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SOLUTION OF COMPLEX PARTIAL DERIVATIVE EQUATIONS WITH CONSTANT COEFFIENTS VIA ELZAKI TRANSFORM METHOD

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Abstract. In this study, the Elzaki Transform method is applied for general nth order complex equations with constant coefficients.

Keywords: Elzaki transform, Complex equation.

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1. Introduction

In R^2, general solutions of some equations, especially of elliptic types, cannot be found. A real partial differential equation system, of which number of independent variables is even, can be transformed to a complex partial differential equation system. Solving a complex equation can be easier with complex methods. For example,

$$u_{xx} + u_{yy} = 0$$

Laplace equation doesn't have general solution in R^2, but it can be written as

$$u_{z\overline{z}} = 0$$

and the solution of this equation is

$$u = f(z) + g(\overline{z})$$

where f is analytic, g is anti analytic arbitrary function [1] . The most elementary works in the theory of complex differential equations are "Theory of Pseudo Analytic Functions" [3], and "Generalized Analytic Functions" by [4]. First order linear complex differential equations can be solved by using Elzaki transform, Fourier Transform and Laplace transform [1, 2, 5]. Higher order linear complex differential equations can be solved by approximate

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solution methods like Taylor and Chebyshev expansion method [6, 7]. In this study, to obtain a solution for equations in the form (1) is studied.

$$\begin{aligned} & A_{n,0}\frac{\partial^n w}{\partial z^n}+A_{n-1,1}\frac{\partial^n w}{\partial z^{n-1}\partial \overline{z}}+...+A_{0,n}\frac{\partial^n w}{\partial \overline{z}^n} \\ & +A_{n-1,0}\frac{\partial^{n-1} w}{\partial z^{n-1}}+A_{n-2,1}\frac{\partial^{n-1} w}{\partial z^{n-2}\partial \overline{z}}+...+A_{0,n-1}\frac{\partial^{n-1} w}{\partial \overline{z}^{n-1}} \\ & +...+A_{1,0}\frac{\partial w}{\partial z}+A_{0,1}\frac{\partial w}{\partial \overline{z}}+A_{0,0}w \\ = \; & F\left(z,\overline{z}\right) \end{aligned} \tag{1}$$

where w is dependant variable, $z, \overline{z}$ are independant variables and $A_{i,j}\,(1 \leq i \leq n, 1 \leq j \leq n)$ are real constants. Elzaki transform has been used for the solution of (1). This study presents generalization of [1, 2, 5]. This paper is organized as follows: In section 2, basic definitions and theorems are given. In section 3, formulization is obtained to solve the n th order complex differential equations with constant coefficients and some examples are given.

2. Basic Definitions and Theorems

Definition 2.1. *Let $F(t)$ be a function for $t > 0$. Elzaki transform of $F(t)$ is defined as follows:*

$$E(F(t)) = v\int_0^\infty e^{-\frac{t}{v}}.f\left(t\right)dt$$

Theorem 2.1. [8, 9] *Elzaki transforms of some functions are*

$F(t)$	$E(F(t))$
1	v^2
t^n	$n!v^{n+2}$
e^{at}	$\frac{v^2}{1-av}$
$\cos\, at$	$\frac{v^2}{1+a^2v^2}$
$\sin\, at$	$\frac{av^3}{1+a^2v^2}$

Theorem 2.2. [10]*Elzaki transforms of first order partial derivatives of $f(x,t)$ are*

$$\begin{aligned} \textbf{\textit{i)}}\; E\left[\frac{\partial f}{\partial t}\right] &= \frac{1}{v}T(x,s)-vf(x,0), \\ \textbf{\textit{ii)}}\; E\left[\frac{\partial f}{\partial x}\right] &= \frac{\partial T(x,v)}{\partial x}, \end{aligned} \tag{2}$$

where $T(x,v) = E\left[f(x,t)\right]$.

