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AN APPROACH TO BIPOLAR FUZZY SUBMODULES

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Abstract. We introduce the notion of bipolar fuzzy submodule of a given classical module and study fundamental properties and characterizations.

Keywords: Bipolar valued fuzzy set, Bipolar fuzzy subgroup (resp. subring), Bipolar fuzzy submodule.

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1. Introduction

In 1965, Zadeh [11] proposed the concept of fuzzy set theory. There are several extensions of fuzzy set theory, for example, intuitionistic fuzzy sets, interval- valued fuzzy sets, neutrosophic sets, etc. In fuzzy sets, the membership degree of element range on [0,1]. In 2000, Lee [5] defined bipolar-valued fuzzy set as an extension of fuzzy set. In this set theory interval of membership value is [-1,1]. The bipolar valued fuzzy set have positive and negative memberships. The membership degree 0 means that elements are not satisfying the specific property, the membership degrees on (0,1] indicate that elements somewhat satisfy the property and the membership degrees on [-1,0) indicate that elements satisfying implicit counter property. At present, studies on bipolar valued fuzzy set and its applications are progressing rapidly. In 2009, K. J. Lee [7] applied the concept of bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI algebras. In 2013, M. S. Anitha et. al [1] introduced the notion of bipolar valued fuzzy subgroup and studied some properties. In 2018, S.P. Subbian et. al. [10] worked on bipolar valued fuzzy ideals of ring. The topological structure of bipolar valued fuzzy set was introduced by M. Azhagappan and M. Kamaraj [2] in 2016. Then, in 2019, J. H. Kim et. al. [4] defined the concepts of bipolar fuzzy base, subbase and neighborhood structure.

In this paper, we have initiated the concept of bipolar fuzzy submodule of a given classical module and study some basic properties.

2. Preliminaries

In this section, we give some definitions and several results on bipolar valued fuzzy set theory.

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Definition 2.1 [5] Let X be a non-empty set. A bipolar- valued fuzzy set A on X is an object having the form $A = \{< x, \mu_A^+(x), \mu_A^-(x) >: x \in X\}$ where $\mu_A^+ : X \to [0,1]$ and $\mu_A^- : X \to [-1,0]$ are mappings. The positive membership degree $\mu_A^+(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar valued fuzzy set $A = \{< x, \mu_A^+(x), \mu_A^-(x) >: x \in X\}$ and the negative membership degree $\mu_A^-(x)$ denotes the satisfaction degree of x to some implicit counter property of bipolar valued fuzzy set $A = \{< x, \mu_A^+(x), \mu_A^-(x) >: x \in X\}$.

If $\mu_A^+(x) \neq 0$ and $\mu_A^-(x) = 0$, it is the situation that x is regarded as having only positive satisfaction $A = \{< x, \mu_A^+(x), \mu_A^-(x) >: x \in X\}$.

If $\mu_A^+(x) = 0$ and $\mu_A^-(x) \neq 0$, it is the situation that x does not satisfy property of $A = \{< x, \mu_A^+(x), \mu_A^-(x) >: x \in X\}$ but somewhat satisfies the counter property of $A = \{< x, \mu_A^+(x), \mu_A^-(x) >: x \in X\}$.

It is possible for element x to be such that $\mu_A^+(x) \neq 0$ and $\mu_A^-(x) \neq 0$ when the membership function of the property overlaps that of its counter property over some portion of X.

Example 2.2 Let $X = \{a, b, c\}$. $A = \{< a, 0.4, -0.2 >, < b, 0.6, -0.1 >, < c, 0.3, -0.3 > \}$ is a bipolar valued fuzzy set of X.

Definition 2.3[2] The empty bipolar valued fuzzy set, denoted by $0_{bp} = (0_{bp}^+, 0_{bp}^-)$, is a bipolar valued fuzzy set in X defined by $0_{bp}^+(x) = 0 = 0_{bp}^-(x)$, for each $x \in X$.

The whole bipolar valued fuzzy set, denoted by $1_{bp} = (1_{bp}^+, 1_{bp}^-)$, is a bipolar valued fuzzy set in X defined by $1_{bp}^+(x) = 1$ and $1_{bp}^-(x) = -1$, for each $x \in X$.

