TWMS J. App. and Eng. Math. V.11, Special Issue, 2021, pp. 75-87

DIAMETRAL PATHS IN EXTENDED TRANSFORMATION GRAPHS

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Abstract. In a graph, diametral path is shortest path between two vertices which has length equal to diameter of the graph. Number of diametral paths plays an important role in computer science and civil engineering. In this paper, we introduce the concept of extended transformation graphs. There are 64 extended transformation graphs. We obtain number of diametral paths in some of the extended transformation graphs and we also study the semi-complete property of these extended transformation graphs. Further, a program is given for obtaining number of diametral paths.

Keywords: Diameter, diametral path, transformation graph, semi-complete graph.
AMS Subject Classification: 05C12, 05C76.

1. Introduction

All shortest paths between two vertices in a graph which have length equal to diameter of the graph are diametral paths. Counting number of diametral paths is useful in computer science. Finding number of diametral paths is also significant in civil engineering, like optimally using steel bars. The total number of diametral paths incident at a vertex v in graph G is called diametral reachable index of that vertex v, denoted by DRI(v). Walikar & Shindhe [13] introduced concept of diametral reachable index and gave its algorithm. Diameter of a graph is studied in [2, 3, 10]. Deogun & Kratsch [5] introduced diametral path graphs. Mangam & Kureethara [7–9] have given diametral paths in total graphs.

In a simple graph, for any two vertices v_i and v_j, if there exist a third vertex v_k which is adjacent to both the vertices v_i and v_j then the graph is known as semi-complete. It was introduced by Rao & Raju [12] to solve some defence problems. Further, Amiripalli & Bobba [1] defined *trimet graph optimization topology* on scalable networks, which follows semi-complete property.

Line graph $L(G)$ of graph $G = (V, E)$, is a graph whose vertex set is E and two vertices in $L(G)$ are adjacent if and only if they are adjacent edges in G. Transformation graph is $\mathbb{G}^{\,x_1x_2x_3}(G) = (V_{\mathbb{G}}, E_{\mathbb{G}})$ where $x_r \in \{+, -\}$ $r = 1, 2, 3$ and $V_{\mathbb{G}} = V \cup E$ and for $v_iv_j \in E_{\mathbb{G}}$ if and only if one of the following conditions holds:
(i) $v_i, v_j \in V$ and v_i, v_j are adjacent in G (v_i, v_j are not adjacent in G) for $x_1 = +(-)$.
(ii) $v_i, v_j \in E$ and v_i, v_j are adjacent in G (v_i, v_j are not adjacent in G) for $x_2 = +(-)$.
(iii) $v_i \in V, v_j \in E$ and v_j is incident at v_i in G (v_j is not incident at v_i in G) for

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§ Manuscript received: September 27, 2019; accepted: April 02, 2020.

$x_3 = +(-)$.
Some studies [4, 6, 11] have explored line graph of complete graph and several researchers have worked on the transformation graphs. $G^{+++}(G)$ is total graph $T(G)$ of graph G and $G^{-++}(G)$ is quasi-total graph of graph G.

Let $G = (V, E)$ be a graph with $\mid V \mid = n$. Now we introduce notion of **extended transformation graph**, $\mathbb{G}_{\mathrm{e}}{}^{x_1x_2x_3}_{x_4x_5x_6}(G) = (V_{\mathbb{G}_{\mathrm{e}}}, E_{\mathbb{G}_{\mathrm{e}}})$ where $x_r \in \{+, -\}$ $r = 1, \ldots, 6$ and $V_{\mathbb{G}_{\mathrm{e}}} = V \cup E \cup \overline{E}$ and for $v_i, v_j \in V_{\mathbb{G}_{\mathrm{e}}}$,$v_iv_j \in E_{\mathbb{G}_{\mathrm{e}}}$ if and only if one of the following conditions holds:
(i) $v_i, v_j \in V$ and v_i, v_j are adjacent in G (v_i, v_j are not adjacent in G) for $x_1 = +(-)$.
(ii) $v_i, v_j \in E$ and v_i, v_j are adjacent in G (v_i, v_j are not adjacent in G) for $x_2 = +(-)$.
(iii) $v_i \in V, v_j \in E$ and v_j is incident at v_i in G (v_j is not incident at v_i in G) for $x_3 = +(-)$.
(iv) $v_i, v_j \in \overline{E}$ and v_i, v_j are adjacent in $\overline{G}$ (v_i, v_j are not adjacent in $\overline{G}$) for $x_4 = +(-)$.
(v) $v_i \in V, v_j \in \overline{E}$ and v_j is incident at v_i in $\overline{G}$ (v_j is not incident at v_i in $\overline{G}$) for $x_5 = +(-)$.
(vi) $v_i \in E, v_j \in \overline{E}$ and v_i, v_j are adjacent in K_n (v_i, v_j are not adjacent in K_n) for $x_6 = +(-)$.

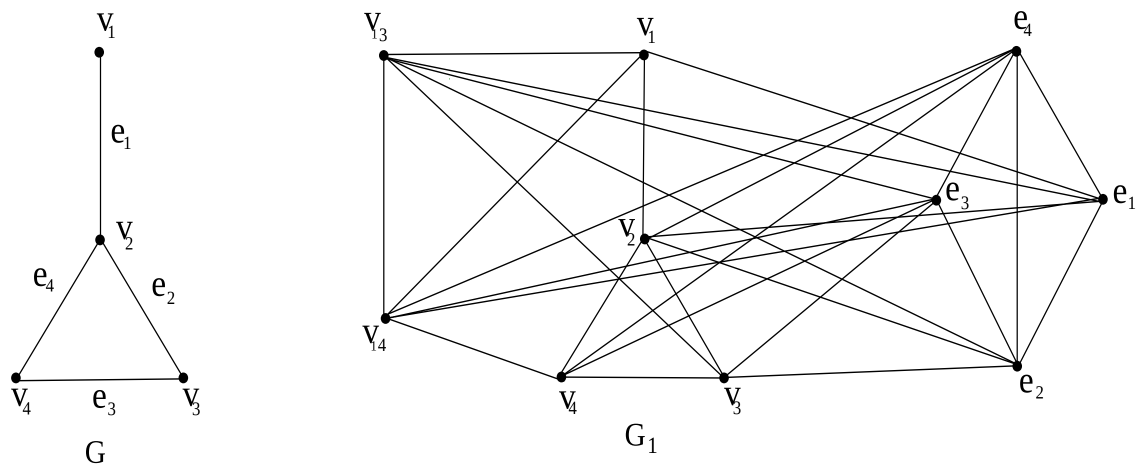

FIGURE 1. Showing the extended transformation graph G_1=$\mathbb{G}_{\mathrm{e}}{}^{+++}_{+++}(G)$ of the graph G