Lemma 2.1. [11]*Elzaki transforms of nth order partial derivatives of $f(x,t)$ are*

$$\begin{aligned} i)E\left[\frac{\partial^n f}{\partial t^n}\right] &= \frac{1}{v^n}T(x,v)-\frac{1}{v^{n-2}}f(x,0)-...-\frac{\partial^{n-2} f}{\partial t^{n-2}}\left(x,0\right)-v\frac{\partial^{n-1} f}{\partial t^{n-1}}\left(x,0\right) \\ ii)E\left[\frac{\partial^n f}{\partial x^n}\right] &= \frac{\partial^n T(x,v)}{\partial x^n} \end{aligned}$$

Theorem 2.3. [11]*Elzaki transforms of $(n+m)$th order partial derivatives of $f(x,t)$ are*

$$E\left[\frac{\partial^{n+m} f}{\partial x^n \partial t^m}\right] = \frac{\partial^n}{\partial x^n}\left(\frac{1}{v^m}T(x,v)-\frac{1}{v^{m-2}}f(x,0)-...-\frac{\partial^{m-2} f}{\partial t^{m-2}}\left(x,0\right)-v\frac{\partial^{m-1} f}{\partial t^{m-1}}\left(x,0\right)\right) \tag{3}$$

3. Solution of constant coeffients partial derivative equations from nth order

Definition 3.1. *Derivative operators*

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right),\frac{\partial}{\partial \overline{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$$

are called complex derivative operators.

Lemma 3.1. *Let n and r be positive integer numbers and $n \geq r$, then*

$$\binom{n}{r}+\binom{n}{r-1}=\binom{n+1}{r}.$$

Lemma 3.2. $\sum_{k=0}^{n} a_k \sum_{h=0}^{r} b_h=\sum_{k=0}^{n}\sum_{h=0}^{r} a_k b_h$

Theorem 3.1. *Let $w=w\left(z\right)$ be a complex valued function with complex variables. Then,*

$$\frac{\partial^n w}{\partial z^n}=\frac{1}{2^n}\sum_{k=0}^{n}(-i)^k\binom{n}{k}\frac{\partial^n w}{\partial x^{n-k}\partial y^k}$$

Proof. Proof can be made by induction.

For $n=1$, following equality can be written from the Definition 3.1

$$\frac{\partial w}{\partial z}=\frac{1}{2}\left(\frac{\partial w}{\partial x}-i\frac{\partial w}{\partial y}\right)=\frac{1}{2}\left[(-i)^0\binom{1}{0}\frac{\partial w}{\partial x}+(-i)\binom{1}{1}\frac{\partial w}{\partial y}\right]=\frac{1}{2}\sum_{k=0}^{1}(-i)^k\binom{1}{k}\frac{\partial w}{\partial x^{1-k}\partial y^k}$$

As a result, it is true for $n=1$.

Assume that it is true for $n=r$. Therefore, following equality can be written.

$$\frac{\partial^r w}{\partial z^r}=\frac{1}{2^r}\sum_{k=0}^{r}(-i)^k\binom{r}{k}\frac{\partial^r w}{\partial x^{r-k}\partial y^k}$$

Then, accuracy of equality must be seen for $n=r+1$.