Definition 2.4 [6] Let A and B be two bipolar- valued fuzzy sets of X. Then

(1) $A \subseteq B$ if and only if $\mu_A^+(x) \leq \mu_B^+(x)$ and $\mu_A^-(x) \geq \mu_A^+(x)$, for all $x \in X$.

(2) $A = B$ if and only if $\mu_A^+(x) = \mu_B^+(x)$ and $\mu_A^-(x) = \mu_B^-(x)$, for all $x \in X$.

(3) $A \cap B = \{< x, \mu_{A\cap B}^+(x), \mu_{A\cap B}^-(x) >: x \in X\}$, where $\mu_{A\cup B}^+(x) = min\{\mu_A^+(x), \mu_B^+(x)\}$ and $\mu_{A\cap B}^-(x) = max\{\mu_A^-(x), \mu_B^-(x)\}$

(4) $A \cup B = \{< x, \mu_{A\cup B}^+(x), \mu_{A\cup B}^-(x) >: x \in X\}$, where $\mu_{A\cup B}^+(x) = max\{\mu_A^+(x), \mu_B^+(x)\}$ and $\mu_{A\cup B}^-(x) = min\{\mu_A^-(x), \mu_B^-(x)\}$

(5) $A^c = \{< x, 1 - \mu_A^+(x), -1 - \mu_A^-(x) >: x \in X\}$

Proposition 2.1. [4] *Let A, B and C be bipolar valued fuzzy sets on the common universe X. Then we have followings:*

(1) $A \cup B = B \cup A, A \cap B = B \cap A$.

(2) $A \cup (B \cup C) = (A \cup B) \cup C, A \cap (B \cap C) = (A \cap B) \cap C$.

(3) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C), A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(4) $A \cap B \subset A$ *and* $A \cap B \subset B$

(5) $A \subset A \cup B$ *and* $B \subset A \cup B$

(6) $(A \cup B)^c = A^c \cap B^c, (A \cap B)^c = A^c \cup B^c$.

Definition 2.5 [4] Let $g : X \to Y$ be a function and A, B be the bipolar valued fuzzy sets on X and Y, respectively. The image of a bipolar valued fuzzy set A is a bipolar valued fuzzy set on Y and it is defined as by

$g(A)(y) = (\mu_{g(A)}^+(y), \mu_{g(A)}^-(y)) = (g(\mu_A^+)(y), g(\mu_A^-)(y)), \forall y \in Y$

where

$$g(\mu_A^+)(y) = \begin{cases} \bigvee \mu_A^+(x), & \text{if } x \in g^{-1}(y); \\ 0, & \text{otherwise} \end{cases},$$

$$g(\mu_A^-)(y) = \begin{cases} \bigwedge \mu_A^-(x), & \text{if } x \in g^{-1}(y); \\ 0, & \text{otherwise.} \end{cases}$$

The preimage of a bipolar fuzzy set B is a bipolar valued fuzzy set on X and it is defined by

$g^{-1}(B)(x) = (\mu^+_{g^{-1}(B)}(x), \mu^-_{g^{-1}(B)}(x)) = (\mu^+_B(g(x)), \mu^-_B(g(x))), \forall x \in X.$

Definition 2.6 [1] A bipolar valued fuzzy set $A = \{< x, \mu^+_A(x), \mu^-_A(x) >: x \in X\}$ of classical group G is called bipolar fuzzy subgroup of G if

(i) $\mu^+_A(xy) \geq \mu^+_A(x) \wedge \mu^+_A(y)$ and $\mu^-_A(xy) \leq \mu^-_A(x) \vee \mu^-_A(y)$

(ii) $\mu^+_A(x^{-1}) \geq \mu^+_A(x)$ and $\mu^-_A(x^{-1}) \leq \mu^-_A(x)$

for all $x, y \in G$.

Definition 2.7 [10] A bipolar valued fuzzy set $A = \{< x, \mu^+_A(x), \mu^-_A(x) >: x \in X\}$ of classical ring R is called bipolar fuzzy subring of R if

(i) $\mu^+_A(x+y) \geq \mu^+_A(x) \wedge \mu^+_A(y)$ and $\mu^-_A(x+y) \leq \mu^-_A(x) \vee \mu^-_A(y)$

(ii) $\mu^+_A(-x) \geq \mu^+_A(x)$ and $\mu^-_A(-x) \leq \mu^-_A(x)$

(iii) $\mu^+_A(xy) \geq \mu^+_A(x) \wedge \mu^+_A(y)$ and $\mu^-_A(xy) \leq \mu^-_A(x) \vee \mu^-_A(y)$

for all $x, y \in R$.

