \int_x^{\infty} (p-x)^{m-1} f(p)\ dp$$

Which also gives as

$$\frac{x^{\alpha}}{\Gamma(m-\gamma)}\int_x^{\infty}\sum_{s=0}^{\infty}(t-x)^{m-\gamma+s-1}\frac{x^{\alpha'-s}(-1)^s(-\alpha)_s(m-\beta)_s}{(m-\gamma)_s s!}$$

$$\times F\left(-\alpha', m-\gamma+s+\beta'; m-\gamma+s; \frac{(t-x)}{x}\right) g(t)\, dt$$

$$=\frac{1}{(m-1)!}\int_x^{\infty}(p-x)^{m-1} f(p)\ dp \tag{17}$$

Differentiate m times, we obtain

$$f(x)=\frac{d^m}{dx^m}\left(\frac{x^{\alpha}}{\Gamma(m-\gamma)}\int_x^{\infty}(t-x)^{m-\gamma-1}t^{\alpha'}\right)$$

$$\times\left(F_3\left(-\alpha', -\alpha, -\beta', m-\beta, m-\gamma; \frac{(t-x)}{t}, \frac{(x-t)}{x}\right)\ g(t)\ dt\right) \tag{18}$$

FRACTIONAL CALCULUS OPERATOR ASSOCIATED WITH F_3 FUNCTION

The pair of integral equations (7) and (18) permits us to define new forms of generalised fractional calculus operator involving the third Appell function defined by (1). In view of equation (7), the generalised fractional integral operator $\left(I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma}\right)$ of a function $f(x)$ is defined as

Let $\alpha,\ \alpha',\ \beta,\ \beta' \in \mathbb{C}$ and $\gamma \in \mathbb{R}_+$ and ($0<\gamma<1$), ($\mathbb{R}_+(\gamma)>0$)

$$I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma} f(x)=\frac{x^{-\alpha'}}{\Gamma\gamma}\int_x^{\infty}(t-x)^{\gamma-1}t^{\alpha}F_3\left(\alpha', \alpha, \beta, \beta', \gamma; \frac{(t-x)}{t}, \frac{(x-t)}{x}\right) f(t)\ dt. \tag{19}$$

where $\left(I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma}\right)=\left(I^{\alpha,\alpha',\beta,\beta',\gamma}\right)$ and

$$I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma} f(x)=\left(\frac{-d}{dx}\right)^m\left(I_{x,\infty}^{\alpha,\alpha',\beta,\beta'+m,\gamma+m}\right) f(x),\ (\mathbb{R}_+(\gamma)\le 0, m\ =\ [-\mathbb{R}_+(\gamma)+1]). \tag{20}$$

Based upon the solution (18) of the integral equation (7), the generalised fractional derivative $\left(D_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma}\right)$ of a function $f(x)$ can be defined by operator

$$D_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma} f(x)=\left(-\tfrac{d}{dx}\right)^m\left(I_{x,\infty}^{\alpha,\alpha',\beta,\beta'+m,m-\gamma}\right) f(x);$$

$$(\mathbb{R}_+(\gamma)>0, (m-1)\le\gamma\le m; m=[\mathbb{R}_+(\gamma)+1], m\in N)$$

$$=\frac{d^m}{dx^m}\frac{x^{\alpha}}{\Gamma(m-\gamma)}\left(\int_x^{\infty}(t-x)^{m-\gamma-1}t^{\alpha'}\right)$$

$$\times\left(F_3\left(-\alpha', -\alpha, -\beta', m-\beta, m-\gamma; \frac{(t-x)}{t}, \frac{(x-t)}{x}\right) g(t)\ dt\right) \tag{21}$$

were earlier defined by Saigo and Maeda [6] and Kiryakova [3] as the generalised operators of fractional integral and fractional derivative of a function $f(x)$ involving the third Appell function, respectively.

The power function x^ρ under The Saigo-Maeda operators (20) and (21) are given by [6]:

$$I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1} = \frac{\Gamma(1-\beta-\rho)\,\Gamma(1+\alpha+\alpha'-\gamma-\rho)\,\Gamma(1+\alpha+\beta'-\gamma-\rho)}{\Gamma(1+\alpha+\alpha'+\beta'-\gamma-\rho)\,\Gamma(1+\alpha-\beta-\rho)\,\Gamma(1-\rho)}\, x^{(\rho-\alpha-\alpha'+\gamma-1)}, \tag{22}$$

$$\mathbb{R}_+(\gamma)>0,\ \mathbb{R}_+(\rho)<1+min\left[0,\mathbb{R}_+(-\beta),\mathbb{R}_+\left(\alpha+\alpha'-\gamma\right),\mathbb{R}_+\left(\alpha+\beta'-\gamma\right)\right].$$

and $I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma}(x)^{\rho-1} = \left(-\frac{d}{dx}\right)^m \left(I_{x,\infty}^{\alpha,\alpha',\beta,\beta'+m,\gamma+m}\right)(x)^{\rho-1}$

$$= \frac{\Gamma(1-\beta-\rho)\,\Gamma(1+\alpha+\alpha'-\gamma-\rho)\,\Gamma(1+\alpha+\beta'-\gamma-\rho)}{\Gamma(1+\alpha+\alpha'+\beta'-\gamma-\rho)\,\Gamma(1+\alpha-\beta-\rho)\,\Gamma(1-\rho)}\, x^{(\rho-\alpha-\alpha'+\gamma-1)}, \tag{23}$$

$$\left(\mathbb{R}_+(\gamma)>0,\ \mathbb{R}_+(\rho)<1+min\left[0,\mathbb{R}_+(-\beta),\mathbb{R}_+\left(\alpha+\alpha'-\gamma\right),\mathbb{R}_+\left(\alpha+\beta'-\gamma\right)\right]\right)$$

On the other hand, it is worth nothing here that our generalised fractional derivative operator (21) gives the following image formula for the power function $(x)^{\rho-1}$.

If $((m-1)\le\gamma\le m; m=[\mathbb{R}_+(\gamma)+1], m\in N)$,

$$D_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma} f(x) = \left(\frac{-d}{dx}\right)^m \left(I_{x,\infty}^{\alpha,\alpha',\beta,\beta'+m,m-\gamma}\right) f(x)$$

$$= \frac{\Gamma(1-\beta'-\rho)\,\Gamma(1-\alpha-\alpha'+\gamma-\rho)\,\Gamma(1-\alpha'-\beta+\gamma-\rho)}{\Gamma(1-\alpha-\alpha'-\beta+\gamma-\rho)\,\Gamma(1-\alpha'+\beta'-\rho)\,\Gamma(1-\rho)}\, x^{(\rho-\alpha-\alpha'+\gamma-1)} \tag{24}$$

$$\mathbb{R}_+(\gamma)>0,\ \mathbb{R}_+(\rho)<1+min\left[0,\mathbb{R}_+\left(-\beta'\right),\mathbb{R}_+\left(-\alpha-\alpha'+\gamma\right),\mathbb{R}_+\left(-\alpha'-\beta+\gamma\right)\right].$$

The operator (19) and (21) satisfy the following relationship

$$\left(I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma}\right)^{-1} = D_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma} = I_{x,\infty}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} \tag{25}$$

Which provide improvement to similar type of operational relationship given in [5]. It may be observed that when $\alpha' = 0$ in equation (25), we get the following Saigo type fractional integral and differential operators relationship [6].

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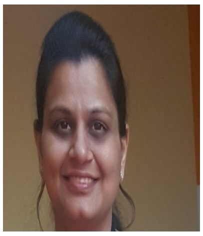

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