TWMS J. App. and Eng. Math. V.11, N.1, 2021, pp. 9-19

EXACT SOLUTION WITH DUST AND SHELL-CROSSINGS FOR LTB INHOMOGENEOUS COSMOLOGICAL MODELS

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Abstract. We describe the solution of the Lemaitre-Tolman-Bondi (LTB) inhomogeneous cosmological models for a spherically symmetric with dust and an interaction term modeled as anisotropic pressure is also studied and a differential equation governing the time evolution is derived. The field equations are fully integrated for all parameter subcases and compared with analogous subcases of LTB dust solutions of general relativity (GR).

Keywords: LTB cosmology model, Spherically symmetric space-time, Inhomogeneous universe, Exact solutions, Hubble parameter.

AMS Subject Classification: 83D05, 83F05, 83C15

1. Introduction

An inhomogeneous cosmological models plays an important role in understanding some essential features of the universe such as the formation of galaxies during the early stages of evolution and process of homogenization. Many papers have been published in which various properties of the LTB model were discussed. The early attempt at the construction of such models has done by R. C. Tolman (1934) and H. Bondi (1947) who consider spherically symmetric models [2, 20]. A. H. Taub (1951,1956) and later by N. Tomimura (1978), P. Szekeres (1975), C. B. Collins and D. A. Szafron (1979) was first considered inhomogeneous plane symmetric model [3, 11, 18, 19, 21].
Krasinski (1997) gave a very extensive survey of inhomogeneous cosmological models and their physical properties [9]. He fitted the many specific models discussed in the literature into a relatively small number of families, including only those classes which contain an Friedmann Lemaitre Robertson Walker (FLRW) or Kantowski-Sachs (KSs) metric, which

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§ Manuscript received: March 16, 2019; accepted: May 29, 2109.
TWMS Journal of Applied and Engineering Mathematics, Vol.11, No.1
The first and the second authors are supported by the University Grant Commission under UGC-SAP-DRS(III).
The first author is supported by Al-Baydha University, and the Government of the Republic of Yemen.

excludes a few models of physical importance. Stephani et al. (2003) give the ones with a perfect fluid energy content [5, 17].
The LTB model (Lemaitre, 1933b, Tolman, 1934, Bondi, 1947), which is inhomogeneous in the radial direction only. In its application later in this paper, we assume the Earth is at or near the centre of isotropy [2, 5, 20].
The spherically symmetric cases are the simplest inhomogeneous models. The perfect fluid cases include stellar models and collapse solutions (see e.g. Misner, Thorne, and Wheeler (1973)): Bolejko et al. (2010) refer to these as the Lemaitre models, since they were discussed in Lemaitre (1933b) [1, 10]. The dust solutions form the LTB class. A further family that has been extensively studied is the self-similar subclass [9].
A LTB models are the simplest inhomogeneous expanding models, spherically symmetric about a center. They have been used to give exact nonlinear models of inhomogeneous cosmologies where no dark energy is needed the apparent acceleration of the universe seen in the supernova data is not a consequence of dark energy (as in an FLRW model), but it is due to spatial inhomogeneity. This is an important alternative to the standard interpretation and is discussed in [5].
The evolution of LTB voids was studied in much more detail by Occhionero and colleagues and Sato and colleagues (see Occhionero, Santangelo, and Vittorio (1983), Sato (1984), and papers cited therein) [12, 16].
In this paper, we have considered inhomogeneous cosmological model with dust. Exact values of parameters are calculated by using explicit solutions of the necessary field equations and solved them to get various cosmological models.

2. The Metric and the Field Equations

The LTB metric for a spherically symmetric, inhomogeneous line element is given by [6, 14]

$$ds^2 = dt^2 - A_1^2\, dr^2 - A_2^2\, d\Omega^2, \tag{1}$$

where

$$d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2.$$

Because of the signature $(-,-,-,+)$, and the functions $A_1 = A_1(r,t)$ and $A_2 = A_2(r,t)$ have both temporal spatial dependencies on space and radial coordinates (r,t) respectively. The homogeneous FRW metric is a special case and obtained by letting