3. Bipolar fuzzy submodules

In this section, we introduce the concept of bipolar fuzzy submodule of a given classical module over a ring and also investigate its elementary properties. Throughout this paper, R denotes a commutative ring with unity 1.

Definition 3.1 Let M be a module over a ring R. A bipolar valued fuzzy set A on M is called a bipolar fuzzy submodule of M if

(M1)$A(0) = \widetilde{X}$, i.e.,

$\mu^+_A(0) = 1,\ \mu^-_A(0) = -1.$

(M2)$A(x+y) \geq A(x) \wedge A(y)$, for each $x, y \in M$ i.e.,

$\mu^+_A(x+y) \geq \mu^+_A(x) \wedge \mu^+_A(y)$ and $\mu^-_A(x+y) \leq \mu^-_A(x) \wedge \mu^-_A(y)$

(M3)$A(rx) \geq A(x)$, for each $x \in M,\ r \in R$, i.e.,

$\mu^+_A(rx) \geq \mu^+_A(x)$ and $\mu^-_A(rx) \leq \mu^-_A(x)$.

The collection of all bipolar fuzzy submodules of M is denoted by $BFM(M)$.

Example 3.2 Let $R = \mathbb{Z}_4 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$. Let consider $M = \mathbb{Z}_4$ as a classical module. Define the bipolar valued fuzzy set A by

$A = \{< 1, -1 > /\overline{0} + < 0.6, -0.6 > /\overline{1} + < 0.8, -0.4 > /\overline{2} + < 0.6, -0.6 > /\overline{3}\}.$

Hence the bipolar valued fuzzy set A is a bipolar fuzzy submodule of the module M.

Definition 3.3 Let A and B be bipolar valued fuzzy sets on M. Then we define their sum $A + B$ as the bipolar valued fuzzy set on M by

$\mu^+_{A+B}(x) = \vee\{\mu^+_A(y) \wedge \mu^+_B(z) \mid x = y + z,\ y, z \in M\},$

and

$\mu^-_{A+B}(x) = \wedge\{\mu^-_A(y) \vee \mu^-_B(z) \mid x = y + z,\ y, z \in M\}.$

Definition 3.4 Let A be a bipolar valued fuzzy set on M, then $-A$ is a bipolar valued fuzzy set on M, defined by

$\mu^+_{-A}(x) = \mu^+_A(-x)$ and $\mu^-_{-A}(x) = \mu^-_A(-x)$, for each $x \in M$.

Definition 3.5 Let A be a bipolar valued fuzzy set on M and $r \in R$. Define bipolar valued fuzzy set rA on M by

$\mu^+_{rA}(x) = \vee\{\mu^+_A(y) \mid y \in M,\ x = ry\}$ and $\mu^-_{rA}(x) = \wedge\{\mu^-_A(y) \mid y \in M,\ x = ry\}$.

Proposition 3.1. *If A is a bipolar valued fuzzy submodule of an R-module M, then $1.A = A$ and $(-1)A = -A$.*

Proof. Let $x \in M$.

$\mu^+_{(-1)A}(x) = \bigvee\{\mu^+_A(y) : y \in M,\ x = (-1)y\} = \bigvee\{\mu^+_A(y) : y \in M,\ y = -x\} = \mu^+_A(-x) = \mu^+_{-A}(x)$

Similarly $\mu^-_{(-1)A}(x) = \mu^-_{-A}(x)$, for all $x \in M$.