$$\begin{aligned}
\frac{\partial^{r+1} w}{\partial z^{r+1}} &= \frac{\partial}{\partial z}\frac{\partial^r w}{\partial z^r}=\frac{\partial}{\partial z}\frac{1}{2^r}\sum_{k=0}^{r}(-i)^k\binom{r}{k}\frac{\partial^r w}{\partial x^{r-k}\partial y^k}\\
&= \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{1}{2^r}\sum_{k=0}^{r}(-i)^k\binom{r}{k}\frac{\partial^r w}{\partial x^{r-k}\partial y^k}\right)-\frac{i}{2}\frac{\partial}{\partial y}\left(\frac{1}{2^r}\sum_{k=0}^{r}(-i)^k\binom{r}{k}\frac{\partial^r w}{\partial x^{r-k}\partial y^k}\right)\\
&= \frac{1}{2^{r+1}}\sum_{k=0}^{r}(-i)^k\binom{r}{k}\left(\frac{\partial^{r+1} w}{\partial x^{r+1-k}\partial y^k}-i\frac{\partial^{r+1} w}{\partial x^{r-k}\partial y^{k+1}}\right)\\
&= \frac{1}{2^{r+1}}\left[\left(\frac{\partial^{r+1} w}{\partial x^{r+1}}-i\frac{\partial^{r+1} w}{\partial x^{r}\partial y}\right)-i\binom{r}{1}\left(\frac{\partial^{r+1} w}{\partial x^{r}\partial y}-i\frac{\partial^{r+1} w}{\partial x^{r-1}\partial y^2}\right)\right]\\
&\quad+\frac{1}{2^{r+1}}\left[(-i)^2\binom{r}{2}\left(\frac{\partial^{r+1} w}{\partial x^{r-1}\partial y^2}-i\frac{\partial^{r+1} w}{\partial x^{r-2}\partial y^3}\right)+...+(-i)^r\binom{r}{r}\left(\frac{\partial^{r+1} w}{\partial x\partial y^r}-i\frac{\partial^{r+1} w}{\partial y^{r+1}}\right)\right]
\end{aligned}$$

If the above equality takes the common multiplier in parenthesis, then the following equality is obtained.

$$\begin{aligned}
= & \frac{1}{2^{r+1}}\left[\frac{\partial^{r+1}w}{\partial x^{r+1}} - i\left(\frac{\partial^{r+1}w}{\partial x^r \partial y} + \binom{r}{1}\frac{\partial^{r+1}w}{\partial x^r \partial y}\right)\right] \\
& +\frac{1}{2^{r+1}}\left[(-i)^2\left(\binom{r}{1}\frac{\partial^{r+1}w}{\partial x^{r-1}\partial y^2} + \binom{r}{2}\frac{\partial^{r+1}w}{\partial x^{r-1}\partial y^2}\right)\right] \\
& +... \\
& +\frac{1}{2^{r+1}}\left[(-i)^r\left(\binom{r}{r-1}\frac{\partial^{r+1}w}{\partial x \partial y^r} + \binom{r}{r}\frac{\partial^{r+1}w}{\partial x \partial y^r}\right) + (-i)^{r+1}\frac{\partial^{r+1}w}{\partial y^{r+1}}\right]
\end{aligned}$$

From Lemma 3.1

$$\begin{aligned}
= & \frac{1}{2^{r+1}}\left[\frac{\partial^{r+1}w}{\partial x^{r+1}} - i\binom{r+1}{1}\frac{\partial^{r+1}w}{\partial x^r \partial y} + (-i)^2\binom{r+1}{2}\frac{\partial^{r+1}w}{\partial x^{r-1}\partial y^2}\right] \\
& +\frac{1}{2^{r+1}}\left[+...+(-i)^{r+1}\binom{r+1}{r+1}\frac{\partial^{r+1}w}{\partial y^{r+1}}\right] \\
= & \frac{1}{2^{r+1}}\sum_{k=0}^{r+1}(-i)^k\binom{r+1}{k}\frac{\partial^{r+1}w}{\partial x^{r+1-k}\partial y^k}
\end{aligned}$$

As a result, proof is completed. □

Theorem 3.2. *Let $w = w(z, \overline{z})$ be a complex valued function with complex variables. Then,*

$$\frac{\partial^n w}{\partial \overline{z}^n} = \frac{1}{2^n}\sum_{k=0}^{n} i^k \binom{n}{k}\frac{\partial^n w}{\partial x^{n-k}\partial y^k}$$

