TWMS J. App. and Eng. Math. V. 11, Special Issue, 2021, pp. 144-153

THE MINIMUM MEAN MONOPOLY ENERGY OF A GRAPH

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ABSTRACT. The motivation for the study of the graph energy comes from chemistry, where the research on the so-called total π - electron energy can be traced back until the 1930s. This graph invariant is very closely connected to a chemical quantity known as the total π - electron energy of conjugated hydro carbon molecules. In recent times analogous energies are being considered, based on Eigen values of a variety of other graph matrices. In 1978, I.Gutman [1] defined energy mathematically for all graphs. Energy of graphs has many mathematical properties which are being investigated. The ordinary energy of an undirected simple finite graph G is defined as the sum of the absolute values of the Eigen values of its associated matrix. i.e. if $\mu_1, \mu_2, ..., \mu_n$ are the Eigen values of adjacency matrix A(G), then energy of graph is E(G) = $\sum_{i=1}^{n} |\mu_i|$. Laura Buggy, Amalia Culiuc, Katelyn Mccall and Duyguyen [9] introduced the more general M-energy or Mean Energy of G is then defined as E^M (G) = $\sum_{i=1}^{n} |\mu_i - \bar{\mu}|$, where $\bar{\mu}$ is the average of $\mu_1, \mu_2, ..., \mu_n$.
A subset M ⊆ V (G), in a graph G (V, E), is called a monopoly set of G if every vertex v ∈ (V - M) has at least $\frac{d(v)}{2}$ neighbors in M. The minimum cardinality of a monopoly set among all monopoly sets in G is called the monopoly size of G, denoted by mo(G).Ahmed Mohammed Naji and N.D.Soner [7] introduced minimum monopoly energy E_{MM} [G] of a graph G. In this paper we are introducing the minimum mean monopoly energy, denoted by E^M_{MM} (G), of a graph G and computed minimum monopoly energies of some standard graphs. Upper and lower bounds for E^M_{MM} (G)are also established.

Keywords: Monopoly Set, Monopoly Size, Minimum Monopoly Matrix, Minimum Monopoly Eigenvalues, Minimum Monopoly Energy and Minimum Mean Monopoly Energy of a graph
AMS Subject Classification: 05C50, 05C99

1. INTRODUCTION

In this paper, a graph G(V, E) mean a simple non-empty undirected graph, having no loops and no multiple edges. Let n and m be the number of vertices and edges, respectively, of G. The degree of a vertex v in a graph G, denoted by d(v), is the number of edges incident on v. For any vertex v of a graph G, the open neighborhood of v is the set

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§ Manuscript received: October, 19, 2019; accepted: April 2, 2020.
TWMS Journal of Applied and Engineering Mathematics, Vol.11, Special Issue

N(v) = $\{u \in V :< u, v >\in E(G)\}$. For a subset S $\subseteq$ V (G) the degree of a vertex v $\in$ V (G) with respect to a subset S is d_s (V) = | N(v) $\cap$ S |. For graph theoretic terminology we refer to Harary book [5]

A subset M of V in G = (V, E) is called a monopoly set if for every vertex v $\in$ (V - M) has at least $\frac{d(v)}{2}$ neighbors in M, the monopoly size of a graph G, denoted by mo(G), is a minimum cardinality of a monopoly set in G.

In particular, monopolies are dynamic monopoly (dynamos) that, when colored black at a certain time step, will cause the entire graph to be colored black in the next time step under an irreversible majority conversion process.

The energy of a graph was introduced by I. Gutman [1], [6] in the year 1978. Let G be a graph with n vertices { $v_1, v_2, ..., v_n$ } and m edges. Let A = (a_{ij}) be the adjacency matrix of the graph. The Eigen values $m_1, m_2, ..., m_n$ of A, assumed in non-increasing order, are the Eigen values of the graph G.
The energy of graph is defined as E(G) = $\sum_{i=1}^{n} |m_i|$. The minimum mean energy of G is then defined, by Laura Buggy, Amaliailiuc, Katelyn Mccall, Duyguyen [9], as E^M(G) =$\sum_{i=1}^{n} |m_i - \bar{m}|$, where $\bar{m}$ is the mean of the Eigen values. To know further basic concepts about energy of graphs refer [2], [3], [4].