The terminologies used in this paper are taken from West [14]. All graphs are undirected, simple and unweighted. We will use notations v_i and v_{ij} for vertices in extended transformation graph. The vertices $v_i \in V$ and $v_{ij} \in E \cup \overline{E}$, where $v_{ij} = v_iv_j; 1 \leq i, j \leq n, i \neq j$ and $v_{ij} = v_{ji}$. In Section 2, we determine number of diametral paths in various extended transformation graphs. In Corollary 3, we will give an alternate proof of number of diametral paths in total graph of complete graph which is previously given by Mangam & Kureethara [7]. Moreover, we study the semi-complete property of these extended transformation graphs. In Section 3, we will give a program for finding number of diametral paths.

2. Diametral paths in $\mathbb{G}_e{}^{+++}_{+++}(G)$, $\mathbb{G}_e{}^{++-}_{+-+}(G)$, $\mathbb{G}_e{}^{+-+}_{-+-}(G)$ and $\mathbb{G}_e{}^{+--}_{---}(G)$

In this section, we determine number of diametral paths and semi-complete property in $\mathbb{G}_e{}^{+++}_{+++}(G)$, $\mathbb{G}_e{}^{++-}_{+-+}(G)$, $\mathbb{G}_e{}^{+-+}_{-+-}(G)$ and $\mathbb{G}_e{}^{+--}_{---}(G)$. We also determine semi-complete property and number of diametral paths in $\mathbb{G}_e{}^{-++}_{+++}(G)$, $\mathbb{G}_e{}^{-+-}_{+-+}(G)$, $\mathbb{G}_e{}^{--+}_{-+-}(G)$ and $\mathbb{G}_e{}^{---}_{---}(G)$ as corollary. Vertex set of $\mathbb{G}_e{}^{+++}_{+++}(G)$ and $\mathbb{G}_e{}^{++-}_{+-+}(G)$ can be partitioned into two parts such that one part induces G and the other part induces $L(K_n)$, where n is the order of G. Vertex set of $\mathbb{G}_e{}^{+-+}_{-+-}(G)$ and $\mathbb{G}_e{}^{+--}_{---}(G)$ can also be partitioned into two parts such that one part induces G and the other part induces $\overline{L(K_n)}$, where n is the order of G

Let $G_1(V_1, E_1)$ and $G_2(V_2 = E_1 \cup \overline{E_1}, E_2)$ be two simple graphs such that vertex set of $V_1 = \{v_i; 1 \le i \le n\}$ and vertex set of $V_2 = \{v_{ij}; 1 \le i, j \le n, i \ne j \text{ and } v_{ij} = v_{ji}\}$. Now we define two new operations as:

(i) $G_1 \oplus_1 G_2 = G(V, E)$,
$V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup E_3$ where $E_3 = \{v_i v_{ij}\}$.

(ii) $G_1 \oplus_2 G_2 = G(V, E)$,
$V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup E_3$ where $E_3 = \{v_i v_{jk}; i \ne j, k\}$.

Proposition 1. *If $|V(G)| = n$ then $\mathbb{G}_e{}^{+++}_{+++}(G) \cong G \oplus_1 L(K_n)$.*

Proof. The proof is obvious from the definition of $\mathbb{G}_e{}^{+++}_{+++}(G)$ and the definition of operation $\oplus_1$. □

Theorem 2.1. *The extended transformation graph $\mathbb{G}_e{}^{+++}_{+++}(G)$ of a graph G is semi-complete except G is $K_1 \cup K_1$.*

Proof. Since $\mathbb{G}_e{}^{+++}_{+++}(K_1) \cong K_1$, $\mathbb{G}_e{}^{+++}_{+++}(P_2) \cong K_3$ and $\mathbb{G}_e{}^{+++}_{+++}(K_1 \cup K_1) \cong P_3$. Now K_1 is trivially semi-complete, K_3 is semi-complete and P_3 is not semi-complete. Therefore using Proposition 1 in $\mathbb{G}_e{}^{+++}_{+++}(G)$ for $|G| = n > 2$, we determine the semi-complete property by analysing the following cases:

Case1− : Let $v_i, v_j \in V(G)$ then $v_{ij} \in V(L(K_n))$. Therefore between any two vertices in G, there always exists a third vertex v_{ij}, which is adjacent to both the vertices v_i and v_j.

Case2− : Let $v_{ij}, v_{kl} \in V(L(K_n))$,
Since from Theorem $IV.7$ of Rao & Raju [12], line graph $L(G)$ is semi-complete if and only if G is complete. So between any two vertices of $L(K_n)$, we always have a third vertex in $L(K_n)$ which is adjacent to both the vertices of $L(K_n)$.

Case3− : Let $v_i \in V(G)$ and $v_{jk} \in V(L(K_n))$ then consider the following two subcases:
Subcase 3.1 -: If $i = j$ then $v_i \in V(G)$ and $v_{ik} \in V(L(K_n))$. Now third vertex $v_{il} \in V(L(K_n))(l \ne i, k)$, which is adjacent to both the vertices v_i and v_{ik}.
Subcase 3.2 -: If $i \ne \{j, k\}$ then $v_i \in V(G)$ and $v_{jk} \in V(L(K_n))$. Now third vertex $v_{ik} \in V(L(K_n))$, which is adjacent to both the vertices v_i and v_{jk}.
Therefore between any two vertices in $\mathbb{G}_e{}^{+++}_{+++}(G)$ for $|G| = n > 2$, there always exists a third vertex, which is adjacent to both the vertices.
Hence, $\mathbb{G}_e{}^{+++}_{+++}(G)$ is semi-complete except G is $K_1 \cup K_1$.