Proof. Proof is similar to the proof of the previous theorem. □

Theorem 3.3. *Let $w = w(z, \overline{z})$ be a complex valued function with complex variables. Then,*

$$\frac{\partial^{n+r}w}{\partial z^n \partial \overline{z}^r} = \frac{1}{2^{n+r}}\sum_{h=0}^{n}\sum_{k=0}^{r}(-i)^h i^k\binom{n}{h}\binom{r}{k}\frac{\partial^{n+r}w}{\partial x^{n+r-(h+k)}\partial y^{h+k}}$$

Proof. From Theorem 3.1 and Theorem 3.2, following equality is obtained

$$\begin{aligned}
\frac{\partial^{n+r}w}{\partial z^n \partial \overline{z}^r} & = \frac{\partial}{\partial z^n}\frac{\partial^r w}{\partial \overline{z}^r} = \frac{\partial}{\partial z^n}\left(\frac{1}{2^r}\sum_{k=0}^{r} i^k\binom{r}{k}\frac{\partial^r w}{\partial x^{r-k}\partial y^k}\right) \\
& = \frac{1}{2^{n+r}}\sum_{h=0}^{n}(-i)^h\binom{n}{h}\frac{\partial^n}{\partial x^{n-h}\partial y^h}\left(\sum_{k=0}^{r} i^k\binom{r}{k}\frac{\partial^r w}{\partial x^{r-k}\partial y^k}\right)
\end{aligned}$$

From Lemma 3.2

$$= \frac{1}{2^{n+r}}\sum_{h=0}^{n}\sum_{k=0}^{r}(-i)^h i^k\binom{n}{h}\binom{r}{k}\frac{\partial^{n+r}w}{\partial x^{n+r-(h+k)}\partial y^{h+k}}$$

□

Theorem 3.4. *A special solution of the following complex equation*

$$\begin{aligned} & A_{n,o}\frac{\partial^n w}{\partial z^n}+A_{n-1,1}\frac{\partial^n w}{\partial z^{n-1}\partial \overline{z}}+...+A_{0,n}\frac{\partial^n w}{\partial \overline{z}^n} \\ & +A_{n-1,o}\frac{\partial^{n-1} w}{\partial z^{n-1}}+A_{n-2,1}\frac{\partial^{n-1} w}{\partial z^{n-2}\partial \overline{z}}+...+A_{0,n-1}\frac{\partial^{n-1} w}{\partial \overline{z}^{n-1}} \\ & +...+A_{1,0}\frac{\partial w}{\partial z}+A_{0,1}\frac{\partial w}{\partial \overline{z}}+A_{0,0}w \\ = \; & F\left(z,\overline{z}\right) \end{aligned}$$

$$\frac{\partial^k w}{\partial y^k}\left(x,0\right)=f_k\left(x\right),0\leq k\leq n-1$$

is

$$w\left(z,\overline{z}\right)=E^{-1}\left[T\left(x,v\right)\right]$$

where is

$$\begin{aligned} & T\left(x,v\right) \\ = \; & \frac{E\left[F_1\left(x,y\right)+iF_2\left(x,y\right)\right]}{P\left(D\right)} \\ & +\frac{\sum\limits_{k=0}^{n}\sum\limits_{l=0}^{n-k}A_{n-k-l}\frac{1}{2^{n-k}}\sum\limits_{m=0}^{n-k-l}\sum\limits_{h=0}^{l}\left(-i\right)^m.i^h\binom{n-l-k}{m}\binom{l}{h}\frac{\partial^{n-l-m-h}A(x,v)}{\partial x^{n-l-m-h}}}{P\left(D\right)}. \end{aligned}$$

$$P(D)=\sum_{k=0}^{n}\sum_{l=0}^{n-k}A_{n-k-l}\frac{1}{2^{n-k}}\sum_{m=0}^{n-k-l}\sum_{h=0}^{l}\frac{(-i)^m.i^h}{v^{h+m}}\binom{n-l-k}{m}\binom{l}{h}D^{n-l-m-h},$$

$$A(x,v)=\left(\frac{w(x,0)}{v^{h+m-2}}+\frac{1}{v^{h+m-3}}\frac{\partial w}{\partial y}\left(x,0\right)+...+v\frac{\partial^{m+h-1}w}{\partial y^{m+h-1}}\left(x,0\right)\right)$$