We have $(-1)A = -A$. □

Proposition 3.2. *If A is a bipolar valued fuzzy set on M, then $r(sA) = (rs)A$, for each $r, s \in R$.*

Proof. Let $x \in M$ and $r, s \in R$.

$$\mu^-_{r(sA)}(x) = \bigwedge_{x=ry} \mu^-_{sA}(y) = \bigwedge_{x=ry}\bigwedge_{y=sz} \mu^-_A(z) = \bigwedge_{x=r(sz)} \mu^-_A(z) = \bigwedge_{x=(rs)z} \mu^-_A(z) = \mu^-_{(rs)A}(x).$$

Similarly we get the other equality, so $r(sA) = (rs)A$. □

Proposition 3.3. *If A and B are bipolar valued fuzzy sets on M, then $r(A+B) = rA+rB$, for each $r \in R$.*

Proof. Let A and B are bipolar valued fuzzy sets on M, $x \in M$ and $r \in R$.

$$\begin{aligned}
\mu^+_{r(A+B)}(x) &= \bigvee_{x=ry} \mu^+_{A+B}(y) \\
&= \bigvee_{x=ry}\bigvee_{y=y_1+y_2} (\mu^+_A(y_1) \wedge \mu^+_B(y_2)) \\
&= \bigvee_{x=ry_1+ry_2} (\mu^+_A(y_1) \wedge \mu^+_B(y_2)) \\
&= \bigvee_{x=x_1+x_2} ((\bigvee_{x_1=ry_1} \mu^+_A(y_1)) \wedge (\bigvee_{x_2=ry_2} \mu^+_B(y_2))) \\
&= \bigvee_{x=x_1+x_2} (\mu^+_{rA}(x_1) \wedge \mu^+_{rB}(x_2)) = \mu^+_{rA+rB}(x).
\end{aligned}$$

Similarly, we show that $\mu^-_{r(A+B)}(x) = \mu^-_{rA+rB}(x)$, $\forall x \in M$.

So, $r(A+B) = rA + rB$. □

Proposition 3.4. *If A is a bipolar valued fuzzy set on M, then $\mu^+_{rA}(rx) \geq \mu^+_A(x)$ and $\mu^-_{rA}(rx) \leq \mu^-_A(x)$.*

Proof. Straightforward. □

Proposition 3.5. *Let A and B are bipolar valued fuzzy sets on M. Then we obtain followings:*

(1) $\mu^+_B(rx) \geq \mu^+_A(x)$, $\forall x \in M \Leftrightarrow \mu^+_{rA} \leq \mu^+_B$.

(2) $\mu^-_B(rx) \leq \mu^-_A(x)$, $\forall x \in M$, $\Leftrightarrow \mu^-_{rA} \geq \mu^-_B$.

Proof. (1) Let $\mu^+_B(rx) \geq \mu^+_A(x)$, for each $x \in M$, then $\mu^+_{rA}(x) = \bigvee_{x=ry, y\in M} \mu^+_A(y)$. Hence, $\mu^+_{rA} \leq \mu^+_B$.

Conversely, let $\mu^+_{rA} \leq \mu^+_B$. Then $\mu^+_{rA}(x) \leq \mu^+_B(x)$, for each $x \in M$. By Proposition 3.4 we have $\mu^+_B(rx) \geq \mu^+_{rA}(rx) \geq \mu^+_A(x)$, for each $x \in M$.

(2) Straightforward. □

Proposition 3.6. *Let A and B are bipolar valued fuzzy sets on M, then have followings:*

(1) $\mu^+_{rA+sB}(rx + sy) \geq \mu^+_A(x) \wedge \mu^+_B(y)$,

(2) $\mu^-_{rA+sB}(rx + sy) \leq \mu^-_A(x) \vee \mu^-_B(y)$, $\forall x, y \in M, r, s \in R$.

Proof. Straightforward. □

Proposition 3.7. *Let A be a bipolar valued fuzzy set on M and $r, s \in R$. Then*
(1) $\mu^+_{rA} \leq \mu^+_A \Leftrightarrow \mu^+_A(rx) \geq \mu^+_A(x)$ *and* $\mu^-_{rA} \geq \mu^-_A \Leftrightarrow \mu^-_A(rx) \leq \mu^-_A(x)$, $\forall x \in M$.
(2) $\mu^+_{rA+sA} \leq \mu^+_A \Leftrightarrow \mu^+_A(rx+sy) \geq \mu^+_A(x) \wedge \mu^+_A(y)$ *and* $\mu^-_{rA+sA} \geq \mu^-_A \Leftrightarrow \mu^-_A(rx+sy) \leq \mu^-_A(x) \vee \mu^-_A(y)$.