□

Proposition 2. *The equation $\mathbb{G}_e{}^{-++}_{+++}(G) \cong \mathbb{G}_e{}^{+++}_{+++}(H)$ holds if and only if G is complement graph of H.*

Proof. Since $\mathbb{G}_e {}^{+++}_{+++}(G) \cong G \oplus_1 L(K_n)$ and $\mathbb{G}_e {}^{-++}_{+++}(H) \cong \overline{H} \oplus_1 L(K_n)$. Therefore, subgraph of $\mathbb{G}_e {}^{-++}_{+++}(G)$ and $\mathbb{G}_e {}^{+++}_{+++}(H)$ obtained by removing edges of G and H are identical. So G is complement graph of H. □

Corollary 1. *The extended transformation graph* $\mathbb{G}_e {}^{-++}_{+++}(G)$ *of a graph* G *is semi-complete except* G *is* P_2.

Proof. The proof follows from Theorem 2.1 and Proposition 2. □

Corollary 2. *Diameter of the extended transformation graph* $\mathbb{G}_e {}^{+++}_{+++}(G)$ *of a graph* G *is 2 except* G *is* K_1 *and* P_2.

Proof. From Observation (3) of Rao & Raju [12], the distance between any two vertices of a semi-complete graph is at most 2 (distance is 1 when vertices are adjacent and distance is 2 when vertices are non adjacent). Since $\mathbb{G}_e {}^{+++}_{+++}(G)$ is semi-complete except G is $K_1 \cup K_1$ and there are non adjacent vertices in $\mathbb{G}_e {}^{+++}_{+++}(G)$ for $n > 2$. So the diameter of $\mathbb{G}_e {}^{+++}_{+++}(G)$ is 2 for $n > 2$. Now $\mathbb{G}_e {}^{+++}_{+++}(K_1) \cong K_1$, $\mathbb{G}_e {}^{+++}_{+++}(P_2) \cong K_3$ and $\mathbb{G}_e {}^{+++}_{+++}(K_1 \cup K_1) \cong P_3$ and diameters of K_1, K_3, P_3 are $0, 1, 2$ respectively. Therefore the diameter of $\mathbb{G}_e {}^{+++}_{+++}(G)$ is 2 except G is K_1 and P_2.

□

Theorem 2.2. *If* G *is a graph of* n *vertices with vertex set* $\{v_i; 1 \leq i \leq n\}$, m *edges,* t *number of triangles and each vertex* v_i *has degree* d_i *and* $G \neq K_1, G \neq P_2$ *then number of diametral paths in the extended transformation graph* $\mathbb{G}_e {}^{+++}_{+++}(G)$ *is*
$(\sum_{i=1}^{n} \binom{d_i}{2}) - 3t + m(2n-5) + \frac{n(n-1)(n^2-3n+3)}{2}$.

Proof. Since $\mathbb{G}_e {}^{+++}_{+++}(G) \cong G \oplus_1 L(K_n)$ and the diameter of $\mathbb{G}_e {}^{+++}_{+++}(G)$ is 2 except G is K_1 and P_2. So we determine the number of diametral paths in $\mathbb{G}_e {}^{+++}_{+++}(G)$ for $G \neq K_1, G \neq P_2$ as follows:

(i) Number of diametral paths between vertices of G through a vertex in G = Number of shortest paths of length 2 in $G = (\sum_{i=1}^{n} \binom{d_i}{2}) - 3t$.

(ii) Number of diametral paths between vertices of G through a vertex in $L(K_n) = ({}^nC_2 - m) \times 1 = ({}^nC_2 - m)$.

(iii) Number of diametral paths between vertices of G and $L(K_n)$ through a vertex in $G = \sum_{i=1}^{n} d_i(n-2) = (n-2)\sum_{i=1}^{n} d_i = 2m(n-2)$.

(iv) Number of diametral paths between vertices of G and $L(K_n)$ through a vertex in $L(K_n) = n \times ({}^nC_2 - (n-1)) \times 2 = n(n-1)(n-2)$.

(v) Number of diametral paths between vertices of $L(K_n)$ through a vertex in $L(K_n)$ [Since through a vertex in G is not possible] $= {}^nC_2 \times ({}^nC_2 - (2n-2-1)) \times 4 \times \frac{1}{2} = \dfrac{n(n-1)(n-2)(n-3)}{2}$.

Hence total number of diametral paths in $\mathbb{G}_e {}^{+++}_{+++}(G)$ is
$\sum_{i=1}^{n} \binom{d_i}{2} - 3t + (\binom{n}{2} - m) + 2m(n-2) + n(n-1)(n-2) + \frac{n(n-1)(n-2)(n-3)}{2} = (\sum_{i=1}^{n} \binom{d_i}{2}) - 3t + m(2n-5) + \frac{n(n-1)(n^2-3n+3)}{2}$ for $G \neq K_1, G \neq P_2$.

□

Corollary 3. *The number of diametral paths in total graph of complete graph* $T(K_n)$ *is* $\frac{n(n-1)(n^2-n-2)}{2}$.

Proof. If G is K_n in $\mathbb{G}_e {}^{+++}_{+++}(G)$ then we have $(\sum_{i=1}^{n} \binom{d_i}{2}) - 3t = 0$ and $m = {}^nC_2$ in Theorem 2.2. Since $\mathbb{G}_e {}^{+++}_{+++}(K_n) \cong T(K_n)$. So the number of diametral paths in $T(K_n)$ is

$=0 + (\binom{n}{2} -^n C_2) + 2\dfrac{n(n-1)}{2}(n-2) + n(n-1)(n-2) + \frac{n(n-1)(n-2)(n-3)}{2}$
$= n(n-1)(n-2)(2 + \frac{n-3}{2}) = \frac{(n+1)n(n-1)(n-2)}{2} = \frac{n(n-1)(n^2-n-2)}{2}.$

□

Corollary 4. *If G is a graph of n vertices, m edges, $\overline{G}$ has vertex set $\{v_i; 1 \leq i \leq n\}$ such that each vertex v_i has degree d_i, t number of triangles in $\overline{G}$ and $G \neq K_1, G \neq K_1 \cup K_1$ then number of diametral paths in $\mathbb{G}_{\mathrm{e}}\,{}^{-++}_{+++}(G)$ is*
$(\sum_{i=1}^{n} \binom{d_i}{2}) - 3t - m(2n-5) + \frac{(n+1)n(n-1)(n-2)}{2}$

Proof. From Proposition 2, we replace m by $(\binom{n}{2} - m)$ in Theorem 2.2 and also the number shortest paths of length 2 is taken in $\overline{G}$ in place of G in Theorem 2.2. In this way, we get the required result.