2. The Minimum Mean Monopoly Energy of Graphs

Let G be a graph of order n with vertex set V (G) = { $v_1, v_2, ..., v_n$ } and edge set E. Let M be a minimum monopoly set of G. The minimum monopoly matrix of G is the n × n matrix, denoted by $A_{MM}(G) = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 & if \ < v_i, v_j >\in E \\ 1 & if \ i = j, v_i \in M \\ 0 & otherwise \end{cases}$$

Since A_{MM} (G) is real and symmetric, its Eigen values $m_1, m_2, ..., m_n$ are real and positive. The Minimum Monopoly energy of graph G is defined as E_{MM} (G) = $\sum_{i=1}^{n} |m_i|$, where $m_1, m_2, ..., m_n$ are Eigen values of minimum monopoly matrix A_{MM} (G).The Minimum Mean Monopoly Energy of G is then defined as E_{MM}^{M} (G) = $\sum_{i=1}^{n} |m_i - \bar{m}|$, where $\bar{m}$ is the mean of $m_1, m_2, ..., m_n$. For further basic concepts about mean energy refer [8], [9].

Remark 2.1. *The minimum monopoly energy of a graph G depends [7] on the choice of the minimum monopoly set. i.e. the minimum monopoly energy is not a graph invariant.*

3. Minimum Mean Monopoly Energy of Some Standard Graphs

Theorem 3.1. *For n ≥ 2, the minimum mean monopoly energy of complete graph K_n is*

$$\begin{cases} (n-2) + \sqrt{n^2+1}, & if \ n \ is \ even \\ \\ \frac{(n-1)^2}{n} + \sqrt{n^2-1}, & if \ n \ is \ odd \end{cases}$$

Proof. K_n is complete graph with vertex set V = $\{v_1, v_2, ..., v_n\}$. Then the minimum monopoly size is mo(K_n) = $\lfloor \frac{n}{2} \rfloor = \begin{cases} \frac{n}{2}, & if \ n \ is \ even \\ \\ \frac{(n-1)}{2}, & if \ n \ is \ odd \end{cases}$ The minimum monopoly (MM) set is D = $\{v_1, v_2, ..., v_{\frac{n}{2}}\}$, when n is even or

D $= \{v_1, v_2, ..., v_{\frac{n-1}{2}}\}$, when n is odd.

Case I: When n is even
Then the MM matrix $A_{MM}(K_n) =$

$$\begin{bmatrix} 1 & 1 & 1 & .. & 1 & 1 & 1 & 1 & .. & 1 \\ 1 & 1 & 1 & .. & 1 & 1 & 1 & 1 & .. & 1 \\ 1 & 1 & 1 & .. & 1 & 1 & 1 & 1 & .. & 1 \\ . & . & . & .. & . & . & . & . & .. & . \\ . & . & . & .. & . & . & . & . & .. & . \\ 1 & 1 & 1 & .. & 1 & 1 & 1 & 1 & .. & 1 \\ 1 & 1 & 1 & .. & 1 & 0 & 1 & 1 & .. & 1 \\ 1 & 1 & 1 & .. & 1 & 1 & 0 & 1 & .. & 1 \\ 1 & 1 & 1 & .. & 1 & 1 & 1 & 0 & .. & 1 \\ . & . & . & .. & . & . & . & . & .. & . \\ . & . & . & .. & . & . & . & . & .. & . \\ 1 & 1 & 1 & .. & 1 & 1 & 1 & 1 & .. & 0 \end{bmatrix}_{n \times n}$$

Then the characteristic equation $|A_{MM}(K_n) - \mu I| = 0$ is

$\mu^{\frac{n-2}{2}} \; (\mu + 1)^{\frac{n-2}{2}} \; (\mu^2 - (n-1)\mu - \frac{n}{2}) = 0$

Then $\mu = 0$; $\frac{n-2}{2}$ times, $\mu =$ -1; $\frac{n-2}{2}$ times and $\mu = \frac{(n-1) \pm \sqrt{n^2+1}}{2}$.