$$A_1 = \frac{a(t)}{\sqrt{1-kr^2}}, \tag{2}$$
$$A_2 = a(t)r. \tag{3}$$

The energy-momentum tensor for the above metric takes a diagonal form, and is given by $T_4^4 = -\rho, T_i^i = p$ with $(i = 1,2,3)$, where $\rho = \rho(r,t)$ is the proper energy density, $p = p(r,t)$ is the isotropic pressure. The Einstein field Equations (EFEs), $G_j^i = 8\pi T_j^i$, are given by

$$\begin{aligned}
\frac{\dot{A_2}^2}{A_2^2}+\frac{2\ddot{A}_2}{A_2}+\frac{1}{A_2^2}-\frac{A_2'^2}{A_1^2A_2^2} &= -8\pi p, &(4)\\
\frac{\dot{A}_2'}{A_2}-\frac{\dot{A}_1A_2'}{A_1A_2} &= 0, &(5)\\
\frac{\ddot{A_2}^2}{A_2^2}+\frac{\ddot{A}_1}{A_1}+\frac{\dot{A}_1\dot{A}_2}{A_1A_2}+\frac{A_1'A_2'}{A_1^3A_2}-\frac{A_2''}{A_1^2A_2} &= -8\pi p, &(6)\\
\frac{\dot{A_2}^2}{A_2^2}+\frac{1}{A_2^2}+2\frac{\dot{A}_1\dot{A}_2}{A_1A_2}++2\frac{A_1'A_2'}{A_1^3A_2}-\frac{A_2'^2}{A_1^2A_2^2}-2\frac{A_2''}{A_1^2A_2} &= -8\pi\rho, &(7)
\end{aligned}$$

where dot and dash are, respectively denoted the derivative with respect to time t and radial coordinate r. Solving the Equation (5) gives

$$A_2' = C(r)A_1, \tag{8}$$

where the function $C(r)$ is an arbitrary function only of the radial coordinate r. We can thus write the LTB metric Equation (1) in its usual form

$$ds^2 = dt^2 - A_2'^2C^{-2}\ dr^2 - A_2^2\ d\Omega^2, \tag{9}$$

The average scale-factor $a(t)$ and spatial volume V are given by

$$V = \sqrt{-g} = a^3 = A_2'A_2^2. \tag{10}$$

We define two Hubble parameters (HPs) and deceleration parameter (DP) as

$$\begin{aligned}
H_t &= \frac{\dot{A}_2}{A_2},\ H_r = \frac{\dot{A_2}'}{A_2'}, &(11)\\
q(t) &= -\frac{\ddot{A}_2}{A_2}H_t^{-2} = -\left(1+\frac{\dot{H}_t}{H_t^2}\right). &(12)
\end{aligned}$$

The independent Equations for the above metric are given by

$$\begin{aligned}
H_t^2 + 2(\dot{H}_t + H_t^2) - \frac{K(r)}{A_2^2} &= -8\pi p, &(13)\\
H_t^2 + 2H_tH_r - \frac{K(r)}{A_2^2} - \frac{K(r)'}{A_2A_2'} &= -8\pi\rho, &(14)
\end{aligned}$$

where $K(r) = C^2 - 1$. The cosmological parameters such as effective HP (H_{eff}), scalar expansion (θ) and shear scalar (σ^2) and anisotropy parameter (A_m) are given by

$$\begin{aligned}
H_{eff} &= \frac{1}{3}\left[2H_t + H_r\right], &(15)\\
\theta_{eff} &= 3H_{eff} = 2H_t + H_r, &(16)\\
\sigma_{eff}^2 &= \frac{1}{3}\left[H_r - H_t\right]^2, &(17)\\
A_{m(eff)} &= \frac{1}{3}\sum_{i=1}^{3}\left[\frac{H_i - H_{eff}}{H_{eff}}\right]^2 = \frac{1}{3}\left(2\left[\frac{H_t - H_{eff}}{H}\right]^2 + \left[\frac{H_r - H_{eff}}{H_{eff}}\right]^2\right). &(18)
\end{aligned}$$