Proof. Using Theorem 3.3, the complex equation, which is stated in the theorem, can be written as follows

$$\begin{aligned} & A_{n,o}\frac{1}{2^n}\sum_{k=0}^{n}\left(-i\right)^k\binom{n}{k}\frac{\partial^n w}{\partial x^{n-k}\partial y^k}+A_{n-1,1}\frac{1}{2^n}\sum_{h=0}^{n-1}\sum_{k=0}^{1}\left(-i\right)^h i^k\binom{n-1}{h}\binom{1}{k}\frac{\partial^n w}{\partial x^{n-h-k}\partial y^{h+k}} \\ +... \quad & +A_{o,n}\frac{1}{2^n}\sum_{k=0}^{n}i^k\binom{n}{k}\frac{\partial^n w}{\partial x^{n-k}\partial y^k}+A_{n-1,o}\frac{1}{2^{n-1}}\sum_{k=0}^{n-1}\left(-i\right)^k\binom{n-1}{k}\frac{\partial^{n-1}w}{\partial x^{n-1-k}\partial y^k} \\ & +A_{n-2,1}\frac{1}{2^{n-1}}\sum_{h=0}^{n-2}\sum_{k=0}^{1}\left(-i\right)^h i^k\binom{n-1}{h}\binom{1}{k}\frac{\partial^{n-1}w}{\partial x^{n-1-h-k}\partial y^{h+k}} \\ & +...+A_{o,n-1}\frac{1}{2^{n-1}}\sum_{k=0}^{n-1}i^k\binom{n-1}{k}\frac{\partial^{n-1}w}{\partial x^{n-1-k}\partial y^k} \\ & +...+A_{1,0}\frac{1}{2}\sum_{k=0}^{1}\left(-i\right)^k\frac{\partial^n w}{\partial x^{n-k}\partial y^k}+A_{0,1}\frac{1}{2}\sum_{k=0}^{1}i^k\binom{1}{k}\frac{\partial^n w}{\partial x^{n-k}\partial y^k}+A_{0,0}w \\ = \; & F_1\left(x,y\right)+iF_2\left(x,y\right) \end{aligned}$$

If elzaki transform is used for the equation above, the following equality is obtained by using Theorem 2.4.