Mean $\bar{\mu} = \frac{1}{2}$.

Minimum Monopoly Energy $= E_{MM}(K_n) = \frac{n-2}{2} + \sqrt{n^2 + 1}$.

Minimum Mean Monopoly(MMM) Energy $= E^M_{MM}(K_n)$

$= \sum_{i=1}^{\frac{n-2}{2}} |0 - \frac{1}{2}| + \sum_{i=1}^{\frac{n-2}{2}} |-1 - \frac{1}{2}| + |\frac{(n-1) \pm \sqrt{n^2+1}}{2} - \frac{1}{2}|$

$= \frac{(n-2)}{4} + \frac{3(n-2)}{4} + |\frac{(n-2) \pm \sqrt{n^2+1}}{2}|$

$= \frac{(n-2)}{4} + \frac{3(n-2)}{4} + \sqrt{n^2 + 1}$

$=(n-2) + \sqrt{n^2 + 1}$.
Case II: When n is odd

Then the MM matrix $A_{MM}(K_n) =$

$$\begin{bmatrix} 1 & 1 & 1 & .. & 1 & 1 & 1 & 1 & .. & 1 \\ 1 & 1 & 1 & .. & 1 & 1 & 1 & 1 & .. & 1 \\ 1 & 1 & 1 & .. & 1 & 1 & 1 & 1 & .. & 1 \\ . & . & . & .. & . & . & . & . & .. & . \\ . & . & . & .. & . & . & . & . & .. & . \\ 1 & 1 & 1 & .. & 1 & 1 & 1 & 1 & .. & 1 \\ 1 & 1 & 1 & .. & 1 & 0 & 1 & 1 & .. & 1 \\ 1 & 1 & 1 & .. & 1 & 1 & 0 & 1 & .. & 1 \\ 1 & 1 & 1 & .. & 1 & 1 & 1 & 0 & .. & 1 \\ . & . & . & .. & . & . & . & . & .. & . \\ . & . & . & .. & . & . & . & . & .. & . \\ 1 & 1 & 1 & .. & 1 & 1 & 1 & 1 & .. & 0 \end{bmatrix}_{n\times n}$$

Then the characteristic equation $|A_{MM}(K_n) - \mu I| = 0$ is

$\mu^{\frac{n-3}{2}}\ (\mu+1)^{\frac{n-1}{2}}\ (\mu^2 - (n-1)\mu - \frac{n-1}{2}) = 0.$

Then $\mu = 0$; $\frac{n-3}{2}$ times, $\mu =$ -1; $\frac{n-1}{2}$ times and $\mu = \frac{(n-1)\pm\sqrt{n^2-1}}{2}$.

Mean $\bar{\mu} = \frac{n-1}{2n}$.

Minimum Monopoly Energy $= E_{MM}(K_n) = \frac{n-1}{2} + \sqrt{n^2-1}$.

Minimum Mean Monopoly(MMM) Energy $= E^M_{MM}(K_n)$

$= \sum_{i=1}^{\frac{n-3}{2}} |0 - \frac{n-1}{2n}| + \sum_{i=1}^{\frac{n-1}{2}} |-1-\frac{n-1}{2n}| + |\frac{(n-1)\pm\sqrt{n^2-1}}{2} - \frac{n-1}{2n}|$

$= \frac{(n-1)(n-3)}{4n} + \frac{(3n-1)(n-1)}{4n} + |\frac{n(n-1)\pm n\sqrt{n^2-1}-(n-1)}{2n}|$

$= \frac{(n-1)(n-3)}{4n} + \frac{(3n-1)(n-1)}{4n} + |\frac{(n-1)^2\pm n\sqrt{n^2-1}}{2n}|$

$= \frac{(n-1)^2}{n} + \sqrt{n^2+1}$.