By integrating Equation (13), it follows that

$$H_t^2 = \frac{E(r)}{A_2^3} + \frac{8\pi p}{3} + \frac{K(r)}{A_2^2}, \tag{19}$$

where $E = E(r)$ is a non-negative function. Substitution Equation (19) into EFEs (13), (14) respectively we get

$$\rho = \frac{E'}{8\pi A_2' A_2^2} = \frac{E'}{8\pi V},\ p = -\frac{\dot{E}}{8\pi \dot{A}_2 A_2^2}. \tag{20}$$

Hence a physically realistic model must have $E' > 0$. The energy density ρ in the LTB model becomes infinite where $A_2 = A_2' = 0 \neq E'$. This singularity is called shell-crossing singularity (SCS), because at those locations the radial distance between two adjacent. The SCS, where the energy density goes to infinity and changes sign to become negative. Equation (20) implies that, with $E' = 0$, the LTB model becomes vacuum. Being spherically symmetric, when $E' = 0$, it must coincide with the Schwarzschild solution or its extension through the event horizon and indeed it does as shown in [14].
An interesting thing happens when $A_2' = 0$. The metric (9) is then an inhomogeneous perturbation of the KSs (1966) metric: it is spherically symmetric but does not contain the centers of symmetry in the hyper surfaces $t = const$. In Russian literature, such solutions are called "T-models" (Novikov 1963, Ruban 1968, 1969 and 1983) [9].
Now we will explain that the universe is spatially flat to within a few percent we can take $C(r) = 1$ such that the field equations finally reduce to the following two independent Equations for the metric are given by

$$H_t^2 + 2(\dot{H}_t + H_t^2) = -8\pi p, \tag{21}$$

$$H_t^2 + 2H_t H_r = -8\pi\rho. \tag{22}$$

The matter satisfies the following conservation Equation

$$\nabla_i T^{ij} = 0. \tag{23}$$

The conservation Equation gives above implies that the test particles describes geodesics as in the case of GR which in turn yields

$$\dot{\rho} + \frac{1}{A_2' A_2^2} \frac{d}{dt} \left(A_2' A_2^2 \right) (\rho + p) = 0. \tag{24}$$

If we further assume that the perfect fluid obeys the barotropic Equation of state of the form

$$p = \omega\rho,\ 0 \leq \omega \leq 1,$$

with the equation of state parameter ω as time-independent. In this case, Equation (24) can be integrated for the energy density to yield

$$\rho = N(r) \left(A_2' A_2^2 \right)^{-(1+\omega)}, \tag{25}$$

where $N(r)$ is an arbitrary function of r. Plugging in the expression of ρ from equation (22) we finally get

$$H_t^2 + 2H_t H_r = -8\pi N(r) \left(A_2' A_2^2 \right)^{-(1+\omega)}. \tag{26}$$

3. Dust Dominated LTB Line Element

We shall now consider LTB line element with dust only, putting $p = 0$. By using Equation (13), we obtain the DP

$$q = \frac{1}{2}\left[1 - \frac{K}{\dot{A_2}^2}\right]. \tag{27}$$

The condition for accelerated expansion now takes the form

$$K > \dot{A_2}^2 > 0 \; or \; C^2 > 1 + \dot{A_2}^2. \tag{28}$$

Inserting the expression for the DP, Equation (13) takes the form

$$\dot{A_2}^2 + 2A_2\ddot{A}_2 - K = 0. \tag{29}$$

Integration leads to

$$A_2\dot{A_2}^2 = KA_2 + E(r) \; or \; \dot{A}_2 = \pm\sqrt{\frac{E(r)}{A_2} + K}. \tag{30}$$

Hence, the dynamical effects of K and E are similar to that of curvature and dust respectively. Therefore $E(r)$ is regarded as a gravitational energy function. Substituting Equation (30) into Equation (27) we find

$$q = \frac{E}{2A_2\dot{A_2}^2} \; or \; \ddot{A}_2 = -\frac{E}{2A_2^2}. \tag{31}$$

Allowing for inhomogeneity with $-KA_2 < E(r) < 0$ seems to allow accelerated expansion even for dust dominated universe models. The dynamical effect of $E < 0$ corresponds to that of dust with negative density in a homogeneous universe model. It should be noted, however, that the inequality above forbids accelerated expansion in a big bang (BB) model where the scale factor has the initial value $A_2(r, 0) = 0$ which implies $E(r) \geq 0$. However this initial condition may not be physically realistic. The universe may have started with a finite scale factor, or maybe has collapsed and reached a finite minimum radius. In such models accelerated expansion does not seem to be forbidden [6].