□

Proposition 3. *If $|V(G)| = n$ then $\mathbb{G}_e\,{}^{++-}_{+-+}(G) \cong G \oplus_2 L(K_n)$.*

Proof. The proof is obvious from the definition of $\mathbb{G}_e\,{}^{++-}_{+-+}(G)$ and the definition of operation $\oplus_2$. □

Theorem 2.3. *Let G be a graph of n vertices then the extended transformation graph $\mathbb{G}_{\mathrm{e}}\,{}^{++-}_{+-+}(G)$ is semi-complete for $n \geq 4$.*

Proof. Since $\mathbb{G}_e\,{}^{++-}_{+-+}(K_1) \cong K_1$, $\mathbb{G}_e\,{}^{++-}_{+-+}(K_1 \cup K_1) \cong K_1 \cup K_1 \cup K_1$, $\mathbb{G}_e\,{}^{++-}_{+-+}(K_2) \cong K_1 \cup P_2$ and if $|V(G)| = 3$ then there does not exist a third vertex, which is adjacent to both the vertices v_i and v_{jk} $(i \neq \{j,k\})$ in $\mathbb{G}_e\,{}^{++-}_{+-+}(G)$. Therefore, $\mathbb{G}_{\mathrm{e}}\,{}^{++-}_{+-+}(G)$ is not semi-complete for $|V(G)| = n \leq 3$. Now using Proposition 3 in $\mathbb{G}_{\mathrm{e}}\,{}^{++-}_{+-+}(G)$ for $|V(G)| = n > 3$, we determine the semi-complete property by analysing the following cases:
Case1− : Let $v_i, v_j \in V(G)$ then $v_{kl} \in V(L(K_n))$ $(i \neq j \neq k \neq l)$. Therefore between any two vertices in G, there always exists a third vertex v_{kl}, which is adjacent to both the vertices v_i and v_j.
Case2− : Let $v_{ij}, v_{kl} \in V(L(K_n))$,
Since from Theorem IV.7 of Rao & Raju [12], line graph $L(G)$ is semi-complete if and only if G is complete. So between any two vertices of $L(K_n)$, we always have a third vertex in $L(K_n)$ which is adjacent to both the vertices of $L(K_n)$.
Case3− : Let $v_i \in V(G)$ and $v_{jk} \in V(L(K_n))$ then consider the following two subcases:
Subcase 3.1 -: If $i = j$ then $v_i \in V(G)$ and $v_{ik} \in V(L(K_n))$. Now third vertex $v_{kl} \in V(L(K_n))(i \neq k \neq l)$, which is adjacent to both the vertices v_i and v_{ik}.
Subcase 3.2 -: If $i \neq \{j,k\}$ then $v_i \in V(G)$ and $v_{jk} \in V(L(K_n))$. Now third vertex $v_{kl} \in V(L(K_n))$ $(i \neq j \neq k \neq l)$, which is adjacent to both the vertices v_i and v_{jk}.
Therefore, between any two vertices in $\mathbb{G}_e\,{}^{++-}_{+-+}(G)$ for $|V(G)| = n \geq 4$, there always exists a third vertex, which is adjacent to both the vertices.
Hence, $\mathbb{G}_{\mathrm{e}}\,{}^{++-}_{+-+}(G)$ is semi-complete for $n \geq 4$.

□

Proposition 4. *The equation $\mathbb{G}_{\mathrm{e}}\,{}^{-+-}_{+-+}(G) \cong \mathbb{G}_{\mathrm{e}}\,{}^{++-}_{+-+}(H)$ holds if and only if G is complement graph of H.*

Proof. Similar proof as in Proposition 2. □

Corollary 5. *Let G be a graph of n vertices then the extended transformation graph $\mathbb{G}_{\mathrm{e}}\,{}^{-+-}_{+-+}(G)$ is semi-complete for $n \geq 4$.*

Proof. The proof follows from Theorem 2.3 and Proposition 4. □

Corollary 6. *Diameter of the extended transformation graph* $\mathbb{G}_e{}^{++-}_{+-+}(G)$ *of a graph* G *is* 2 *except* G *is* K_1, $K_1 \cup K_1$, P_2, $K_1 \cup K_1 \cup K_1$ *and* $K_1 \cup P_2$.

Proof. Similar proof as in corollary 2. □

Theorem 2.4. *If* G *is a graph of* n *vertices with vertex set* $\{v_i; 1 \leq i \leq n\}$, m *edges,* t *number of triangles and each vertex* v_i *has degree* d_i *and* $G \neq K_1, G \neq K_1 \cup K_1, G \neq P_2, G \neq K_1 \cup K_1 \cup K_1, G \neq K_1 \cup P_2$ *then number of diametral paths in the extended transformation graph* $\mathbb{G}_e{}^{++-}_{+-+}(G)$ *is* $(\sum_{i=1}^{n} \binom{d_i}{2}) - 3t - \frac{m(n-2)(n-7)}{2} + \frac{n(n-1)(n-2)(n^2-n+2)}{8}$.

Proof. Since $\mathbb{G}_e{}^{++-}_{+-+}(G) \cong G \oplus_2 L(K_n)$ and the diameter of $\mathbb{G}_e{}^{++-}_{+-+}(G)$ is 2 except G is K_1, $K_1 \cup K_1$ and P_2. So we determine the number of diametral paths in $\mathbb{G}_e{}^{++-}_{+-+}(G)$ for $G \neq K_1$, $G \neq K_1 \cup K_1$ and $G \neq P_2$ as follows:

(i) Number of diametral paths between vertices of G through a vertex in G = Number of shortest paths of length 2 in $G = (\sum_{i=1}^{n} \binom{d_i}{2}) - 3t$.