□

Theorem 3.2. *For n ≥ 2, the minimum mean monopoly energy of star graph $K_{1,n-1}$ is* $\frac{n-2}{n} + \sqrt{4n-3}$.

Proof. $K_{1,n-1}$ is a star graph with vertex set V = $\{ v_1, v_2, ..., v_n \}$.
The MM set $= \{v_1\}$. (Assume that v_1 is the centre vertex).

$A_{MM}(K_{1,n-1}) =$

$$\begin{bmatrix} 1 & 1 & 1 \ldots & 1 & 1 \\ 1 & 0 & 0 \ldots & 0 & 0 \\ 1 & 0 & 0 \ldots & 0 & 0 \\ . & . & . & \ldots & . \\ . & . & . & \ldots & . \\ 1 & 0 & 0. \ldots & 0 & 0 \end{bmatrix}_{n\times n}$$

Then the characteristic equation $|A_{MM}(K_{1,n-1}) - \mu I| = 0$ is

$\mu^{n-2}\ (\mu^2 - \mu - (n-1)) = 0$

Then $\mu = 0$; (n-2) times, $\mu = \frac{1 \pm \sqrt{4n-3}}{2}$.
Mean $\bar{\mu} = \frac{1}{n}$.
The Minimum Monopoly(MM) Energy of star graph is

$E_{MM}(K_{1,n-1}) = \sum_{i=1}^{n-2} |0|(n-2) + |\frac{1 \pm \sqrt{4n-3}}{2}| = \sqrt{4n-3}.$
Minimum Mean Monopoly(MMM) Energy of star graph is $E_{MM}^{M}(K_{1,n-1})$

$= \sum_{i=1}^{n-2} |0 - \frac{1}{n}| + |\frac{1 \pm \sqrt{4n-3}}{2} - \frac{1}{n}|$

$= \frac{(n-2)}{n} + |\frac{(n-2) \pm n\sqrt{4n-3}}{2n}|$

$= \frac{(n-2)}{n} + \sqrt{4n-3}.$

□

Theorem 3.3. *For the complete bipartite graph $K_{m,n}$, for $m \leq n$, the minimum mean monopoly energy is equal to* $\frac{2mn-(n+m)}{m+n} + \sqrt{4mn+1}$.

Proof. For the complete bipartite graph $K_{m,n}$, for m ≤ n, with vertex set V = { v_1, v_2,...,v_m , u_1, u_2,...,u_n}, the MM set = { v_1,v_2,...,v_m } .

Then the MM matrix is $A_{MM}(K_{m,n}) =$

$$\begin{bmatrix} 1 & 0 & 0 & .. & 0 & 1 & 1 & 1 & .. & 1 \\ 0 & 1 & 0 & .. & 0 & 1 & 1 & 1 & .. & 1 \\ 0 & 0 & 1 & .. & 0 & 1 & 1 & 1 & .. & 1 \\ . & . & . & .. & . & . & . & . & .. & . \\ . & . & . & .. & . & . & . & . & .. & . \\ 0 & 0 & 0 & .. & 1 & 1 & 1 & 1 & .. & 1 \\ 1 & 1 & 1 & .. & 1 & 0 & 0 & 0 & .. & 0 \\ 1 & 1 & 1 & .. & 1 & 0 & 0 & 0 & .. & 0 \\ 1 & 1 & 1 & .. & 1 & 0 & 0 & 0 & .. & 0 \\ . & . & . & .. & . & . & . & . & .. & . \\ . & . & . & .. & . & . & . & . & .. & . \\ 1 & 1 & 1 & .. & 1 & 0 & 0 & 0 & .. & 0 \end{bmatrix}_{(m+n)\times(m+n)}$$

Then the characteristic equation $|A_{MM}(K_{m,n}) - \mu I| = 0$ is

$\mu^{m-1}(\mu - 1)^{n-1}(\mu^2 - \mu - mn) = 0$.