The solution of Equation (30) will contain one more arbitrary function ($t_B(r)$), that will appear in the combination ($t - t_B(r)$). It is called the bang-time function [14]. After a separation of variables, Equation (30) can be integrated in time [15]

$$t - t_B(r) = \int_0^{A_2(t_0,r)} \frac{1}{\sqrt{K + \frac{E(r)}{\widetilde{A}_2}}} d\widetilde{A}_2, \tag{32}$$

where t_0 denotes the present time. In general, $t_B = t_B(r)$ is the third arbitrary function, describing the time of the BB at the comoving radius r, which means that the BB does not need to occur synchronously.

The solutions to $A_2(r, t)$ can be categorized into three classes,

(1) For $K = 0$, the parabolic evolution gives

$$A_2 = \left(\frac{9}{4}E\left(t - t_B(r)\right)^2\right)^{\frac{1}{3}}. \tag{33}$$

(2) For $K > 0$, the hyperbolic evolution gives

$$A_2 = \frac{E(cosh\eta - 1)}{2K}, \; sinh\eta - \eta = \frac{(2K)^{\frac{3}{2}}\left(t - t_B(r)\right)}{E}. \tag{34}$$

(3) For $K < 0$, the elliptic evolution gives

$$A_2 = \frac{E(1 - cos\eta)}{(-2K)}, \ \eta - sin\eta = \frac{(-2K)^{\frac{3}{2}} \left(t - t_B(r)\right)}{E}, \tag{35}$$

where η is a parameter. The angular part of the line element is $A_2^2 d\Omega^2$ where $d\Omega$ is a solid angle element. It represents the area of a surface extending a certain solid angle.
At the origin, $r = 0$, this area must vanish, and thus $A_2(0, t) = 0$. From Equation (20) we then get $E(0) = 0$. Since $E' > 0$, it follows that $E(r) > 0$ for all r. But accelerated expansion is only possible for models with $E < 0$. Hence the dust dominated LTB universe models have decelerated expansion [6].
The LTB model is characterized by three arbitrary functions $K(r)$, $E(r)$ and $t_B(r)$ of the coordinate radius r. $K(r) \geq -1$ has a geometrical role, determining the local embedding angle of spatial slices, and also a dynamical role, determining the local energy per unit energy of the dust particles, and, hence, the type of evolution of A_2. $E(r)$ is the effective gravitational energy within comoving radius r. $t_B(r)$ is the local time at which $A_2 = 0$, i.e. the local time of the BB. if $t_B \neq const$, we have a non-simultaneous bang surface. Specification of these three arbitrary functions fully determines the model, and while each of them can be given some type of interpretation for arbitrary choice of the radial coordinate r, there is still a freedom to choose this coordinate, leaving two physically meaningful free functions, e.g. two of $r = r(E)$, $K = K(E)$, and $t_B = t_B(E)$. For more details of the dynamics of these models and its relation to initial data, see Bolejko et al. (2010).
A physical limitation on the choices of the arbitrary functions is that if $A_2' \neq 0$ we may have a SCS, where comoving shells of distinct r collide (Hellaby and Lake, 1985). This Equation also holds at an extremum of density if E' and K have zeros of the same order [5].
Adjacent contours of small constant A_2 must have a similar shape. Hence, if either of the two equations was not fulfilled, either the upper branch or the lower branch of some contours would be a non-monotonic function, whose derivative by E would change the sign somewhere. At the changeover points, the tangents to the contours would be horizontal, and these would be the SCs [14].
Now we come to the conditions for avoiding SCs at those points where $E' \neq 0$. We wish to translate the condition $A_2' \neq 0$ into properties of the functions $E(r)$, $K(r)$ and $t_B(r)$. The cases $A_2' > 0$ and $A_2' < 0$ have to be considered separately. We will write out the conditions only for $A_2' > 0$.
In all cases, in those regions where $A_2' > 0$, in order that the energy density is positive, we must have

$$E' > 0. \tag{36}$$

From here on, the three types of models have to be considered separately.