(ii) Number of diametral paths between vertices of G through a vertex in $L(K_n) = (\binom{n}{2} - m) \times \binom{n-2}{2}$

(iii) Number of diametral paths between vertices of G and $L(K_n)$ through a vertex in $G = \sum_{i=1}^{n} d_i(n-2) = (n-2)\sum_{i=1}^{n} d_i = 2m(n-2)$

(iv) Number of diametral paths between vertices of G and $L(K_n)$ through a vertex in $L(K_n) = 3 \times \binom{n}{3} \times 2 = n(n-1)(n-2)$

(v) Number of diametral paths between vertices of $L(K_n)$ through a vertex in $G = (n-4) \times \binom{n}{4} \times 3 = \frac{n(n-1)(n-2)(n-3)(n-4)}{8}$

(vi) Number of diametral paths between vertices of $L(K_n)$ through a vertex in $L(K_n)$ $= {}^nC_2 \times ({}^nC_2 - (2n-2-1)) \times 4 \times \frac{1}{2} = \dfrac{n(n-1)(n-2)(n-3)}{2}$.

Hence total number of diametral paths in $\mathbb{G}_e{}^{++-}_{+-+}(G)$ is
$(\sum_{i=1}^{n} \binom{d_i}{2}) - 3t + (\binom{n-2}{2} - m) \times \binom{n-2}{2} + 2m(n-2) + n(n-1)(n-2) + \frac{n(n-1)(n-2)(n-3)(n-4)}{8} + \frac{n(n-1)(n-2)(n-3)}{2}$.
$= (\sum_{i=1}^{n} \binom{d_i}{2}) - 3t - \frac{m(n-2)(n-7)}{2} + \frac{n(n-1)(n-2)(n^2-n+2)}{8}$. for $G \neq K_1, G \neq K_1 \cup K_1, G \neq P_2, G \neq K_1 \cup K_1 \cup K_1, G \neq K_1 \cup P_2$.

□

Corollary 7. *If* G *is a graph of* n *vertices,* m *edges,* $\overline{G}$ *has vertex set* $\{v_i; 1 \leq i \leq n\}$ *such that each vertex* v_i *has degree* d_i, t *number of triangles in* $\overline{G}$ *and* $G \neq K_1, G \neq K_1 \cup K_1, G \neq P_2, G \neq K_3, G \neq P_3$ *then number of diametral paths in* $\mathbb{G}_e{}^{-+-}_{+-+}(G)$ *is* $(\sum_{i=1}^{n} \binom{d_i}{2}) - 3t + \frac{m(n-2)(n-7)}{2} + \frac{n(n-1)(n-2)(n^2-3n+16)}{8}$.

Proof. The proof follows from Theorem 2.4 and Proposition 4. □

Proposition 5. *If* $|V(G)| = n$ *then* $\mathbb{G}_e{}^{+-+}_{-+-}(G) \cong G \oplus_1 \overline{L(K_n)}$.

Proof. The proof is obvious from the definition of $\mathbb{G}_e{}^{+-+}_{-+-}(G)$ and the definition of operation $\oplus_2$. □

Theorem 2.5. *Let* G *be a graph of* n *vertices then the extended transformation graph* $\mathbb{G}_e{}^{+-+}_{-+-}(G)$ *is semi-complete if and only if* G *is* K_1, K_2, K_3 *and* K_n *for* $n \geq 6$.

Proof. Since $\mathbb{G}_e{}^{+-+}_{-+-}(K_1) \cong K_1$, $\mathbb{G}_e{}^{+-+}_{-+-}(K_1 \cup K_1) \cong P_3$, $\mathbb{G}_e{}^{+-+}_{-+-}(K_2) \cong K_3$ and if $|V(G)| = 3$ then only $\mathbb{G}_e{}^{+-+}_{-+-}(K_3)$ is semi-complete. Therefore, if $|V(G)| = n \leq 3$ then $\mathbb{G}_e{}^{+-+}_{-+-}(G)$ is semi-complete if and only if G is K_1, K_2, K_3. Now using Proposition 3 in $\mathbb{G}_e{}^{++-}_{+-+}(G)$ for $|V(G)| = n > 3$, we determine the semi-complete property by analysing

the following cases:
Case1$-$: Let $v_i, v_j \in V(G)$ then $v_{ij} \in V(\overline{L(K_n)})$. Therefore between any two vertices in G, there always exists a third vertex v_{ij}, which is adjacent to both the vertices v_i and v_j.
Case2$-$: Let $v_{ij}, v_{kl} \in V(\overline{L(K_n)})$, then consider the following two subcases:
Subcase 2.1 -: If $i = k$ then $v_{ij}, v_{il} \in V(\overline{L(K_n)})$. Now third vertex $v_i \in V(G)$, which is adjacent to both the vertices v_{ij} and v_{il}.
Subcase 2.2 -: If $i \neq j \neq k \neq l$ then $v_{ij}, v_{kl} \in V(\overline{L(K_n)})$. Now for $n \geq 6$, there exist a third vertex $v_{ab} \in V(\overline{L(K_n)})$ $(a \neq b \neq i \neq j \neq k \neq l)$, which is adjacent to both the vertices v_{ij} and v_{kl}. Since there does not exist a third vertex in vertex set $V(G)$, which is adjacent to both the vertices v_{ij} and v_{kl}. Therefore $n \neq 4$, $n \neq 5$ and $n \geq 6$.
Case3$-$: Let $v_i \in V(G)$ and $v_{jk} \in V(\overline{L(K_n)})$ then consider the following two subcases:
Subcase 3.1 -: If $i = j$ then $v_i \in V(G)$ and $v_{ik} \in V(\overline{L(K_n)})$. Now there does not exist a third vertex in vertex set $V(\overline{L(K_n)})$, which is adjacent to both the vertices v_i and v_{ik}. So to have third vertex in vertex set $V(G)$ for each pair of v_i and v_{ik}, G should be a complete graph.
Subcase 3.2 -: If $i \neq \{j, k\}$ then $v_i \in V(G)$ and $v_{jk} \in V(\overline{L(K_n)})$. Now third vertex $v_{il} \in V(\overline{L(K_n)})$ $(i \neq j \neq k \neq l)$, which is adjacent to both the vertices v_i and v_{jk}.
Since from Subcase 2.2, $n \neq 4$, $n \neq 5$ and $n \geq 6$ and from Subcase 3.1, G should be a complete graph. Therefore $\mathbb{G}_{\mathrm{e}}{}^{+-+}_{-+-}(G)$ is semi-complete if and only if G is K_1, K_2, K_3 and K_n for $n \geq 6$. □

Proposition 6. *The equation* $\mathbb{G}_{\mathrm{e}}{}^{--+}_{-+-}(G) \cong \mathbb{G}_{\mathrm{e}}{}^{+-+}_{-+-}(H)$ *holds if and only if* G *is complement graph of* H.

