TWMS J. App. and Eng. Math. V.11, N.1, 2021, pp. 160-167

RIGHT $(\sigma,\tau)-$LIE IDEALS AND ONE SIDED GENERALIZED DERIVATIONS

E. GÜVEN, §

Abstract. Let R be a prime ring with characteristic not 2 and $\sigma,\tau,\alpha,\beta,\lambda,\mu,\gamma$ automorphisms of R. Let $h: R\longrightarrow R$ be a nonzero left(resp.right)-generalized $(\alpha,\beta)-$derivation, $a,b\in R$. and U,V nonzero right $(\sigma,\tau)-$Lie ideals of R. The main object in this article is to study the situations. (1) $a[U,b]_{\lambda,\mu}=0$ or $[U,b]_{\lambda,\mu}a=0$, (2) $a(U,b)_{\lambda,\mu}=0$ or $(U,b)_{\lambda,\mu}a=0$, (3) $bh(I)\subset C_{\lambda,\mu}(U)$ or $h(I)b\subset C_{\lambda,\mu}(U)$, (4) $(b,U)_{\lambda,\mu}=0$ or $[b,U]_{\lambda,\mu}=0$, (5) $(U,b)_{\lambda,\mu}\subset C_{\lambda,\mu}(R)$, (6) $bV\subset C_{\lambda,\mu}(U)$ or $Vb\subset C_{\lambda,\mu}(U)$. Also, some characteristics of left and right generalized $(\alpha,\beta)-$derivation satisfying several conditions on ideals are given.

Keywords: Prime ring, generalized derivation, $(\sigma,\tau)-$Lie Ideal.

AMS Subject Classification: 16W25, 16U80.

1. Introduction

Let R be a ring and σ,τ two mappings of R. For each $r,s\in R$ we set $[r,s]_{\sigma,\tau}=r\sigma(s)-\tau(s)r$ and $(r,s)_{\sigma,\tau}=r\sigma(s)+\tau(s)r$. Let U be an additive subgroup of R. If $[U,R]\subset U$ then U is called a Lie ideal of R. The definition of $(\sigma,\tau)-$Lie ideal of R is introduced in [8] as follows: (i) U is called a right $(\sigma,\tau)-$Lie ideal of R if $[U,R]_{\sigma,\tau}\subset U$, (ii) U is called a left $(\sigma,\tau)-$Lie ideal if $[R,U]_{\sigma,\tau}\subset U$, (iii) U is called a $(\sigma,\tau)-$Lie ideal if U is both right and left $(\sigma,\tau)-$Lie ideal of R. Every Lie ideal of R is a $(1,1)-$Lie ideal of R, where $1: R\rightarrow R$ is the identity mapping.

If $R=\{\ \left(\begin{smallmatrix} x & y\\ 0 & 0\end{smallmatrix}\right)\mid : x$ and y are integers$\}$, $U=\{\ \left(\begin{smallmatrix} x & 0\\ 0 & 0\end{smallmatrix}\right)\mid x$ is integer$\}$, $\sigma\left(\begin{smallmatrix} x & y\\ 0 & 0\end{smallmatrix}\right)=\left(\begin{smallmatrix} x & 0\\ 0 & 0\end{smallmatrix}\right)$ and $\tau\left(\begin{smallmatrix} x & y\\ 0 & 0\end{smallmatrix}\right)=\left(\begin{smallmatrix} x & -y\\ 0 & 0\end{smallmatrix}\right)$ then U is $(\sigma,\tau)-$right Lie ideal but not a Lie ideal of R.

A derivation d is an additive mapping on R which satisfies $d(rs)=d(r)s+rd(s)$,for all $r,s\in R$. The notion of generalized derivation was introduced by Brešar [2] as follows. An additive mapping $h: R\rightarrow R$ will be called a generalized derivation if there exists a derivation d of R such that $h(xy)=h(x)y+xd(y)$, for all $x,y\in R$.

An additive mapping $d: R\rightarrow R$ is said to be a $(\sigma,\tau)-$derivation if $d(rs)=d(r)\sigma(s)+\tau(r)d(s)$ for all $r,s\in R$. Every derivation $d: R\rightarrow R$ is a $(1,1)-$derivation. Chang [3] gave

Kocaeli University, Faculty of Art and Sciences, Department of Mathematics, Kocaeli, Turkey.
e-mail: evrim@kocaeli.edu.tr; ORCID: http://orcid.org/0000-0001-5256-4447.