Then $\mu = 0$; (m-1) times, μ =1 ; (n-1) times, $\mu = \frac{1 \pm \sqrt{4mn+1}}{2}$.

Mean $\bar{\mu} = \frac{n}{m+n}$.

Then, MM Energy = $E_{MM}(K_{m,n})$ = (m-1) + $\sqrt{4mn+1}$.

MMM Energy = $E_{MM}^{M}(K_{m,n})$
$= \sum_{i=1}^{m-1} |(0 - \frac{n}{m+n})| + \sum_{i=1}^{n-1} |(1 - \frac{n}{m+n})| + |\frac{1 \pm \sqrt{4mn+1}}{2} - \frac{n}{m+n}|$

$= \frac{n(m-1)}{m+n} + \frac{m(n-1)}{m+n} + |\frac{(m-n)\pm(m+n)\sqrt{4mn+1}}{2(m+n)}|$

$=\frac{2mn-(m+n)}{m+n} + \sqrt{4mn+1}$. □

Theorem 3.4. *For a double star graph $S_{n,n}$, the minimum mean monopoly energy is equal to* $\frac{2(n-2)}{n} + 2(\sqrt{n} + \sqrt{n-1})$.

Proof. For a double star graph $S_{n,n}$ with vertex set V $= \{v_0, v_1, v_2, ..., v_{n-1}, u_0, u_1, u_2, ..., u_{n-1}\}$, the MM set = { u_0,v_0 }.

The MM matrix $A_{MM}(S_{n,n}) =$

$$\begin{bmatrix} 1 & 1 & 1 & .. & 1 & 1 & 0 & 0 & .. & 0 \\ 1 & 0 & 0 & .. & 0 & 0 & 0 & 0 & .. & 0 \\ 1 & 0 & 0 & .. & 0 & 0 & 0 & 0 & .. & 0 \\ . & . & . & .. & . & . & . & . & .. & . \\ . & . & . & .. & . & . & . & . & .. & . \\ 1 & 0 & 0 & .. & 0 & 0 & 0 & 0 & .. & 0 \\ 1 & 0 & 0 & .. & 0 & 1 & 1 & 1 & .. & 1 \\ 0 & 0 & 0 & .. & 0 & 1 & 0 & 0 & .. & 0 \\ 0 & 0 & 0 & .. & 0 & 1 & 0 & 0 & .. & 0 \\ . & . & . & .. & . & . & . & . & .. & . \\ . & . & . & .. & . & . & . & . & .. & . \\ 0 & 0 & 0 & .. & 0 & 1 & 0 & 0 & .. & 0 \end{bmatrix}_{2n\times 2n}$$

The characteristic equation $|A_{MM}(S_{n,n}) - \mu I| = 0$ is

$\mu^{2n-4}(\mu^2 - (n-1))(\mu^2 - 2\mu - (n-1)) = 0.$

Then $\mu = 0$; $(2n-4)$ times , $\mu = \pm\sqrt{n-1}$, $\mu = 1 \pm\sqrt{n}$.

Mean $\bar{\mu} = \frac{1}{n}$.

Then MM Energy = $E_{MM}(S_{n,n}) = |0|(2n-4) + |\pm\sqrt{n-1}| + |1 \pm \sqrt{n}|$

= (2n-4) + 2 $(\sqrt{n} + \sqrt{n-1})$.