3.1. **The Parabolic Evolution for $K = 0$.** We calculate from the Equation (33) for getting

$$\frac{A_2'}{A_2} = \frac{E'}{3E} - \frac{\sqrt{2E}\ t_B'}{A_2^{\frac{3}{2}}}. \tag{37}$$

As $A_2 \to 0$, the second term dominates, so, in order that $A_2' > 0$ everywhere, we must have $t_B' < 0$. Together with the Equation $E' > 0$ this is then seen to be the necessary and sufficient condition. If we take $E(r) = \frac{12}{9} r^3$, it is clear that

$$A_2 = r\left(t - t_B(r)\right)^{\frac{2}{3}}. \tag{38}$$

If at this stage, we assume that $t_B(r)$ vanishes or losing its space dependence becomes a true constant, the line element reduces to

$$ds^2 = dt^2 - t^{\frac{4}{3}} \left(dr^2 - r^2 \, d\Omega^2\right). \tag{39}$$

This is a new solution and may be termed as the generalized Einstein-de Sitter metric for the inhomogeneous space-time. From equation (25), we get the expression of density as

$$\rho = N(r) \left(r^2 \left(t - t_B(r)\right)^2 + \frac{2}{3} r^3 \left(t - t_B(r)\right) t'_B \right)^{-(1+\omega)}. \tag{40}$$

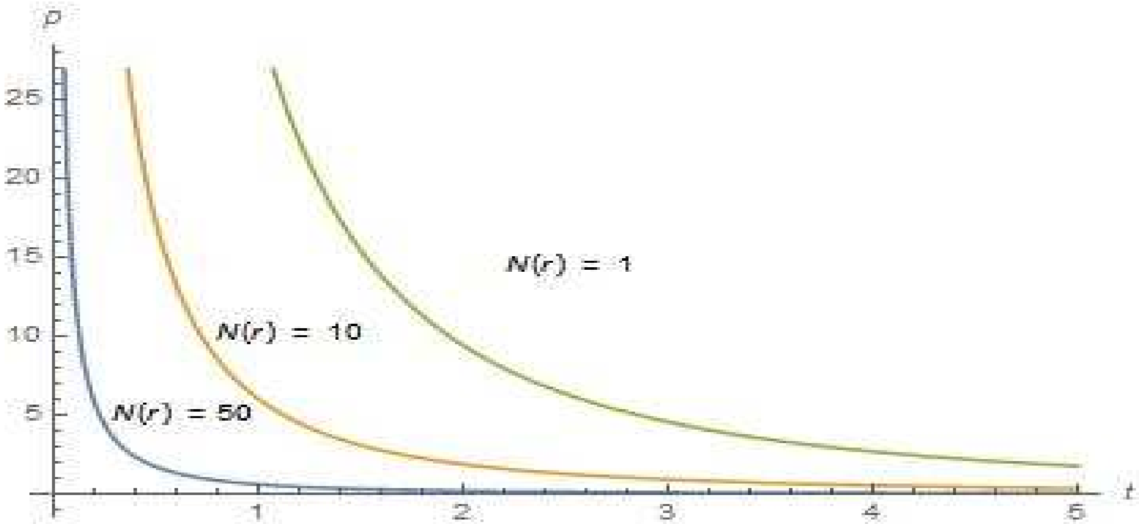

FIGURE 1. The behavior of $\rho(r,t)$ versus t for different values of $N(r)$ is shown. The graphs clearly show that the density increases for greater $N(r)$ i.e., greater inhomogeneity. Taking $r = 1$ & $\omega = 0$.

Figure (1) shows that the density depends on $N(r)$ when $t_B(r)$ vanishes, which represents the inhomogeneity. So $N(r)$ may be the measure of inhomogeneity.