§ Manuscript received: December 16, 2019; accepted: March 22, 2020.

the following definition. Let R be a ring, σ and τ automorphisms of R and $d : R \to R$ a $(\sigma,\tau)-$derivation. An additive mapping $h : R \to R$ is said to be a right generalized $(\sigma,\tau)-$derivation of R associated with d if $h(xy) = h(x)\sigma(y) + \tau(x)d(y)$, for all $x, y \in R$ and h is said to be a left generalized $(\sigma,\tau)-$derivation of R associated with d if $h(xy) = d(x)\sigma(y) + \tau(x)h(y)$, for all $x, y \in R$. h is said to be a generalized $(\sigma,\tau)-$derivation of R associated with d if it is both a left and right generalized $(\sigma,\tau)-$derivation of R associated with d. Every $(\sigma,\tau)-$derivation $d : R \to R$ is a generalized $(\sigma,\tau)-$derivation with d. Based on this definition of Chang, every $(\sigma,\tau)-$derivation $d : R \to R$ is a generalized $(\sigma,\tau)-$derivation associated with d and every derivation $d : R \to R$ is a generalized $(1,1)-$derivation associated with d. A generalized $(1,1)-$derivation is simply called a generalized derivation. It is clear that the generalized derivation defined by [2] is the right generalized derivation in the definition given by Chang.

The mapping $h(r) = (a,r)_{\sigma,\tau}, \forall r \in R$ is a left-generalized $(\sigma,\tau)-$derivation with $(\sigma,\tau)-$derivation $d_1(r) = [a,r]_{\sigma,\tau}, \forall r \in R$ and right-generalized $(\sigma,\tau)-$derivation with $(\sigma,\tau)-$derivation $d(r) = -[a,r]_{\sigma,\tau}, \forall r \in R$.

Throughout the paper, R will be a prime ring with center Z, characteristic not 2 and $\sigma,\tau,\alpha,\beta,\lambda,\mu,\gamma$ automorphisms of R. We set $C_{\sigma,\tau}(R) = \{c \in R \mid c\sigma(r) = \tau(r)c, \forall\, r \in R\}$, and shall use the following relations frequently:

$[rs,t]_{\sigma,\tau} = r[s,t]_{\sigma,\tau} + [r,\tau(t)]s = r[s,\sigma(t)] + [r,t]_{\sigma,\tau}s,$
$[r,st]_{\sigma,\tau} = \tau(s)[r,t]_{\sigma,\tau} + [r,s]_{\sigma,\tau}\sigma(t),$
$(rs,t)_{\sigma,\tau} = r(s,t)_{\sigma,\tau} - [r,\tau(t)]s = r[s,\sigma(t)] + (r,t)_{\sigma,\tau}s,.$
$(r,st)_{\sigma,\tau} = \tau(s)(r,t)_{\sigma,\tau} + [r,s]_{\sigma,\tau}\sigma(t) = -\tau(s)[r,t]_{\sigma,\tau} + (r,s)_{\sigma,\tau}\sigma(t).$

2. Results

Lemma 2.1. [1] *Let R be a prime ring and $d : R \longrightarrow R$ a $(\sigma,\tau)-$derivation. If U is a right ideal of R and $d(U) = 0$ then $d = 0$.*

Lemma 2.2. [6] *Let U be a nonzero right $(\sigma,\tau)-$Lie ideal of R and $a \in R$. If $[U,a]_{\alpha,\beta} = 0$ then $a \in Z$ or $U \subset C_{\sigma,\tau}(R)$.*

Lemma 2.3. [5] *Let $h : R \longrightarrow R$ be a nonzero left-generalized $(\sigma,\tau)-$derivation associated with a nonzero $(\sigma,\tau)-$derivation $d : R \longrightarrow R$ and I, J be nonzero ideals of R. If $h(I) \subset C_{\lambda,\mu}(J)$ then R is commutative.*

The following Lemma is a generalization of [7].

Lemma 2.4. *Let I be a nonzero ideal of R and $a, b \in R$. If $b, ba \in C_{\lambda,\mu}(I)$ or $(b, ab \in C_{\lambda,\mu}(I))$ then $b = 0$ or $a \in Z$.*

Proof. If $b, ba \in C_{\lambda,\mu}(I)$ then we have

$$0 = [ba,x]_{\lambda,\mu} = b[a,\lambda(x)] + [b,x]_{\lambda,\mu}a = b[a,\lambda(x)]$$

and so $b[a,\lambda(x)] = 0$ for all $x \in I$. Replacing x by $xr, r \in R$ we get $b\lambda(I)[a,R] = 0$. This gives that $b = 0$ or $a \in Z$.