The MMM Energy = $E^M_{MM}(S_{n,n}) = \sum_{i=1}^{2n-4} |(0 - \frac{1}{n})| + |\pm\sqrt{n-1} - \frac{1}{n}| + |1 \pm \sqrt{n} - \frac{1}{n}|$

$= \frac{(2n-4)}{n} + |\frac{(-1\pm n\sqrt{n-1})}{n}| + |\frac{((n-1)\pm n\sqrt{n})}{n}|$

$= \frac{2(n-2)}{n} + 2(\sqrt{n} + \sqrt{n-1})$. □

Theorem 3.5. *For any integer $n \geq 3$, the minimum mean monopoly energy of the crown graph S_n^0 is* $\sqrt{5}(n-1) + \sqrt{4n^2 - 8n + 5}$.

Proof. Let S_n^0 be the Crown graph with vertex set $V=$ { u_1, u_2,...,u_n, v_1,v_2,...,v_n }.

Then minimum monopoly set = { u_1, u_2,...,u_n }.

Then the MMmatrix $A_{MM}(S_n^0) =$

$$\begin{bmatrix} 1 & 0 & 0 & .. & 0 & 0 & 1 & 1 & .. & 1 \\ 0 & 1 & 0 & .. & 0 & 1 & 0 & 1 & .. & 1 \\ 0 & 0 & 1 & .. & 0 & 1 & 1 & 0 & .. & 1 \\ . & . & . & .. & . & . & . & . & .. & . \\ . & . & . & .. & . & . & . & . & .. & . \\ 0 & 0 & 0 & .. & 1 & 1 & 1 & 1 & .. & 1 \\ 0 & 1 & 1 & .. & 1 & 0 & 0 & 0 & .. & 0 \\ 1 & 0 & 1 & .. & 1 & 0 & 0 & 0 & .. & 0 \\ 1 & 1 & 0 & .. & 1 & 0 & 0 & 0 & .. & 0 \\ . & . & . & .. & . & . & . & . & .. & . \\ . & . & . & .. & . & . & . & . & .. & . \\ 1 & 1 & 1 & .. & 0 & 0 & 0 & 0 & .. & 0 \end{bmatrix}_{2n \times 2n}$$

Then the characteristic equation $|A_{MM}(S_n^0) - \mu I| = 0$ is

$(\mu^2 - \mu - 1)^{n-1}(\mu^2 - \mu - (n-1)^2) = 0$.

Then the Eigen values are $\mu = \frac{1 \pm \sqrt{5}}{2}$ of multiplicity$(n-1)$ and $\mu = \frac{1 \pm \sqrt{4n^2 - 8n + 5}}{2}$ two simple roots.

Then, Mean $\bar{\mu} = \frac{1}{n}$.

Then minimum monopoly energy is $E_{MM}(S_n^0) = \sqrt{5}(n-1) + \sqrt{4n^2 - 8n + 5}$.

Then minimum mean monopoly energy is

$E_{MM}^M(G) = \sum_{i=1}^{n-1} |\frac{1 \pm \sqrt{5}}{2} - \frac{1}{n}| + |\frac{1 \pm \sqrt{4n^2-8n+5}}{2} - \frac{1}{n}|$

$=$ (n-1) $|\frac{(n-2) \pm n\sqrt{5}}{2n}| + |\frac{(n-2) \pm n\sqrt{4n^2-8n+5}}{2n}|$

$= \sqrt{5}(n-1) + \sqrt{4n^2 - 8n + 5}$. □

4. Bounds of Minimum Mean Monopoly Energy

Result 4.1. *Let G be a connected graph with vertex set* $V = \{v_1, v_2, ..., v_n\}$ *and size m. Then* $\sqrt{2m + mo(G)} \le E_{MM}(G) \le \sqrt{n(2m + mo(G))}$.