Proof. Straightforward. □

Theorem 3.1. *Let A be a bipolar valued fuzzy set on M. Then A is a bipolar fuzzy submodule of M iff*
(i) $\mu^+_A(0) = 1$, $\mu^-_A(0) = -1$
(ii) $\mu^+_A(rx+sy) \geq \mu^+_A(x) \wedge \mu^+_A(y)$ *and* $\mu^-_A(rx+sy) \leq \mu^-_A(x) \vee \mu^-_A(y)$, *for each* $x, y \in M$, $r, s \in R$.

Proof. Let A be a bipolar fuzzy submodule of M and $x, y \in M$. Since $A \in BFM(M)$, we have (i). By (M2) and (M3), we have followings,
$\mu^+_A(rx+sy) \geq \mu^+_A(rx) \wedge \mu^+_A(sy) \geq \mu^+_A(x) \wedge \mu^+_A(y)$,
and
$\mu^-_A(rx+sy) \leq \mu^-_A(rx) \vee \mu^-_A(sy) \leq \mu^-_A(x) \vee \mu^-_A(y)$ for each $x, y \in M$, $r, s \in R$.
Conversely, let A satisfies (i) and (ii). So we have
$\mu^+(0) = 1$, $\mu^+(0) = -1$.
$\mu^+_A(x+y) = \mu^+_A(1.x+1.y) \geq \mu^+_A(x) \wedge \mu^+_A(y)$ and
$\mu^-_A(x+y) = \mu^-_A(1.x+1.y) \leq \mu^-_A(x) \vee \mu^-_A(y)$.
So, the condition (M2) is satisfied.
By the hypothesis,
$\mu^+_A(rx) = \mu^+_A(rx+r0) \geq \mu^+_A(x) \wedge \mu^+_A(0) = \mu^+_A(x)$ and $\mu^-_A(rx) = \mu^-_A(rx+r0) \leq \mu^-_A(x) \vee \mu^-_A(0) = \mu^-_A(x)$, for each $x, y \in M$, $r \in R$.
Hence, A is a bipolar fuzzy submodule of M. □

Theorem 3.2. *If A and B are bipolar fuzzy submodules of a classical module M, then the intersection $A \cap B$ is also a bipolar fuzzy submodule of M.*

Proof. Let $A, B \in BFM(M)$. It is enough to show that Theorem 3.1 is satisfied.
We have $\mu^+_A(0) = 1$, $\mu^-_A(0) = -1$ and $\mu^+_B(0) = 1$, $\mu^-_B(0) = -1$.
$\mu^+_{A\cap B}(0) = \mu^+_A(0) \wedge \mu^+_B(0) = 1$
$\mu^-_{A\cap B}(0) = \mu^-_A(0) \vee \mu^-_B(0) = -1$.
Let $x, y \in M$, $r, s \in R$.
$\mu^+_{A\cap B}(rx+sy) \geq \mu^+_{A\cap B}(x) \wedge \mu^+_{A\cap B}(y)$ and $\mu^-_{A\cap B}(rx+sy) \leq \mu^-_{A\cap B}(x) \vee \mu^-_{A\cap B}(y)$.

$$\begin{aligned}\mu^+_{A\cap B}(rx+sy) &= \mu^+_A(rx+sy) \wedge \mu^+_B(rx+sy)\\ &\geq (\mu^+_A(x) \wedge \mu^+_A(y)) \wedge (\mu^+_B(x) \wedge \mu^+_B(y))\\ &= (\mu^+_A(x) \wedge \mu^+_B(x)) \wedge (\mu^+_A(y) \wedge \mu^+_B(y)) = \mu^+_{A\cap B}(x) \wedge \mu^+_{A\cap B}(y).\end{aligned}$$

The other inequality is similarly obtained. So, $A \cap B \in BFM(M)$. □

Definition 3.6 [12] Let $\lambda \in [0,1], \beta \in [-1,0]$. Define the level sets of A:
$A^+_\lambda = \{x \in X : \mu^+_A(x) \geq \lambda\}$ is called positive λ-cut of A.
$A^-_\beta = \{x \in X : \mu^-_A(x) \leq \beta\}$ is called negative β- cut of A.
For all $\gamma \in [0,1]$, the set $A^+_\gamma \cap A^-_{-\gamma}$ is called the γ- cut of A.