If $b, ab \in C_{\lambda,\mu}(I)$ then the relation

$$0 = [ab,x]_{\lambda,\mu} = a[b,x]_{\lambda,\mu} + [a,\mu(x)]b = [a,\mu(x)]b \text{ for all } x \in I$$

gives that $[a,\mu(I)]b = 0$. Considering as above we get $b = 0$ or $a \in Z$. □

Lemma 2.5. *Let $h : R \longrightarrow R$ be a nonzero right-generalized $(\alpha,\beta)-$derivation associated with a nonzero $(\alpha,\beta)-$derivation $d : R \longrightarrow R$. Let I be a nonzero ideal of R and $a, b \in R$. If $a[h(I),b]_{\lambda,\mu} = 0$ or $a(h(I),b)_{\lambda,\mu} = 0$ then $d\alpha^{-1}\lambda(b) = 0$ or $a[a,\mu(b)] = 0$.*

Proof. If $a[h(I), b]_{\lambda,\mu} = 0$ then we get, for all $x \in I$

$$\begin{aligned} 0 &= a[h(x\alpha^{-1}\lambda(b)), b]_{\lambda,\mu} = a[h(x)\lambda(b) + \beta(x)d\alpha^{-1}\lambda(b), b]_{\lambda,\mu} \\ &= ah(x)[\lambda(b), \lambda(b)] + a[h(x), b]_{\lambda,\mu}\lambda(b) + a\beta(x)[d\alpha^{-1}\lambda(b), b]_{\lambda,\mu} \\ &+ a[\beta(x), \mu(b)]d\alpha^{-1}\lambda(b) \\ &= a\beta(x)[d\alpha^{-1}\lambda(b), b]_{\lambda,\mu} + a[\beta(x), \mu(b)]d\alpha^{-1}\lambda(b). \end{aligned}$$

That is

$$a\beta(x)[k, b]_{\lambda,\mu} + a[\beta(x), \mu(b)]k = 0 \text{ for all } x \in I. \tag{1}$$

where $k = d\alpha^{-1}\lambda(b)$. Replacing x by $\beta^{-1}(a)x$ in (1) and using (1) we have for all $x \in I$

$$\begin{aligned} 0 &= aa\beta(x)[k, b]_{\lambda,\mu} + a[a\beta(x), \mu(b)]k \\ &= aa\beta(x)[k, b]_{\lambda,\mu} + aa[\beta(x), \mu(b)]k + a[a, \mu(b)]\beta(x)k \\ &= a[a, \mu(b)]\beta(x)k \end{aligned}$$

which gives $a[a, \mu(b)]\beta(I)k = 0$. Since $\beta(I)$ is a nonzero ideal of R and R is a prime ring then the last relation gives that $d\alpha^{-1}\lambda(b)$ or $a[a, \mu(b)] = 0$.

If $a(h(I), b)_{\lambda,\mu} = 0$ then considering as above and using the relation

$$(rs, t)_{\sigma,\tau} = r(s, t)_{\sigma,\tau} - [r, \tau(t)]s = r[s, \sigma(t)] + (r, t)_{\sigma,\tau} \text{ for all } r, s, t \in R. \tag{2}$$

We have the same result. □

Lemma 2.6. *Let $h : R \longrightarrow R$ be a nonzero left-generalized $(\alpha, \beta)-$derivation associated with a nonzero $(\alpha, \beta)-$derivation $d : R \longrightarrow R$. Let I be a nonzero ideal of R and $a, b \in R$. If $[h(I), b]_{\lambda,\mu}a = 0$ or $(h(I), b)_{\lambda,\mu}a = 0$ then $d\beta^{-1}\mu(b) = 0$ or $[a, \lambda(b)]a = 0$.*

Proof. If $[h(I), b]_{\lambda,\mu}a = 0$ then we get for all $x \in I$

$$\begin{aligned} 0 &= [h(\beta^{-1}\mu(b)x), b]_{\lambda,\mu}a = [d\beta^{-1}\mu(b)\alpha(x) + \mu(b)h(x), b]_{\lambda,\mu}a \\ &= d\beta^{-1}\mu(b)[\alpha(x), \lambda(b)]a + [d\beta^{-1}\mu(b), b]_{\lambda,\mu}\alpha(x)a \\ &+ \mu(b)[h(x), b]_{\lambda,\mu}a + [\mu(b), \mu(b)]h(x)a \\ &= d\beta^{-1}\mu(b)[\alpha(x), \lambda(b)]a + [d\beta^{-1}\mu(b), b]_{\lambda,\mu}\alpha(x)a \end{aligned}$$

which gives that

$$k[\alpha(x), \lambda(b)]a + [k, b]_{\lambda,\mu}\alpha(x)a = 0 \text{ for all } x \in I \tag{3}$$

where $k = d\beta^{-1}\mu(b)$. Replacing x by $x\alpha^{-1}(a)$ in (3) and using (3) we have for all $x \in I$

$$\begin{aligned} 0 &= k[\alpha(x)a, \lambda(b)]a + [k, b]_{\lambda,\mu}\alpha(x)aa \\ &= k\alpha(x)[a, \lambda(b)]a + k[\alpha(x), \lambda(b)]aa + [k, b]_{\lambda,\mu}\alpha(x)aa \\ &= k\alpha(x)[a, \lambda(b)]a. \end{aligned}$$

That is $k\alpha(I)[a, \lambda(b)]a = 0$. This relation gives that $d\beta^{-1}\mu(b) = 0$ or $[a, \lambda(b)]a = 0$.