For proof refer [7]

Theorem 4.1. *Let G be a connected graph with vertex set* $V = \{v_1, v_2, ..., v_n\}$ *and size m. Then* $\sqrt{(2m + mo(G)) - 2|\bar{\mu}|\sqrt{n(2m + mo(G))}} \le E_{MM}^M(G) \le n\sqrt{(2m + mo(G)) + \bar{\mu}^2}$.

Proof. By Cauchy Schwartz inequality

$(\sum_{i=1}^n |a_i b_i|)^2 = (\sum_{i=1}^n a_i^2)(\sum_{i=1}^n b_i^2)$.

Take $a_i = 1$, $b_i = |\mu_i - \bar{\mu}|$ then,

$[E_{MM}^M(G)]^2 = (\sum_{i=1}^n 1)(\sum_{i=1}^n (|\mu_i - \bar{\mu}|)^2)$

$= (n)(\sum_{i=1}^{n}(|\mu_i - \bar{\mu}|)^2)$

$\leq n(\sum_{i=1}^{n}\mu_i^2 + \sum_{i=1}^{n}[\bar{\mu}]^2)$

$= n(\sum_{i=1}^{n}\mu_i^2 + n\ \bar{\mu}^2)$

$\leq n(n(2m + mo(G)) + n\bar{\mu}^2)$.

Then $E_{MM}^{M}(G) \leq n\sqrt{(2m + mo(G)) + \bar{\mu}^2}$.

Also $[E_{MM}^{M}(G)]^2 = (\sum_{i=1}^{n}|\mu_i - \bar{\mu}|)^2$

$\geq \sum_{i=1}^{n}|\mu_i - \bar{\mu}|^2$

$\geq \sum_{i=1}^{n}|\mu_i|^2 - 2|\bar{\mu}|\sum_{i=1}^{n}|\mu_i|$

$\geq (2m + mo(G)) - 2|\bar{\mu}|\sqrt{n(2m + mo(G))}$.

Then $E_{MM}^{M}(G) \geq \sqrt{(2m + mo(G)) - 2|\bar{\mu}|\sqrt{n(2m + mo(G))}}$.
Hence the theorem. □

Result 4.2. *Let G be a connected graph with vertex set $V = \{v_1, v_2, ..., v_n\}$ and size m. Then $\sqrt{n+1} \leq E_{MM}(G) \leq n\sqrt{n}$.*

For proof refer [7]

Theorem 4.2. *Let G be a connected graph with vertex set $V = \{v_1, v_2, ..., v_n\}$ and size m. Then $\sqrt{(n+1) - 2|\bar{\mu}|n\sqrt{n}} \leq E_{MM}^{M}(G) \leq \sqrt{n(n^3 + n\bar{\mu}^2)}$.*

Proof. By Cauchy Schwartz inequality

$(\sum_{i=1}^{n}|a_i b_i|)^2 = (\sum_{i=1}^{n}a_i^2)(\sum_{i=1}^{n}b_i^2)$.

Take $a_i = 1$, $b_i = |\mu_i - \bar{\mu}|$ then,

$[E_{MM}^{M}(G)]^2 = (\sum_{i=1}^{n}1)(\sum_{i=1}^{n}(|\mu_i - \bar{\mu}|)^2)$

$= (n)(\sum_{i=1}^{n}(|\mu_i - \bar{\mu}|)^2)$

$\leq n(\sum_{i=1}^{n}\mu_i^2 + \sum_{i=1}^{n}[\bar{\mu}]^2)$

$= n(\sum_{i=1}^{n}\mu_i^2 + n\ \bar{\mu}^2)$

$\leq n(n^3 + n\bar{\mu}^2)$

Then $E_{MM}^{M}(G) \leq \sqrt{n(n^3 + n\bar{\mu}^2)}$.