$$
\begin{aligned}
& A_{n,o}\frac{1}{2^n}\sum_{k=0}^{n}(-i)^k\binom{n}{k}\frac{\partial^{n-k}}{\partial x^{n-k}}\left(\frac{1}{v^k}T(x,v)-...-\frac{\partial^{k-2}w}{\partial y^{k-2}}(x,0)-v\frac{\partial^{k-1}w}{\partial y^{k-1}}(x,0)\right) \\
& +A_{n-1,1}\frac{1}{2^n}\sum_{h=0}^{n-1}\sum_{k=0}^{1}(-i)^h i^k\binom{n-1}{h}\frac{\partial^{n-h-k}}{\partial x^{n-h-k}}\left(\frac{1}{v^{h+k}}T(x,v)-...-v\frac{\partial^{k+h-1}w}{\partial y^{k+h-1}}(x,0)\right) \\
& +...+A_{0,n}\frac{1}{2^n}\sum_{k=0}^{n}i^k\binom{n}{k}\frac{\partial^{n-k}}{\partial x^{n-k}}\left(\frac{1}{v^k}T(x,v)-...-\frac{\partial^{k-2}w}{\partial y^{k-2}}(x,0)-v\frac{\partial^{k-1}w}{\partial y^{k-1}}(x,0)\right) \\
& +A_{n-1,o}\frac{1}{2^{n-1}}\sum_{k=0}^{n-1}(-i)^k\binom{n-1}{k}\frac{\partial^{n-1-k}}{\partial x^{n-1-k}}\left(\frac{1}{v^k}T(x,v)-...-v\frac{\partial^{k-1}w}{\partial y^{k-1}}(x,0)\right) \\
& +A_{n-2,1}\frac{1}{2^{n-1}}\sum_{h=0}^{n-2}\sum_{k=0}^{1}(-i)^h i^k\binom{n-2}{h}\frac{\partial^{n-1-h-k}}{\partial x^{n-1-h-k}}\left(\frac{1}{v^{h+k}}T(x,v)-...-v\frac{\partial^{k+h-1}w}{\partial y^{k+h-1}}(x,0)\right) \\
& +...+A_{0,n-1}\frac{1}{2^{n-1}}\sum_{k=0}^{n-1}i^k\binom{n-1}{k}\frac{\partial^{n-1-k}}{\partial x^{n-1-k}}\left(\frac{1}{v^k}T(x,v)-...-\frac{\partial^{k-2}w}{\partial y^{k-2}}(x,0)-v\frac{\partial^{k-1}w}{\partial y^{k-1}}(x,0)\right) \\
& +...+A_{1,0}\frac{1}{2}\sum_{k=0}^{1}(-i)^k\frac{\partial^{1-k}}{\partial x^{1-k}}\left(\frac{1}{v^k}T(x,v)-...-\frac{\partial^{k-2}w}{\partial y^{k-2}}(x,0)-v\frac{\partial^{k-1}w}{\partial y^{k-1}}(x,0)\right) \\
& +A_{0,1}\frac{1}{2}\sum_{k=0}^{1}i^k\frac{\partial^{1-k}}{\partial x^{1-k}}\left(\frac{1}{v^k}T(x,v)-...-\frac{\partial^{k-2}w}{\partial y^{k-2}}(x,0)-v\frac{\partial^{k-1}w}{\partial y^{k-1}}(x,0)\right)+A_{0,0}T(x,v) \\
= \; & E[F]
\end{aligned}
$$

In the equation above, the terms which are equal to sum of the indices can be written under a single total symbol. If $T(x,v)$ and its derivatives are added to the left side of the equation and by using definition of $A(x,v)$, the following equality is obtained.

$$
\begin{aligned}
& \sum_{k=0}^{n}\left[A_{n-k,k}\frac{1}{2^n}\sum_{m=0}^{n-k}\sum_{h=0}^{k}\frac{(-i)^m .i^h}{v^{h+m}}\binom{n-k}{m}\binom{k}{h}\frac{\partial^{n-m-h}T(x,v)}{\partial x^{n-m-h}}\right] \\
& +\sum_{k=0}^{n-1}\left[A_{n-1-k,k}\frac{1}{2^{n-1}}\sum_{m=0}^{n-1-k}\sum_{h=0}^{k}\frac{(-i)^m .i^h}{v^{h+m}}\binom{n-1-k}{m}\binom{k}{h}\frac{\partial^{n-1-m-h}T(x,v)}{\partial x^{n-1-m-h}}\right] \\
& +....+\sum_{k=0}^{1}\left[A_{1-k,k}\frac{1}{2}\sum_{m=0}^{1-k}\sum_{h=0}^{k}\frac{(-i)^m .i^h}{v^{h+m}}\binom{1-k}{m}\binom{k}{h}\frac{\partial^{1-m-h}T(x,v)}{\partial x^{1-m-h}}\right]+A_{0,0}T(x,v) \\
& = E[F_1(x,y)+iF_2(x,y)]
\end{aligned}
$$