If $(h(I), b)_{\lambda,\mu}a = 0$ then considering as above and using the relation (2) we get the same result. □

Theorem 2.1. *Let U be a nonzero right $(\sigma, \tau)-$Lie ideal of R and $a, b \in R$.*

(i) If $a[U, b]_{\lambda,\mu} = 0$ (or $a(U, b)_{\lambda,\mu} = 0$) then $a[a, \mu(b)] = 0$ or $U \subset C_{\sigma,\tau}(R)$.
(ii) If $[U, b]_{\lambda,\mu}a = 0$ (or $(U, b)_{\lambda,\mu}a = 0$) then $[a, \lambda(b)]a = 0$ or $U \subset C_{\sigma,\tau}(R)$.
(iii) If $(U, b)_{\lambda,\mu} \subset C_{\lambda,\mu}(R)$ then $b^2 \in Z$ or $U \subset C_{\sigma,\tau}(R)$.

Proof. Let u be an element of U. The mapping defined by $d(r)=[u,r]_{\sigma,\tau},\forall r\in R$ is a left(and right)-generalized $(\sigma,\tau)-$derivation associated with d. If $d=0$ then $u\in C_{\sigma,\tau}(R)$ is obtained. Let $d\neq 0$.

(i) If $a[U,b]_{\lambda,\mu}=0$ or $a(U,b)_{\lambda,\mu}=0$ then we have $a[[u,R]_{\sigma,\tau},b]_{\lambda,\mu}=0$ or $a([u,R]_{\sigma,\tau},b)_{\lambda,\mu}=0$. That is $a[d(R),b]_{\lambda,\mu}=0$ or $a(d(R),b)_{\lambda,\mu}=0$. This implies that $d\sigma^{-1}\lambda(b)=0$ or $a[a,\mu(b)]=0$ by Lemma 2.5. That is $[u,\sigma^{-1}\lambda(b)]_{\sigma,\tau}=0$ or $a[a,\mu(b)]=0$. If we consider this argument for all $u\in U$ we get

$$[U,\sigma^{-1}\lambda(b)]_{\sigma,\tau}=0 \text{ or } a[a,\mu(b)]=0.$$

If $[U,\sigma^{-1}\lambda(b)]_{\sigma,\tau}=0$ then we obtain that

$$b\in Z \text{ or } U\subset C_{\sigma,\tau}(R)$$

by Lemma 2.5. Finally we obtain that $a[a,\mu(b)]=0$ or $U\subset C_{\sigma,\tau}(R)$ for any cases.

(ii) If $[U,b]_{\lambda,\mu}a=0$ or $(U,b)_{\lambda,\mu}a=0$ then we have $[[u,R]_{\sigma,\tau},b]_{\lambda,\mu}a=0$ or $([u,R]_{\sigma,\tau},b)_{\lambda,\mu}a$. This means that $[d(R),b]_{\lambda,\mu}a=0$ or $(d(R),b)_{\lambda,\mu}a=0$. Using Lemma 2.6 we get $d\tau^{-1}\mu(b)=0$ or $[a,\lambda(b)]a=0$. That is $[u,\tau^{-1}\mu(b)]_{\sigma,\tau}=0$ or $[a,\lambda(b)]a=0$. Considering as above we get $[a,\lambda(b)]a=0$ or $U\subset C_{\sigma,\tau}(R)$.

(iii) If $(U,b)_{\lambda,\mu}\subset C_{\lambda,\mu}(R)$ then we have $[(U,b)_{\lambda,\mu},R]_{\lambda,\mu}=0$. This gives that, for all $u\in U$

$$\begin{aligned}0=[(u,b)_{\lambda,\mu},b]_{\lambda,\mu}&=[u\lambda(b)+\mu(b)u,b]_{\lambda,\mu}\\&=u\lambda(b)\lambda(b)+\mu(b)u\lambda(b)-\mu(b)u\lambda(b)-\mu(b)\mu(b)u\\&=u\lambda(b)\lambda(b)-\mu(b)\mu(b)u.\end{aligned}$$

That is $[U,b^2]_{\lambda,\mu}=0$. Using Lemma 2.2 we get $b^2\in Z$ or $U\subset C_{\sigma,\tau}(R)$. □

Lemma 2.7. *Let I be a nonzero ideal of R and $b\in R$.*

(i) If $b\in C_{\alpha,\beta}(I)$ then $b\in C_{\alpha,\beta}(R)$.
(ii) If $[b,I]_{\alpha,\beta}\subset C_{\lambda,\mu}(R)$ then $b\in C_{\alpha,\beta}(R)$ or R is commutative.

Proof. The mapping defined by $d(r)=[b,r]_{\alpha,\beta},\forall r\in R$ is a $(\alpha,\beta)-$derivation and so left-generalized $(\alpha,\beta)-$derivation associated with d.