Also $[E_{MM}^{M}(G)]^2 = (\sum_{i=1}^{n}|\mu_i - \bar{\mu}|)^2$

$\geq \sum_{i=1}^{n}|\mu_i - \bar{\mu}|^2$

$\geq \sum_{i=1}^{n} |\mu_i|^2 - 2|\bar{\mu}| \sum_{i=1}^{n} |\mu_i|$

$\geq (n+1) - 2|\bar{\mu}|n\sqrt{n}.$

Then $E_{MM}^{M}(G) \geq \sqrt{(n+1) - 2|\bar{\mu}|n\sqrt{n}}.$

Hence the theorem. □

Result 4.3. *Let G be a connected graph with vertex set $V = \{v_1, v_2, ..., v_n\}$, size m and $D = det\ (A_{MM}(G))$. Then*

$$\sqrt{2m + mo(G) + n(n-1)D^{\frac{2}{n}}} \leq E_{MM}(G) \leq \frac{2m+mo(G)}{n} + \sqrt{(n-1)[2m + mo(G) - (\frac{2m+mo(G)}{n})^2]}.$$

For proof refer [7]

Theorem 4.3. *Let G be a connected graph with vertex set $V = \{v_1, v_2, ..., v_n\}$, size m and $D = det\ (A_{MM}(G))$. Then*

$$\sqrt{2m + mo(G) + n(n-1)D^{\frac{2}{n}} - 2|\bar{\mu}|\frac{2m+mo(G)}{n} + \sqrt{(n-1)[2m + mo(G) - (\frac{2m+mo(G)}{n})^2]}} \leq$$
$$E_{MM}^{M}(G) \leq \sqrt{n[(\frac{2m+mo(G)}{n}) + \sqrt{(n-1)[2m + mo(G) - (\frac{2m+mo(G)}{n})^2]}]^2 + n^2\bar{\mu}^2}.$$

Proof. By Cauchy Schwartz inequality

$(\sum_{i=1}^{n} |a_i b_i|)^2 = (\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2)$.

Take $a_i = 1$, $b_i = |\mu_i - \bar{\mu}|$ then,

$[E_{MM}^{M}(G)]^2 = (\sum_{i=1}^{n} 1)(\sum_{i=1}^{n}(|\mu_i - \bar{\mu}|)^2)$

$= (n)(\sum_{i=1}^{n}(|\mu_i - \bar{\mu}|)^2)$

$\leq n(\sum_{i=1}^{n} \mu_i^2 + \sum_{i=1}^{n}[\bar{\mu}]^2)$

$= n(\sum_{i=1}^{n} \mu_i^2 + n\ \bar{\mu}^2).$

$\leq n[[(\frac{2m+mo(G)}{n}) + \sqrt{(n-1)[2m + mo(G) - (\frac{2m+mo(G)}{n})^2]}]^2 + n\bar{\mu}^2].$

Then $E_{MM}^{M}(G) \leq \sqrt{n\ [(\frac{2m+mo(G)}{n}) + \sqrt{(n-1)[2m + mo(G) - (\frac{2m+mo(G)}{n})^2]}\ ]^2 + n^2\bar{\mu}^2}$

Also $[E_{MM}^{M}(G)]^2 = (\sum_{i=1}^{n} |\mu_i - \bar{\mu}|)^2$

$\geq \sum_{i=1}^{n} |\mu_i - \bar{\mu}|^2$

$\geq \sum_{i=1}^{n} |\mu_i|^2 - 2|\bar{\mu}| \sum_{i=1}^{n} |\mu_i|$

$\geq 2m + mo(G) + n(n-1)D^{\frac{2}{n}} - 2|\bar{\mu}|\frac{2m+mo(G)}{n} + \sqrt{(n-1)[2m + mo(G) - (\frac{2m+mo(G)}{n})^2]}$

Then $E^M_{MM}(G) \geq$

$$\sqrt{2m + mo(G) + n(n-1)D^{\frac{2}{n}} - 2|\bar{\mu}|\frac{2m+mo(G)}{n} + \sqrt{(n-1)[2m + mo(G) - (\frac{2m+mo(G)}{n})^2]}}.$$

Hence the theorem. □

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