$$
\begin{aligned}
& +\sum_{k=0}^{n}\left[A_{n-k,k}\frac{1}{2^n}\sum_{m=0}^{n-k}\sum_{h=0}^{k}(-i)^m .i^h\binom{n-k}{m}\binom{k}{h}\frac{\partial^{n-m-h}A(x,v)}{\partial x^{n-m-h}}\right] \\
& +\sum_{k=0}^{n-1}\left[A_{n-1-k,k}\frac{1}{2^{n-1}}\sum_{m=0}^{n-1-k}\sum_{h=0}^{k}(-i)^m .i^h\binom{n-1-k}{m}\binom{k}{h}\frac{\partial^{n-1-m-h}A(x,v)}{\partial x^{n-1-m-h}}\right] \\
& +...+\sum_{k=0}^{1}\left[A_{1-k,k}\frac{1}{2}\sum_{m=0}^{1-k}\sum_{h=0}^{k}(-i)^m .i^h\binom{1-k}{m}\binom{k}{h}\frac{\partial^{1-k-m-h}A(x,v)}{\partial x^{1-k-m-h}}\right]
\end{aligned}
$$

All terms can be written on the right and left side of the equation inside a single parenthesis and the following equation is obtained.

$$\left[\sum_{k=0}^{n}\sum_{l=0}^{n-k}A_{n-k-l}\frac{1}{2^{n-k}}\sum_{m=0}^{n-k-l}\sum_{h=0}^{l}\frac{(-i)^m.i^h}{v^{h+m}}\binom{n-l-k}{m}\binom{l}{h}D^{n-l-m-h}\right]T(x,v)=E\left[F_1(x,y)+iF_2(x,y)\right]$$

$$+\sum_{k=0}^{n}\sum_{l=0}^{n-k}A_{n-k-l}\frac{1}{2^{n-k}}\sum_{m=0}^{n-k-l}\sum_{h=0}^{l}(-i)^m.i^h\binom{n-l-k}{m}\binom{l}{h}\frac{\partial^{n-l-m-h}A(x,v)}{\partial x^{n-l-m-h}}$$

As a result, using the inverse operator method, by the definition of $P(D)$ in theorem, $T(x,v)$ is obtained as follows

$$T(x,v)=\frac{E\left[F_1(x,y)+iF_2(x,y)\right]}{P(D)}$$

$$+\frac{\sum_{k=0}^{n}\sum_{l=0}^{n-k}A_{n-k-l}\frac{1}{2^{n-k}}\sum_{m=0}^{n-k-l}\sum_{h=0}^{l}(-i)^m.i^h\binom{n-l-k}{m}\binom{l}{h}\frac{\partial^{n-l-m-h}A(x,v)}{\partial x^{n-l-m-h}}}{P(D)}$$

Thus, solution of the equation is found from inverse elzaki transform as $w(x,y)=E^{-1}(T(x,v))$. □

Example 3.1. *Find the solution for the following differential equation*

$$\frac{\partial^2 w}{\partial z\partial\overline{z}}=4$$

with the conditions

$$\begin{aligned} w(x,0) &= 5x^2+3x+2\\ \frac{\partial w}{\partial y}(x,0) &= i(2x-1)\end{aligned}$$

Solution 3.1. *Coefficients of the equation are* $A_{1,1}=1, A_{2,0}=A_{0,2}=A_{1,0}=A_{0,1}=A_{0,0}=0$ *and* $n=2$. *Using theorem 3.4*

$$T(x,v)=\frac{16v^2+5x^2+3x+2+iv(2x-1)}{D^2+\frac{1}{v^2}}$$

$$w(x,y)=E^{-1}(T(x,v))$$

$$\begin{aligned} w(z,\overline{z}) &= E^{-1}\left[\frac{16v^2+5x^2+3x+2+iv(2x-1)}{D^2+\frac{1}{v^2}}\right]\\ &= E^{-1}\left[v^2\left(1-v^2D^2+v^4D^4-...\right)\left(16v^2+5x^2+3x+2+iv(2x-1)\right)\right]\\ &= E^{-1}\left[v^2\left(16v^2+5x^2+3x+2+iv(2x-1)-10v^2\right)\right]\\ &= 3y^2+5x^2+3x+2+i(2x-1)y\\ &= z^2+4z\overline{z}+2z+\overline{z}+2\end{aligned}$$