(i) If $b\in C_{\alpha,\beta}(I)$ then $[b,I]_{\alpha,\beta}=0$ and so $d(I)=0$ is obtained. This gives that $d=0$ by Lemma 2.1. That is $b\in C_{\alpha,\beta}(R)$.

(ii) If $[b,I]_{\alpha,\beta}\subset C_{\lambda,\mu}(R)$ then we have $d(I)\subset C_{\lambda,\mu}(R)$. Using that d is a left-generalized $(\alpha,\beta)-$derivation then we have R is commutative by Lemma 2.3. Finally we obtain that $b\in C_{\alpha,\beta}(R)$ or R is commutative for any cases. □

Corollary 2.1. *Let U be a nonzero right $(\sigma,\tau)-$Lie ideal of R and I a nonzero ideal of R.*

(i) If $U\subset C_{\alpha,\beta}(I)$ then $U\subset C_{\alpha,\beta}(R)$.
(ii) If $[U,I]_{\alpha,\beta}\subset C_{\lambda,\mu}(R)$ then $U\subset C_{\alpha,\beta}(R)$ or R is commutative.

Proof. (i) If $U\subset C_{\alpha,\beta}(I)$ then we have $U\subset C_{\alpha,\beta}(R)$ by Lemma 2.7 (i).

(ii) If $[U,I]_{\alpha,\beta}\subset C_{\lambda,\mu}(R)$ then we have $U\subset C_{\alpha,\beta}(R)$ or R is commutative by Lemma 2.7 (ii). □

Theorem 2.2. *Let $d:R\longrightarrow R$ be a nonzero $(\alpha,\beta)-$ derivation and $b\in R$. Let U be a nonzero right $(\sigma,\tau)-$Lie ideal of R.*

(i) If $d(U)=0$ then $U\subset Z$ or $U\subset C_{\sigma,\tau}(R)$.
(ii) If $b\in C_{\lambda,\mu}(U)$ then $b\in C_{\lambda,\mu}(R)$ or $U\subset Z$ or $U\subset C_{\sigma,\tau}(R)$.

Proof. (i) If $d(U) = 0$ then we have, for all $v \in U,\ r, s \in R$

$$\begin{aligned} 0 = d[v, rs]_{\sigma,\tau} &= d(\tau(r)[v, s]_{\sigma,\tau} + [v, r]_{\sigma,\tau}\sigma(s)) \\ &= d\tau(r)\alpha[v, s]_{\sigma,\tau} + \beta\tau(r)d[v, s]_{\sigma,\tau} + d[v, r]_{\sigma,\tau}\alpha\sigma(s) + \beta[v, r]_{\sigma,\tau}d\sigma(s) \\ &= d\tau(r)\alpha[v, s]_{\sigma,\tau} + \beta[v, r]_{\sigma,\tau}d\sigma(s). \end{aligned}$$

That is

$$d\tau(r)\alpha[v, s]_{\sigma,\tau} + \beta[v, r]_{\sigma,\tau}d\sigma(s) = 0 \text{ for all } v \in U, r, s \in R. \tag{4}$$

Replacing s by $\sigma^{-1}[u, s]_{\sigma,\tau}, u \in U$ in (4) and using hypothesis we get

$$d(R)\alpha[U, \sigma^{-1}[U, R]_{\sigma,\tau}]_{\sigma,\tau} = 0.$$

Since $d \neq 0$ then using [1, Lemma 3] we obtain $[U, \sigma^{-1}[U, R]_{\sigma,\tau}]_{\sigma,\tau} = 0$. This gives that $[U, R]_{\sigma,\tau} \subset Z$ or $U \subset C_{\sigma,\tau}(R)$ by Lemma 2.2.

If $[U, R]_{\sigma,\tau} \subset Z$ then we have $U \subset C_{\sigma,\tau}(R)$ or R is commutative by Lemma 2.7 (ii). Finally we obtain that $U \subset Z$ or $U \subset C_{\sigma,\tau}(R)$.

(ii) The mapping defined by $d(r) = [b, r]_{\lambda,\mu}, \forall r \in R$ is a $(\lambda, \mu)-$ derivation. If $d = 0$ then $b \in C_{\lambda,\mu}(R)$ is obtained. Let $d \neq 0$.