3.2. **The Hyperbolic Evolution for $K > 0$.** We calculate from the Equation (34) and making use of Equation (19) for expansion $\dot{A}_2 > 0$ we get

$$\frac{A'_2}{A_2} = \left(\frac{E'}{E} - \frac{K'}{K}\right) + \left(\frac{3K'}{2K} - \frac{E'}{E}\right) \Phi_1(\eta) - \frac{(-2K)^{\frac{3}{2}}}{E} t'_B \, \Phi_2(\eta), \tag{41}$$

where

$$\Phi_1(\eta) = \frac{sinh\eta(sinh\eta - \eta)}{(cosh\eta - 1)^2}, \; \Phi_2(\eta) = \frac{sinh\eta}{(cosh\eta - 1)^2}. \tag{42}$$

The following properties of $\Phi_1(\eta)$ and $\Phi_2(\eta)$ are useful in calculations,

$$\lim_{\eta \to 0} \Phi_1(\eta) = \frac{2}{3}, \; \lim_{\eta \to 0} \Phi_2(\eta) = \infty, \tag{43}$$

$$\lim_{\eta \to \infty} \Phi_1(\eta) = 1, \; \lim_{\eta \to \infty} \Phi_2(\eta) = 0, \tag{44}$$

$$\frac{d\Phi_1}{d\eta} > 0 \; for \; \eta > 0, \; \frac{d\Phi_2}{d\eta} < 0 \; for \; \eta > 0. \tag{45}$$

Hence, taking (41) in the limit $\eta \to 0$, the last term dominates and it will be positive if

$$t'_B < 0. \tag{46}$$

Now we take (41) in the limit $\eta \to \infty$, we easily obtain

$$K' > 0. \tag{47}$$

The Equations (36), (46) and (47) are necessary conditions for $A'_2 > 0$. To see that they are also sufficient, it suffices to rewrite (41) in the form

$$\frac{A'_2}{A_2} = \frac{E'}{E}(1 - \Phi_1) + \frac{K'}{K}\left(\frac{3}{2}\Phi_2 - 1\right) - \frac{(-2K)^{\frac{3}{2}}}{E} t'_B\ \Phi_2(\eta), \tag{48}$$

and take note of Equations (43), (44) and (45).

3.3. **The Elliptic Evolution for $K < 0$.** We calculate from the Equation (35) and making use of Equation (19) for expansion $\dot{A}_2 > 0$ we get

$$\frac{A'_2}{A_2} = \left(\frac{E'}{E} - \frac{K'}{K}\right) + \left(\frac{3K'}{2K} - \frac{E'}{E}\right)\Phi_3(\eta) - \frac{(-2K)^{\frac{3}{2}}}{E} t'_B \Phi_4(\eta) = f(\eta)\ (say), \tag{49}$$

where

$$\Phi_3(\eta) = \frac{sin\eta(\eta - sin\eta)}{(1 - cos\eta)^2},\ \ \Phi_4(\eta) = \frac{sin\eta}{(1 - cos\eta)^2}. \tag{50}$$

The function $f(\eta)$ should be strictly positive in the whole range $\eta \in [0, 2\pi]$. Note that

$$\lim_{\eta \to 0} \Phi_3(\eta) = \frac{2}{3},\ \lim_{\eta \to 0} \Phi_4(\eta) = \infty, \tag{51}$$

$$\lim_{\eta \to 2\pi} \Phi_3(\eta) = -\infty,\ \lim_{\eta \to 2\pi} \Phi_4(\eta) = -\infty, \tag{52}$$

$$\lim_{\eta \to 2\pi} \frac{\Phi_3(\eta)}{\Phi_4(\eta)} = 2\pi. \tag{53}$$

Hence, taking (49) in the limit $\eta \to 0$, we see that the last term becomes unbounded, and it will be positive only if

$$t'_B < 0. \tag{54}$$

Now we take (49) in the limit $\eta \to 2\pi$. Then the last two terms become unbounded. Factoring out Φ_4, which goes to $(-\infty)$, and demanding that the (bounded) coefficient is negative in the limit, we obtain:

$$2\pi\left(\frac{3K'}{2K} - \frac{E'}{E}\right) - \frac{(-2K)^{\frac{3}{2}}}{E} t'_B < 0. \tag{55}$$

The Equations (36), (54) and (55) are the necessary and sufficient conditions for the absence of SCs in the case $A'_2 > 0$.

The meaning of Equation (55) is that the crunch time must be an increasing function of r.