Proof. Similar proof as in Proposition 2. □

Corollary 8. *Let* G *be a graph of* n *vertices then the extended transformation graph* $\mathbb{G}_{\mathrm{e}}{}^{+-+}_{-+-}(G)$ *is semi-complete if and only if* G *is empty graph on* 1 *vertex, empty graph on* 2 *vertices, empty graph on* 3 *vertices and empty graph on* $n \geq 6$ *vertices.*

Proof. The proof follows from Theorem 2.5 and Proposition 6. □

Corollary 9. *Diameter of the extended transformation graph* $\mathbb{G}_{\mathrm{e}}{}^{+-+}_{-+-}(G)$ *of a graph* G *is* 2 *except* G *is* K_1, P_2, $K_1 \cup K_1 \cup K_1$ *and* $K_1 \cup P_2$.

Proof. If $|V(G)| = n \leq 3$ then the $\mathbb{G}_{\mathrm{e}}{}^{+--}_{---}(G)$ has diameter 2 if and only if G is $K_1 \cup K_1$, P_3 and K_3. From Theorem 2.7, v_{ij} and v_{kl} are adjacent in Subcase 2.2 and v_i and v_{ik} are adjacent in Subcase 3.1. Hence diameter of $\mathbb{G}_{\mathrm{e}}{}^{+-+}_{-+-}(G)$ of a graph G is 2 except G is K_1, P_2, $K_1 \cup K_1 \cup K_1$ and $K_1 \cup P_2$. □

Theorem 2.6. *If* G *is a graph of* n *vertices with vertex set* $\{v_i; 1 \leq i \leq n\}$, m *edges,* t *number of triangles and each vertex* v_i *has degree* d_i *and* $G \neq K_1, G \neq P_2, G \neq K_1 \cup K_1 \cup K_1, G \neq K_1 \cup P_2$ *then number of diametral paths in the extended transformation graph* $\mathbb{G}_{\mathrm{e}}{}^{+-+}_{-+-}(G)$ *is*
$(\sum_{i=1}^{n} \binom{d_i}{2}) - 3t \ + m(2n-5) \ + \frac{n(n-1)(n^3-7n^2+18n-14)}{4}$.

Proof. Since diameter of $\mathbb{G}_{\mathrm{e}}{}^{+-+}_{-+-}(G)$ is 2 except G is K_1, P_2, $K_1 \cup K_1 \cup K_1$ and $K_1 \cup P_2$. So we determine the number of diametral paths in $\mathbb{G}_{\mathrm{e}}{}^{++-}_{+-+}(G)$ for $G \neq K_1, G \neq P_2, G \neq K_1 \cup K_1 \cup K_1, G \neq K_1 \cup P_2$ as follows:

(i) Number of diametral paths between vertices of G through a vertex in G
= Number of shortest paths of length 2 in $G = (\sum_{i=1}^{n} \binom{d_i}{2}) - 3t$

(ii) Number of diametral paths between vertices of G through a vertex in $\overline{L(K_n)} = ({}^nC_2 - m) \times 1 = ({}^nC_2 - m)$

(iii) Number of diametral paths between vertices of G and $\overline{L(K_n)}$ through a vertex in $G = \sum_{i=1}^{n} d_i(n-2) = (n-2)\sum_{i=1}^{n} d_i = 2m(n-2)$

(iv) Number of diametral paths between vertices of G and $\overline{L(K_n)}$ through a vertex in $\overline{L(K_n)} = n \times (n-1) \times \ {}^{n-2}C_2 = \frac{n(n-1)(n-2)(n-3)}{2}$

(v) Number of diametral paths between vertices of $\overline{L(K_n)}$ through a vertex in $G = n \times^{n-1} C_2 = \frac{n(n-1)(n-2)}{2}$.

(vi) Number of diametral paths between vertices of $\overline{L(K_n)}$ through a vertex in $\overline{L(K_n)}$ $= {}^nC_3 \times 3 \times^{n-3} C_2 = \frac{n(n-1)(n-2)(n-3)(n-4)}{4}$

Hence total number of diametral paths in $\mathbb{G}_{\mathrm{e}}{}^{+-+}_{-+-}(G)$ is

$(\sum_{i=1}^{n}\binom{d_i}{2}) - 3t + ({}^nC_2 - m) + 2m(n-2) + \frac{n(n-1)(n-2)(n-3)}{2} + \frac{n(n-1)(n-2)}{2} + \frac{n(n-1)(n-2)(n-3)(n-4)}{4}$.

$= (\sum_{i=1}^{n}\binom{d_i}{2}) - 3t + m(2n-5) + \frac{n(n-1)(n^3-7n^2+18n-14)}{4}$ for $G \neq K_1, G \neq P_2, G \neq K_1 \cup K_1 \cup K_1, G \neq K_1 \cup P_2$.

□

Corollary 10. *If G is a graph of n vertices, m edges, $\overline{G}$ has vertex set $\{v_i; 1 \leq i \leq n\}$ such that each vertex v_i has degree d_i, t number of triangles in $\overline{G}$ and $G \neq K_1, G \neq K_1 \cup K_1, G \neq K_3, G \neq P_3$ then number of diametral paths in the extended transformation graph $\mathbb{G}_{\mathrm{e}}{}^{--+}_{-+-}(G)$ is*

$(\sum_{i=1}^{n}\binom{d_i}{2}) - 3t$ - $m(2n-5) + \frac{n(n-1)(n-2)(n^2-5n+12)}{4}$

Proof. The proof follows from Theorem 2.6 and Proposition 6.