If $b \in C_{\lambda,\mu}(U)$ then we have $[b, U]_{\lambda,\mu} = 0$. That is $d(U) = 0$. This means that $U \subset Z$ or $U \subset C_{\sigma,\tau}(R)$ by (i). Finally we obtain that $b \in C_{\lambda,\mu}(R)$ or $U \subset Z$ or $U \subset C_{\sigma,\tau}(R)$. □

Corollary 2.2. *Let U be a nonzero right $(\sigma, \tau)-$Lie ideal of R and $a, b \in R$. If $b, ba \in C_{\lambda,\mu}(U)$ or $b, ab \in C_{\lambda,\mu}(U)$ then $b = 0$ or $a \in Z$ or $U \subset C_{\sigma,\tau}(R)$ or $U \subset Z$.*

Proof. If $b, ba \in C_{\lambda,\mu}(U)$ then using Theorem 2.2 (ii) we get, for all $v \in V$

$$\{(U \subset C_{\sigma,\tau}(R) \text{ or } U \subset Z) \text{ or } b \in C_{\lambda,\mu}(R)\} \text{ and } \{(U \subset C_{\sigma,\tau}(R) \text{ or } U \subset Z) \text{ or } ba \in C_{\lambda,\mu}(R)\}.$$

This means that

$$(U \subset C_{\sigma,\tau}(R) \text{ or } U \subset Z) \text{ or } \{b \in C_{\lambda,\mu}(R) \text{ and } ba \in C_{\lambda,\mu}(R)\}$$

If $\{b \in C_{\lambda,\mu}(R) \text{ and } ba \in C_{\lambda,\mu}(R)\}$ then we have $b = 0$ or $a \in Z$ by Lemma 2.4. Finally we obtain that $b = 0$ or $a \in Z$ or $U \subset C_{\sigma,\tau}(R)$ or $U \subset Z$ for any cases.

If $b, ab \in C_{\lambda,\mu}(U)$ then, considering as above we get the same result. □

Lemma 2.8. *Let I be a nonzero ideal of R and $h : R \longrightarrow R$ a nonzero right-generalized $(\sigma, \tau)-$ derivation associated with a nonzero $(\sigma, \tau)-$derivation d. If $b \in R$ such that $[h(I), b]_{\lambda,\mu} = 0$ then $b \in Z$ or $d\sigma^{-1}\lambda(b) = 0$.*

Proof. Using hypothesis we get for all $x \in I$

$$\begin{aligned} 0 = [h(x\sigma^{-1}\lambda(b)), b]_{\lambda,\mu} &= [h(x)\lambda(b) + \mu(x)d\sigma^{-1}\lambda(b), b]_{\lambda,\mu} \\ &= h(x)[\lambda(b), \lambda(b)] + [h(x), b]_{\lambda,\mu}\ \lambda(b) + \mu(x)[d\sigma^{-1}\lambda(b), b]_{\lambda,\mu} + [\mu(x), \mu(b)]d\sigma^{-1}\lambda(b) \\ &= \mu(x)[d\sigma^{-1}\lambda(b), b]_{\lambda,\mu} + [\mu(x), \mu(b)]d\sigma^{-1}\lambda(b). \end{aligned}$$

That is

$$\mu(x)[d\sigma^{-1}\lambda(b), b]_{\lambda,\mu} + [\mu(x), \mu(b)]d\sigma^{-1}\lambda(b) = 0 \text{ for all } x \in I. \tag{5}$$

Replacing x by $rx, r \in R$ in (5) and using (5) we get for all $x \in I,\ r \in R$

$$\begin{aligned} 0 &= \mu(r)\mu(x)[d\sigma^{-1}\lambda(b), b]_{\lambda,\mu} + \mu(r)[\mu(x), \mu(b)]d\sigma^{-1}\lambda(b)\ + [\mu(r), \mu(b)]\mu(x)d\sigma^{-1}\lambda(b) \\ &= [\mu(r), \mu(b)]\mu(x)d\sigma^{-1}\lambda(b) \end{aligned}$$

which gives

$$[R, \mu(b)]\mu(I)d\sigma^{-1}\lambda(b) = 0.$$

Since $\mu(I)$ is a nonzero ideal and R is prime then we have $b \in Z$ or $d\sigma^{-1}\lambda(b) = 0$. □

Corollary 2.3. *Let $h : R \longrightarrow R$ be a nonzero right-generalized $(\sigma,\tau)-$ derivation associated with a nonzero $(\sigma,\tau)-$derivation d and I, J nonzero ideals of R. If $h(I) \subset C_{\lambda,\mu}(J)$ then R is commutative.*

Proof. If $h(I) \subset C_{\lambda,\mu}(J)$ then we have $[h(I), y]_{\lambda,\mu} = 0$ for all $y \in J$. This gives that, for any $y \in J$,

$$y \in Z \text{ or } d\sigma^{-1}\lambda(y) = 0$$

by Lemma 2.8. Then J is the union of its additive subgroups $K = \{y \in J \mid y \in Z\}$ and $L = \{y \in J \mid d\sigma^{-1}\lambda(y) = 0\}$. Since a group can not be the union of two of its proper subgroups, we have $J = K$ or $J = L$. Since $\sigma^{-1}\lambda(J)$ is a nonzero ideal of R then $d\sigma^{-1}\lambda(J) \neq 0$ by Lemma 2.1. Hence we have $J = K$ and so $J \subset Z$. This means that R is commutative by [9]. □

Theorem 2.3. *Let U be a nonzero right $(\sigma,\tau)-$Lie ideal of R and I a nonzero ideal of R. Let $h : R \longrightarrow R$ be a nonzero right-generalized $(\alpha,\beta)-$derivation associated with nonzero $(\alpha,\beta)-$ derivation d and $b \in R$.*

(i) If $h(I) \subset C_{\lambda,\mu}(U)$ then $U \subset Z$ or $U \subset C_{\sigma,\tau}(R)$.
(ii) If $bh(I) \subset C_{\lambda,\mu}(U)$ then $b \in Z$ or $U \subset Z$ or $U \subset C_{\sigma,\tau}(R)$.