Proposition 3.8. *Let M be a module over R. $A \in BFM(M)$ if and only if*
(i) for all $\lambda \in [0,1]$, $(A^+_\lambda \neq \emptyset)$ A^+_λ *is a classical submodule of* M
(ii) for all $\beta \in [-1,0]$, $(A^-_\beta \neq \emptyset)$ A^-_β *is a classical submodule of* M
where $A(0) = \widetilde{X}$.

Proof. Let $A \in NSM(M)$, $\lambda \in [0,1]$, $x, y \in A^+_\lambda$ and $r, s \in R$. We have $\mu^+_A(x) \geq \lambda$, $\mu^+_A(y) \geq \lambda$ and $\mu^+_A(x) \wedge \mu^+_A(y) \geq \lambda$. By Theorem 3.1, $\mu^+_A(rx + sy) \geq \mu^+_A(x) \wedge \mu^+_A(y) \geq \lambda$. So, we obtain $rx + sy \in A^+_\lambda$. Hence, A^+_λ is a classical submodule of M for each $\lambda \in [0,1]$.

Similarly, for $x, y \in A^-_\beta$ we obtain $rx + sy \in A^-_\beta$ for each $\beta \in [-1, 0]$.

Conversely, assume that (i) and (ii) are valid. Let $x, y \in M$, $\lambda = \mu^+_A(x) \wedge \mu^+_A(y)$. Then $\mu^+_A(x) \geq \lambda$ and $\mu^+_A(y) \geq \lambda$. Hence, $x, y \in A^+_\lambda$. Since A^+_λ is a classical submodule of M, we have $rx + sy \in A^+_\lambda$ for all $r, s \in R$. Then, $\mu^+_A(rx + sy) \geq \lambda = \mu^+_A(x) \wedge \mu^+_A(y)$.

Similarly let $x, y \in M$, $\beta = \mu^-_A(x) \vee \mu^-_A(y)$. Then $\mu^-_A(x) \leq \beta$ and $\mu^-_A(y) \leq \beta$. Hence, $x, y \in A^-_\beta$. Since A^-_β is a submodule of M, we have $rx + sy \in A^-_\beta$ for all $r, s \in R$.

□

Definition 3.7 [1] The cartesian product of A and B which is denoted by $A \times B$ is a bipolar valued fuzzy set on $X \times Y$ and it is defined as

$A \times B = \{< (x, y), \mu^+_{(A\times B)}(x, y), \mu^-_{(A\times B)}(x, y) >: x \in X, y \in Y\}$

where $\mu^+_{(A\times B)}(x, y) = \mu^+_A(x) \wedge \mu^+_B(y)$ and $\mu^-_{(A\times B)}(x, y) = \mu^-_A(x) \vee \mu^-_B(y)$, for all $x \in X, y \in Y$.

Proposition 3.9. *Let A and B be bipolar valued fuzzy sets on X and Y. Then the followings are satisfied:*

$(A \times B)^+_\lambda = A^+_\lambda \times B^+_\lambda$ *and* $(A \times B)^-_\beta = A^-_\beta \times B^-_\beta$.

Proof. Let $(x, y) \in (A \times B)^+_\lambda$. So,

$$\begin{aligned} \mu^+_{A\times B}(x, y) \geq \lambda &\Leftrightarrow \mu^+_A(x) \wedge \mu^+_B(y) \geq \lambda \\ &\Leftrightarrow \mu^+_A(x) \geq \lambda \textit{ and } \mu^+_B(y) \geq \lambda \\ &\Leftrightarrow (x, y) \in A^+_\lambda \times B^+_\lambda. \end{aligned}$$

Let $(x, y) \in (A \times B)^-_\beta$. Hence,

$$\begin{aligned} \mu^-_{A\times B}(x, y) \leq \beta &\Leftrightarrow \mu^-_A(x) \vee \mu^-_B(y) \leq \beta \\ &\Leftrightarrow \mu^-_A(x) \leq \beta,\ \mu^-_B(y) \leq \beta \\ &\Leftrightarrow (x, y) \in A^-_\beta \times B^-_\beta. \end{aligned}$$