4. Cosmological Redshift in LTB

In several applications of a cosmological model, we need to calculate the redshift of light emitted by a source at a given time and location and received by an observer located down a null geodesic from the source. One of them is the following method (copied from Bondi (1947)).
From the symmetry of the situation, it is clear that light can travel radially, that is, there exist geodesics with $d\theta = d\phi = 0$. Moreover, since light always travels along null geodesics, we have $ds^2 = 0$. Inserting these conditions into the equation for the line element, Equation (9), we obtain the constraint equation for light rays

$$\frac{dt}{dr} = \frac{A_2'(r,t)}{\sqrt{1+K(r)}}. \tag{56}$$

Let us consider two light rays be emitted in the same direction, the second one later by a small time-interval τ. Let the equations of the first and second rays are given by

$$t = T(t), \; t = T(r) + \tau(r). \tag{57}$$

Both rays must obey Equation (56), so

$$\frac{dT(r)}{dr} = \frac{A_2'(r,T(r))}{\sqrt{1+K(r)}}, \; \frac{d(T(r)+\tau(r))}{dr} = \frac{A_2'(r,T(r)+\tau(r)\dot{A}_2'(r,T(r))}{\sqrt{1+K(r)}}. \tag{58}$$

Using the first Equation of (58) in the second Equation of (58), we obtain

$$\frac{d\tau(r)}{dr} = -\tau(r)\frac{\dot{A}_2'(r,T(r))}{\sqrt{1+K(r)}}. \tag{59}$$

Differentiating the definition of the redshift, $z \equiv \frac{\tau(0)}{\tau(r)} - 1$, we obtain

$$\frac{1}{1+z}\frac{dz}{dr} = \frac{\dot{A}_2'(r,T(r))}{\sqrt{1+K(r)}}, \tag{60}$$

we can combine Equations (56) and (60) to obtain the pair of differential equations

$$(1+z)\frac{dt}{dz} = \frac{A_2'(r,T(r))}{\dot{A}_2'(r,T(r))}. \tag{61}$$

Hence by using Equation (60), the redshift may be calculated numerically from

$$ln(1+z) = \frac{dt}{dz} = \int_r^0 \frac{\dot{A}_2'(r,T(r))}{\sqrt{1+K(r)}}dr. \tag{62}$$

Now that we have related the redshift to the inhomogeneities, we still need the relation between the redshift and the energy flux S, or the luminosity distance D_L, defined as

$$D_L(z) = \sqrt{\frac{L}{4\pi S}}, \tag{63}$$

where L is the total power radiated by the source. This is given by [4]

$$D_L^{LTB}(z) = (1+z)^2 A(r(z), t(z)), \tag{64}$$

where the angular distance diameter is given by

$$D_A^{LTB}(z) = A(r(z), t(z)). \tag{65}$$

5. Conclusions

The LTB models, with metrics as given containing arbitrary functions $K(r), E(r)$ and $t_B(r)$ of the coordinate radius r. Note that we can have LTB models that are FLRW for certain ranges of r giving concentric shells of differing behavior.
The present work may be looked upon as an extension of one of our recent publications where we examined the possibility in a higher dimensional LTB model if the inclusion of extra space jointly with inhomogeneity can induce late inflation in a dust model. While total volume acceleration is ruled out we found that preferential acceleration in a radial direction is possible if the angular direction decelerates fast enough or vice versa.
Aside from space dependence the mathematical structure and its essentially similar to the works of homogeneous space-time except for the appearance of the term, $N(r)$ in Equation (25), which unlike its homogeneous counterpart is not a true constant but depends on the space coordinate. Its presence introduces all the differences in cosmic evolution. Like FRW models our field equations are amenable to the exact solution only at extreme values. We find that at early stage our solution reduces to an inhomogeneous analog of the Einstein de-Sitter type of solution. In all cases, The SCs for the energy density is positive.

Acknowledgement. The authors are thankful to the anonymous referees for critical comments that improved quality of this manuscript a lot.

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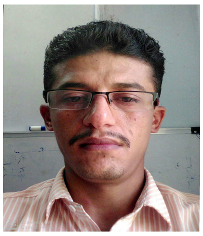

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