□

Proposition 7. *If $|V(G)| = n$ then $\mathbb{G}_e{}^{+--}_{---}(G) \cong G \oplus_2 \overline{L(K_n)}$.*

Proof. The proof is obvious from the definition of $\mathbb{G}_e{}^{+--}_{---}(G)$ and the definition of operation $\oplus_2$. □

Theorem 2.7. *If G is a graph of n vertices then the extended transformation graph $\mathbb{G}_{\mathrm{e}}{}^{+--}_{---}(G)$ is semi-complete for $n \geq 5$.*

Proof. If $|V(G)| = n \leq 3$ then the $\mathbb{G}_{\mathrm{e}}{}^{+--}_{---}(G)$ is not semi-complete. Therefore using Proposition 7 in $\mathbb{G}_{\mathrm{e}}{}^{+--}_{---}(G)$ for $|V(G)| = n > 3$, we determine the semi-complete property by analysing the following cases:

Case1$-$: Let $v_i, v_j \in V(G)$ then $v_{kl} \in V(\overline{L(K_n)})$ $i \neq j \neq k \neq l$. Therefore between any two vertices in G, there always exists a third vertex v_{kl}, which is adjacent to both the vertices v_i and v_j.

Case2$-$: Let $v_{ij}, v_{kl} \in V(\overline{L(K_n)})$, then consider the following two subcases:

Subcase 2.1 -: If $i = k$ then $v_{ij}, v_{il} \in V(\overline{L(K_n)})$. Now third vertex $v_s \in V(G)$ $(i \neq j \neq l \neq s)$, which is adjacent to both the vertices v_{ij} and v_{il}.

Subcase 2.2 -: If $i \neq j \neq k \neq l$ then $v_{ij}, v_{kl} \in V(\overline{L(K_n)})$. Since for $n = 4$, there does not exist a third vertex which is adjacent to both the vertices v_{ij} and v_{kl}. Therefore for $n \geq 5$, third vertex $v_m \in V(\overline{L(K_n)})$ $(m \neq i \neq j \neq k \neq l)$, which is adjacent to both the vertices v_{ij} and v_{kl}.

Case3$-$: Let $v_i \in V(G)$ and $v_{jk} \in V(\overline{L(K_n)})$ then consider the following two subcases:

Subcase 3.1 -: If $i = j$ then $v_i \in V(G)$ and $v_{ik} \in V(\overline{L(K_n)})$. Now third vertex $v_{lm} \in V(G)$ $(i \neq k \neq l \neq m)$, which is adjacent to both the vertices v_i and v_{ik}.

Subcase 3.2 -: If $i \neq \{j,k\}$ then $v_i \in V(G)$ and $v_{jk} \in V(\overline{L(K_n)})$. Since from Subcase 2.2 $n \geq 5$. Therefore there exist a third vertex $v_{lm} \in V(\overline{L(K_n)})$ $(i \neq j \neq k \neq l \neq m)$, which is adjacent to both the vertices v_i and v_{ik}.
Hence $\mathbb{G}_e{}^{+--}_{---}(G)$ is semi-complete for $n \geq 5$.

□

Proposition 8. *The equation* $\mathbb{G}_e{}^{+--}_{---}(G) \cong \mathbb{G}_e{}^{+--}_{---}(H)$ *holds if and only if* G *is complement graph of* H.

Proof. Similar proof as in Proposition 2. □

Corollary 11. *If* G *is a graph of* n *vertices then the extended transformation graph* $\mathbb{G}_e{}^{---}_{---}(G)$ *is semi-complete for* $n \geq 5$.

Proof. The proof follows from Theorem 2.7 and Proposition 8. □

Corollary 12. *Let* G *be a graph of* n *vertices then diameter of the extended transformation graph* $\mathbb{G}_e{}^{+--}_{---}(G)$ *of a graph* G *is 2 for* $n \geq 4$.

Proof. If $|V(G)| = n \leq 3$ then the $\mathbb{G}_e{}^{+--}_{---}(G)$ does not have diameter 2. From Theorem 2.7, v_{ij} and v_{kl} are adjacent in Subcase 2.2 and v_i and v_{jk} are adjacent in Subcase 3.2. Hence diameter of $\mathbb{G}_e{}^{+--}_{---}(G)$ is 2 for $n \geq 4$. □

Theorem 2.8. *If* G *is a graph of* n *vertices with vertex set* $\{v_i; 1 \leq i \leq n\}$, m *edges,* t *number of triangles and each vertex* v_i *has degree* d_i *and* $n \geq 4$ *then number of diametral paths in the extended transformation graph* $\mathbb{G}_e{}^{+--}_{---}(G)$ *is*
$(\sum_{i=1}^{n} \binom{d_i}{2}) - 3t - \frac{m(n-2)(n-7)}{2} + \frac{(n+1)n(n-1)(n-2)(n-3)}{4}$.

Proof. Since diameter of $\mathbb{G}_e{}^{+--}_{---}(G)$ is 2 for $n \geq 4$. So we determine the number of diametral paths in $\mathbb{G}_e{}^{+--}_{---}(G)$ for $n \geq 4$ as follows:
(i) Number of diametral paths between vertices of G through a vertex in G
$= (\sum_{i=1}^{n} \binom{d_i}{2}) - 3t$.
(ii) Number of diametral paths between vertices of G through a vertex in $\overline{L(K_n)} = ({}^nC_2 - m) \times {}^{n-2}C_2$.
(iii) Number of diametral paths between vertices of G and $\overline{L(K_n)}$ through a vertex in G $= \sum_{i=1}^{n} d_i(n-2) = (n-2)\sum_{i=1}^{n} d_i = 2m(n-2)$.
(iv) Number of diametral paths between vertices of G and $\overline{L(K_n)}$ through a vertex in $\overline{L(K_n)} = n \times (n-1) \times {}^{n-2}C_2 = \frac{n(n-1)(n-2)(n-3)}{2}$.
(v) Number of diametral paths between vertices of $\overline{L(K_n)}$ through a vertex in $G = {}^nC_3 \times 3 \times (n-3) = \frac{n(n-1)(n-2)(n-3)}{2}$.
(vi) Number of diametral paths between vertices of $\overline{L(K_n)}$ through a vertex in $\overline{L(K_n)}$ $= {}^nC_3 \times 3 \times {}^{n-3}C_2 = \frac{n(n-1)(n-2)(n-3)(n-4)}{4}$.
Hence total number of diametral paths in $\mathbb{G}_e{}^{+-+}_{-+-}(G)$ is
$(\sum_{i=1}^{n} \binom{d_i}{2}) - 3t + ({}^nC_2 - m) \times {}^{n-2}C_2 + 2m(n-2) + \frac{n(n-1)(n-2)(n-3)(n-4)}{4} + n(n-1)(n-2)(n-3) = (\sum_{i=1}^{n} \binom{d_i}{2}) - 3t - \frac{m(n-2)(n-7)}{2} + \frac{(n+1)n(n-1)(n-2)(n-3)}{4}$ for $n \geq 4$.