Example 3.2. *Find the solution for the following differential equation*

$$\frac{\partial^2 w}{\partial z^2}+2\frac{\partial w}{\partial\overline{z}}=12z+18\overline{z}+9$$

with the conditions

$$\begin{aligned} w(x,0) &= 2x^3+3x^2+8x \\ \frac{\partial w}{\partial y}(x,0) &= i(6x^2-6x+2) \end{aligned}$$

Solution 3.2. *Coefficients of the equation are* $A_{2,0}=1, A_{0,1}=2$, $A_{0,2}=A_{1,1}=A_{1,0}=A_{0,1}=A_{0,0}=0$ *and* $n=2$. *Using theorem 3.4*

$$T(x,v)=\frac{v^2(120x+36)-24iv^3+iv(12x^3+42x-18)-(2x^3+3x^2+8x)}{D^2+(6-\frac{2i}{v})D+\frac{6iv-1}{v^2}}$$

$$w\left(x,y\right)=E^{-1}\left(T\left(x,v\right)\right)$$

Let us assume that,

$$A\left(x,v\right)=-24iv^3+120v^2x+36v^2+12ivx^3+42ivx-18iv-2x^3-3x^2-8x$$

Then, it can be written as,

$$\begin{aligned} w\left(z,\overline{z}\right) &= E^{-1}\left[\frac{A\left(x,v\right)}{D^2+(6-\frac{2i}{v})D+\frac{6iv-1}{v^2}}\right] \\ &= E^{-1}\left[\frac{v^2}{6iv-1}A\left(x,v\right)\right. \\ &\quad \left.\left(1-\frac{v^2}{6iv-1}\left(D^2+(6-\frac{2i}{v})D\right)+\frac{v^4}{(6iv-1)^2}\left(D^2+(6-\frac{2i}{v})D\right)^2+...\right)\right] \\ &= E^{-1}\left[2x^3v^2-\frac{3x^2v^2}{6iv-1}-\frac{6v^3x^2\left(6v-2i\right)}{6iv-1}+\frac{v^2x}{6iv-1}\left(108v^2+42iv-8\right)\right. \\ &\quad \left.+\frac{v^2x\left(36v^2-12iv\right)}{\left(6iv-1\right)^2}\right]+E^{-1}\left[\frac{12v^4}{6iv-1}\left(36v^2-24iv-4\right)\right. \\ &\quad \left.+\frac{v^2}{6iv-1}\left(36v^2-24iv^3-18iv\right)+\frac{6v^4}{\left(6iv-1\right)^2}+\frac{8v^3\left(6v-2i\right)}{\left(6iv-1\right)^2}\right] \\ &\quad +E^{-1}\left[-\frac{v^3\left(6v-2i\right)}{\left(6iv-1\right)^2}\left(120v^2+42iv\right)-\frac{6v^4}{\left(6iv-1\right)^3}\left(36v^2-24iv-4\right)\right] \end{aligned}$$

$$\begin{aligned} w\left(z,\overline{z}\right) &= E^{-1}\left[\left(2x^3v^2-12xv^443v^2\right)x^2-6v^4+8xv^2+i\left(6x^2v^3-12v^5-6xv^3+2v^3\right)\right] \\ &= 2x^3-6xy^2+3x^2-3y^2+8x+i\left(6x^2y-2y^3-6xy+2y\right) \\ &= z^3+3\overline{z}^2+5z+3\overline{z} \end{aligned}$$

4. Conclusion

In this article, it can be seen that the most general linear constant coefficient complex differential equations can be solved by Elzaki transformation. A formula for a specific solution of such equations has been obtained. It can be seen that the results are consistent with the literature.

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Murat DUZ for the photography and short autobiography, see TWMS J. App. Eng. Math., V.7, N.1.