Proof. (i) If $h(I) \subset C_{\lambda,\mu}(U)$ then we have $h(I) \subset C_{\lambda,\mu}(R)$ or $U \subset Z$ or $U \subset C_{\sigma,\tau}(R)$ by Theorem 2.2 (ii).

If $h(I) \subset C_{\lambda,\mu}(R)$ then we get R is commutative by Corollary 2.3 and so $U \subset Z$.

(ii) If $bh(I) \subset C_{\lambda,\mu}(U)$ then using Theorem 2.2 (ii) we get $bh(I) \subset C_{\lambda,\mu}(R)$ or $U \subset Z$ or $U \subset C_{\sigma,\tau}(R)$.

If $bh(I) \subset C_{\lambda,\mu}(R)$ then we have $b \in Z$ by [4]. Finally, we obtain that $b \in Z$ or $U \subset Z$ or $U \subset C_{\sigma,\tau}(R)$. □

Theorem 2.4. *Let U be a nonzero right $(\sigma,\tau)-$Lie ideal of R and I a nonzero ideal of R. Let $h : R \longrightarrow R$ be a nonzero left-generalized $(\alpha,\beta)-$derivation associated with nonzero $(\alpha,\beta)-$ derivation d and $b \in R$.*

(i) If $h(I) \subset C_{\lambda,\mu}(U)$ then $U \subset Z$ or $U \subset C_{\sigma,\tau}(R)$.
(ii) If $h(I)b \subset C_{\lambda,\mu}(U)$ then $b \in Z$ or $U \subset Z$ or $U \subset C_{\sigma,\tau}(R)$.

Proof. (i) If $h(I) \subset C_{\lambda,\mu}(U)$ then using Theorem 2.2 (ii) we get $h(I) \subset C_{\lambda,\mu}(R)$ or $U \subset Z$ or $U \subset C_{\sigma,\tau}(R)$.

If $h(I) \subset C_{\lambda,\mu}(R)$ then we get R is commutative by Lemma 2.3 and so $U \subset Z$.

(ii) If $h(I)b \subset C_{\lambda,\mu}(U)$ then we have $h(I)b \subset C_{\lambda,\mu}(R)$ or $U \subset Z$ or $U \subset C_{\sigma,\tau}(R)$ by Theorem 2.2 (ii).

If $h(I)b \subset C_{\lambda,\mu}(R)$ then using [5] we get $b \in Z$. Finally, we obtain that $b \in Z$ or $U \subset Z$ or $U \subset C_{\sigma,\tau}(R)$. □

Remark 2.1. *Let J be a nonzero ideal of R. If $J \subset C_{\lambda,\mu}(R)$ then $J \subset Z$.*

Proof. If $J \subset C_{\lambda,\mu}(R)$ then we have $[J, R]_{\lambda,\mu} = 0$ and so

$$0 = [xy, r]_{\lambda,\mu} = x[y, r]_{\lambda,\mu} + [x, \mu(r)]y = [x, \mu(r)]y \text{ for all } x, y \in J, r \in R.$$

That is $[J, R]J = 0$. This gives that $J \subset Z$ in prime rings. □

Theorem 2.5. *Let U, V be nonzero right $(\sigma,\tau)-$Lie ideals of R and $a, b \in R$. Let I be a nonzero ideal of R.*

(i) If $[a, I]_{\alpha,\beta} \subset C_{\lambda,\mu}(U)$ then $a \in C_{\alpha,\beta}(R)$ or $U \subset C_{\sigma,\tau}(R)$ or $U \subset Z$.

(ii) If $b[a, I]_{\alpha,\beta} \subset C_{\lambda,\mu}(U)$ or $[a, I]_{\alpha,\beta}b \subset C_{\lambda,\mu}(U)$ then $a \in C_{\alpha,\beta}(R)$ or $b \in Z$ or $U \subset C_{\sigma,\tau}(R)$ or $U \subset Z$.

(iii) If $bV \subset C_{\lambda,\mu}(U)$ or $Vb \subset C_{\lambda,\mu}(U)$ then $b \in Z$ or $V \subset C_{\sigma,\tau}(R)$ or $U \subset C_{\sigma,\tau}(R)$ or $U \subset Z$.