□

Theorem 3.3. *Let $A, B \in BFM(M)$. Then the product $A \times B$ is also a bipolar fuzzy submodule of M.*

Proof. Straightforward. □

Proposition 3.10. *Let A and B be bipolar valued fuzzy sets on X and Y, $g : X \to Y$ be a mapping. Then we have followings:*

(i) $g(A^+_\lambda) \subset (g(A))^+_\lambda$, $g(A^-_\beta) \supset (g(A))^-_\beta$

(ii) $g^{-1}(B^+_\lambda) = (g^{-1}(B))^+_\lambda$, $g^{-1}(B^-_\beta) = (g^{-1}(B))^-_\beta$.

Proof. (i) Let $y \in g(A^+_\lambda)$. Then $\exists x \in A^+_\lambda \ :\ g(x) = y$. So, $\mu^+_A(x) \geq \lambda$. Hence,

$\bigvee_{x\in g^{-1}(y)} \mu^+_A(x) \geq \lambda$, i.e., $g(\mu^+_A)(y) \geq \lambda$ and $y \in (g(A))^+_\lambda$.

Let $y \in g(A^-_\beta)$. Then $\exists x \in A^-_\beta \ :\ g(x) = y$. So, $\mu^-_A(x) \geq \beta$. Hence,

$\bigwedge_{x\in g^{-1}(y)} \mu^-_A(x) \geq \beta$, i.e., $g(\mu^-_A)(y) \geq \beta$ and $y \in (g(A))^-_\beta$.

(ii)

$$\begin{aligned} g^{-1}(B^+_\lambda) &= \{x \in X : g(x) \in B^+_\lambda\} \\ &= \{x \in X : \mu^+_B(g(x)) \geq \lambda\} \\ &= \{x \in X : \mu^+_(g^{-1}(B)(x)) \geq \lambda\} \\ &= (g^{-1}(B))^+_\lambda \end{aligned}$$

□

Theorem 3.4. *Let M, N be the classical modules and $g : M \to N$ be a homomorphism of modules. If $A \in BFM(M)$, then the image $g(A) \in BFM(N)$.*

Proof. Let $y_1, y_2 \in (g(A^+_\lambda)$. Then $\mu^+_{g(A)}(y_1) \geq \lambda$ and $\mu^+_{g(A)}(y_2) \geq \lambda$. Then $\exists x_1, x_2 \in M$: $\mu^+_A(x_1) \geq \mu^+_{g(A)}(y_1) \geq \lambda$ and $\mu^+_A(x_2) \geq \mu^+_{g(A)}(y_2) \geq \lambda$. Hence, $\mu^+_A(x_1) \wedge \mu^+_A(x_2) \geq \lambda$. Since A is a bipolar fuzzy submodule of M, we get $\mu^+_A(rx_1 + sx_2) \geq \mu^+_A(x_1) \wedge \mu^+_A(x_2) \geq \lambda$, for any $r, s \in R$. Therefore,

$rx_1 + sx_2 \in A^+_\lambda \Rightarrow g(rx_1 + sx_2) \in g(A^+_\lambda) \subseteq (g(A))^+_\lambda$

$\Rightarrow rg(x_1) + sg(x_2) \in (g(A))^+_\lambda \Rightarrow ry_1 + sy_2 \in (g(A))^+_\lambda$.

So, $(g(A))^+_\lambda$ is a submodule of N. Similarly, we can show that $g(A^-_\beta)$ is a classical submodules of N for each $\beta \in [-1, 0]$. By Proposition 3.8, $g(A) \in BFM(N)$.

□

Theorem 3.5. *Let M and N be the classical modules and let $g : M \to N$ be a homomophism of modules. If $B \in BFM(N)$, then the preimage $g^{-1}(B) \in BFM(M)$.*

Proof. By Proposition 3.10 (ii) and Proposition 3.8, we obtain the result.

□

4. Conclusions

Our approach in this paper combines the bipolar valued fuzzy set and module structure for defining bipolar fuzzy submodule. We defined bipolar fuzzy submodule of a given classical module and focused on its fundamental properties. Future research may be done to explore further aspects of this structure.

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