□

Corollary 13. *If* G *is a graph of* n *vertices,* m *edges,* $\overline{G}$ *has vertex set* $\{v_i; 1 \leq i \leq n\}$ *such that each vertex* v_i *has degree* d_i, t *number of triangles in* $\overline{G}$ *and* $n \geq 4$ *then number of diametral paths in the extended transformation graph* $\mathbb{G}_e{}^{---}_{---}(G)$ *is*
$(\sum_{i=1}^{n} \binom{d_i}{2}) - 3t + \frac{m(n-2)(n-7)}{2} + \frac{n(n-1)(n-2)(n^2-3n+4)}{4}$.

Proof. The proof follows from Theorem 2.8 and Proposition 8. □

3. Program of number of diametral paths in python

Walikar & Shindhe [13] have given an algorithm for determining diametral recheable index of a vertex in a graph. Based on the algorithm [13], we have given a program in FIGURE 2 for finding number of diametral paths in a simple, undirected and unweighted graph. Since Iterative Deepening Depth First Search (IDDFS) works for infinite graph, so we replace DFS by IDDFS in the algorithm [13].

```python
R = [0 for i in range(40)]
l = 0
L = [0 for i in range(40)]
s = [0 for i in range(40)]
count = [0 for i in range(40)]
lcc = 0
lc = [0 for i in range(40)]
source = 0

def nextadjacent(n, x):

    for i in range(1, n+1):
        if a[x][i] == 1 and (L[i] != 1):
            return i

    return 0

def adjacent(n, j):

    for i in range(1, n+1):
        if a[j][i] == 1 and R[i] != 1:
            return i

    return 0

def get_nextadjacent(n, v):
    global lcc,l,source
    u = nextadjacent(n, v)
    if u == 0:
        l -= 1
        for j in range(1, n+1):
            if a[v][j] == 1 and lc[j] > lc[v] :
                L[j] = 0
                R[j] = 0
                lc[j] = 0
                lcc -= 1

        L[v] = 1
        L[source] = 1

    if v == source:
        l = 0
        for j in range(1, n+1):
            R[j] = 0

    return u

def DFS(n, v, depth_limit, current_depth):
    global lcc,l,source
    if current_depth > depth_limit:
        return
    u = adjacent(n, v)
    if lc[u] == 0:
        lcc = lcc + 1
        lc[u] = lcc

    while u:
        if lc[u] == 0:
            lcc = lcc + 1
            lc[u] = lcc

        if R[u] == 0:
            l += 1
            R[v] = 1
            R[u] = 1
            L[v] = 1
            L[u] = 1

            if l == max:
                if dm[source][u] == max:
                    count[source] += 1
                    l -= 1
                    if lc[u] == 0:
                        lcc = lcc + 1
                        lc[u] = lcc

                    for j in range(1, n+1):
                        if lc[j] > lc[v]:
                            R[j] = 0

                    R[v] = 1
                    L[v] = 1
                    u = get_nextadjacent(n, v)
                    continue

                else:
                    l -= 1
                    L[u] = 1

                    if lc[u] == 0:
                        lcc += 1
                        lc[u] = lcc

                    u = get_nextadjacent(n, v)
                    continue
            else:
                DFS(n, u, depth_limit, current_depth + 1)

        u = get_nextadjacent(n, v)

# Iterative deepening depth-first search
def IDDFS(n, source, max_depth):
    for depth_limit in range(1, max_depth):
        # Do Initialization
```

```python
        for j in range(1, n+1):
            R[j] = 0
            lc[j] = 0
            L[j] = 0

        l = 0
        lcc = 0
        R[source] = 1
        L[source] = 1

        lcc = lcc + 1
        lc[source] = lcc

        DFS(n, source, depth_limit, 0)


#Take input
n = int(input("Enter the number of vertices:\n"))

print("Enter the adjacency matrix for graph:")
a = [[0 for i in range(n+1)] for j in range(n+1)]

for i in range(1,n+1):
    numbers = [int(n) for n in input().split()]
    len = 0
    for j in range(1,n+1):
        a[i][j] = numbers[len]
        len += 1

print("Enter the distance matrix for graph:")
dm = [[0 for i in range(n+1)] for j in range(n+1)]

for i in range(1,n+1):
    numbers = [int(n) for n in input().split()]
    len = 0
    for j in range(1,n+1):
        dm[i][j] = numbers[len]
```

```python
        len += 1
#Find the eccentricity of the graph
max = dm[1][1]
for i in range(1,n+1):
    for j in range(1,n+1):
        if dm[i][j] > max :
            max = dm[i][j]

print("\nEccentricity = %d" % (max))
print("\nThe diametral vertices are:", end = '')

k = 1
for i in range(1, n+1):
    for j in range (1, n+1):
        if dm[i][j] == max:
            s[k] = i
            k += 1
            break

for j in range(1,k):
    print(" %d " % (s[j]), end = '')

print("")

for i in range(1,k):
    source = s[i]
    IDDFS(n, source, max)

print("\nThe DRI values are:\n")
for j in range(1, n+1):
    print("DRI(%d)=%d\n" % (j, count[j]))

print("\nThe number of diametral paths is:")
sum = 0
for j in range(1, n+1):
    sum = sum + count[j]
print(sum/2)
```

Figure 2. Program of number of diametral paths in python

4. Conclusion

In this paper, number of diametral paths in some of the extended transformation graphs are determined and a program in python for finding the number of diametral paths has given. Semi-complete property was developed to solve defence problems and it become useful in various areas of IOT networks by creating TGO topology. In the similar way, we will create a topology based on extended transformation graph. We will take use of these results in future research and explore this work.

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