Proof. The mapping defined by $d(r) = [a, r]_{\alpha,\beta}$ for all $r \in R$ is an $(\alpha, \beta)-$derivation and so right (and left)-generalized $(\alpha, \beta)-$derivation associated with d. If $d = 0$ then $a \in C_{\alpha,\beta}(R)$ is obtained. Let $d \neq 0$.

(i) If $[a, I]_{\alpha,\beta} \subset C_{\lambda,\mu}(U)$ then we have $d(I) \subset C_{\lambda,\mu}(U)$. This means that $U \subset Z$ or $U \subset C_{\sigma,\tau}(R)$ by Theorem 2.3 (i). Finally we obtain that $a \in C_{\alpha,\beta}(R)$ or $U \subset C_{\sigma,\tau}(R)$ or $U \subset Z$ for any cases.

(ii) If $b[a, I]_{\alpha,\beta} \subset C_{\lambda,\mu}(U)$ or $[a, I]_{\alpha,\beta}b \subset C_{\lambda,\mu}(U)$ then we have $bd(I) \subset C_{\lambda,\mu}(U)$ or $d(I)b \subset C_{\lambda,\mu}(U)$. Using Theorem 2.3 (ii) and Theorem 2.4 (ii) we get $b \in Z$ or $U \subset Z$ or $U \subset C_{\sigma,\tau}(R)$. Finally we obtain that $b \in Z$ or $a \in C_{\alpha,\beta}(R)$ or $U \subset C_{\sigma,\tau}(R)$ or $U \subset Z$ for any cases.

(iii) If $bV \subset C_{\lambda,\mu}(U)$ or $Vb \subset C_{\lambda,\mu}(U)$ then we have $b[V, R]_{\sigma,\tau} \subset C_{\lambda,\mu}(U)$ or $[V, R]_{\sigma,\tau}b \subset C_{\lambda,\mu}(U)$. Using Theorem 2.5 (ii) we get $V \subset C_{\sigma,\tau}(R)$ or $b \in Z$ or $U \subset C_{\sigma,\tau}(R)$ or $U \subset Z$. □

Corollary 2.4. *Let V be nonzero right $(\sigma, \tau)-$Lie ideals of R and $b \in R$. Let I be a nonzero ideal of R. If $bV \subset C_{\lambda,\mu}(I)$ or $Vb \subset C_{\lambda,\mu}(I)$ then $b \in Z$ or $V \subset C_{\sigma,\tau}(R)$.*

Proof. Every ideal I of R is a right $(1, 1)-$Lie ideal of R. If $bV \subset C_{\lambda,\mu}(I)$ or $Vb \subset C_{\lambda,\mu}(I)$ then we have $b \in Z$ or $V \subset C_{\sigma,\tau}(R)$ or $I \subset C_{\sigma,\tau}(R)$ or $I \subset Z$ by Theorem 2.5 (iii).

If $I \subset C_{\sigma,\tau}(R)$ then we have $I \subset Z$ by Remark 2.1. On the other hand $J \subset Z$ means that R is commutative by [9]. Finally we obtain that $b \in Z$ or $V \subset C_{\sigma,\tau}(R)$ for any cases. □

References

[1] Aydın, N. and Kaya,K., (1992), Some Generalizations in Prime Rings with $(\sigma, \tau)-$Derivation, Doga-Tr. J. of Math., 16.

[2] Bresar, M., (1991), On the Distance of the Composition of two Derivation to Generalized Derivations, Glasgow Math. J, 33.

[3] Chang, J. C., (2003), On the identity $h(x) = af(x) + g(x)b$, Taiwanese J. Math., pp. 7-1.

[4] Güven, E., (2018), One Sided $(\sigma, \tau)-$ Lie Ideals and Generalized Derivations in Prime Rings, Palestine Journal of Mathematics, Vol. 7(2).

[5] Güven E., One Sided Generalized $(\sigma, \tau)-$Derivations on Rings, Bol. Soc. Paran. Mat., to appear.

[6] Güven, E., Kaya, K. and Soytürk, M., (2007), Some Results on $(\sigma, \tau)-$Lie Ideals, Math. J. Okayama Univ., 49.

[7] Kaya, K., (1988), On Prime Rings with $(\sigma, \tau)-$Derivations, Doga, TU. MAT. D. C., 12-2.

[8] Kaya, K., (1992), $(\sigma, \tau)-$Lie ideals in Prime Rings, An. Univ. Timisoara Stiinte Mat., 30, pp. 2-3.

[9] Mayne, J. H., (1984), Centralizing Mappings of Prime Rings, Canad. Math. Bull.,Vol. 27(1).

Evrim Güven graduated from the Faculty of Arts and Science, Eskişehir Osmangazi University, Eskişehir (1998). She received her M.Sc. (2001) and Ph.D. (2007) degrees from Kocaeli University, Kocaeli. She has been working as an assistant professor in the Faculty of Arts and Science, Kocaeli University. Her research area includes Ring Theory and Applications.