TWMS J. App. Eng. Math. V.11, N.2, 2021, pp. 424-435

NEW REFINEMENTS AND APPLICATIONS OF OSTROWSKI TYPE INEQUALITIES FOR MAPPINGS WHOSE nth DERIVATIVES ARE OF BOUNDED VARIATION

H. BUDAK[1], M. Z. SARIKAYA[1], A. QAYYUM[2], §

Abstract. The main aim of this paper is to establish some Ostrowski type integral inequalities using a newly developed special type of kernel for mappings whose nth derivatives are of bounded variation. We deduce some previous results as a special case. Some new efficient quadrature rules are also introduced.

Keywords: Function of bounded variation, Ostrowski type inequalities, Riemann-Stieltjes integral.

AMS Subject Classification: 26D15, 26A45, 26D10, 41A55

1. Introduction

In 1938, Ostrowski [19] established a following useful inequality:

Theorem 1.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e. $\|f'\|_{\infty}:=\sup\limits_{t\in(a,b)}|f'(t)|<\infty$. Then, we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int\limits_a^b f(t)dt\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\left\|f'\right\|_{\infty}, \tag{1}$$

for all $x\in[a,b]$.

The constant $\frac{1}{4}$ is the best possible.

Ostrowski inequality has potential applications in Mathematical Sciences. In the past, many authors have worked on Ostrowski type inequalities for function of bounded variation, see for example ([1]-[15], [17]). Moreover, Dragomir proved some Ostrowski type inequalities for functions whose nth derivatives are of bounded variation in [16].

The following definitions will be frequently used to prove our results.

[1] Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-TURKEY.
hsyn.budak@gmail.com; ORCID: http://orcid.org/0000-0001-8843-955X.
sarikayamz@gmail.com; ORCID: http://orcid.org/0000-0002-6165-9242.

[2] Department of Mathematics, The Institute of Southern Punjab Multan, Pakistan.
e-mail: atherqayyum@gmail.com; ORCID: https://orcid.org/0000-0003-4149-5305.

§ Manuscript received: January 06, 2017; accepted: July 30, 2017.
TWMS Journal of Applied and Engineering Mathematics, Vol.11, No.2

Definition 1.1. *Let $P : a = x_0 < x_1 < ... < x_n = b$ be any partition of $[a, b]$ and let $\Delta f(x_i) = f(x_{i+1}) - f(x_i)$, then f is said to be of bounded variation if the sum*

$$\sum_{i=1}^{m} |\Delta f(x_i)|$$

is bounded for all such partitions.

Definition 1.2. *Let f be of bounded variation on $[a, b]$, and $\sum \Delta f(P)$ denotes the sum $\sum\limits_{i=1}^{n} |\Delta f(x_i)|$ corresponding to the partition P of $[a, b]$. The number*

$$\bigvee_a^b (f) := \sup\left\{\sum \Delta f(P) : P \in P([a, b])\right\},$$

is called the total variation of f on $[a, b]$. Here $P([a, b])$ denotes the family of partitions of $[a, b]$.

In [13], Dragomir proved the following Ostrowski type inequalities related functions of bounded variation:

Theorem 1.2. *Let $f : [a, b] \to \mathbb{R}$ be a mapping of bounded variation on $[a, b]$. Then*

$$\left|\int_a^b f(t)dt - (b-a) f(x)\right| \leq \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f) \tag{2}$$

holds for all $x \in [a, b]$. The constant $\frac{1}{2}$ is the best possible.

In [?], authors gave the following Ostrowski type inequality:

Theorem 1.3. *Let $f : [a, b] \to \mathbb{R}$ be such that f' is a continuous function of bounded variation on $[a, b]$. Then we have the inequality*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}[f(x) + f(a+b-x)]\right. \\
&\left.+\frac{1}{2}\left(x - \frac{3a+b}{4}\right)[f'(x) - f'(a+b-x)]\right| \\
\leq\ & \frac{1}{16}\left[\frac{5(x-a)^2 - 2(x-a)(b-x) + (b-x)^2}{b-a} + 4\left|x - \frac{3a+b}{4}\right|\right] \bigvee_a^b (f')
\end{aligned}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

In [6], Budak and Sarikaya obtained following Ostrowski type inequality in weighted form for the mappings whose first derivatives are of bounded variation:

Theorem 1.4. *Let $w : [a, b] \to \mathbb{R}$ be nonnegative and continous and let $f : [a, b] \to \mathbb{R}$ be differentiable mapping on $[a, b]$. If f' is of bounded variation on $[a, b]$, then we have the* *weighted inequality*

$$\left|\left(\int_a^b (u-x)\,w(u)du\right)f'(x)+\left(\int_a^b w(u)du\right)f(x)-\int_a^b w(t)f(t)dt\right|$$
$$\leq \left(\int_a^x (u-x)\,w(u)du\right)\bigvee_a^x(f')+\left(\int_x^b (u-x)\,w(u)du\right)\bigvee_x^b(f')$$

for any $x\in[a,b]$.

Recently, Qayyum et. al [20]-[22], proved some Ostrowski inequality using multiple step kernel. In this paper, we obtain some Ostrowski type integral inequalities for functions whose *nth* derivatives are of bounded variation. The results presented here would provide extensions of those given in [**?**]-[11] and [13].

2. Refinements of Ostrowski type integral inequalities

Before we start our main results, we state and prove following lemmas:

Lemma 2.1. *Let* $f:[a,b]\to\mathbb{R}$ *be such that* $f^{(n)}$ *is a continuous function of bounded variation on* $[a,b]$, *then the following identity holds*

$$\int_a^b P_n^1(x,t)df^{(n)}(t) \tag{3}$$
$$= (-1)^n\sum_{k=0}^{n}\frac{1}{(k+1)!}\left[(-1)^k(x-a)^{k+1}+(b-x)^{k+1}\right]f^{(k)}(x)$$
$$+(-1)^{n+1}\int_a^b f(t)dt.$$

where

$$P_n^1(x,t)=\begin{cases}\frac{(t-a)^{n+1}}{(n+1)!}, & a\leq t\leq x\\ \frac{(t-b)^{n+1}}{(n+1)!}, & x<t\leq b\end{cases}$$

for all $x\in[a,b]$.

Proof. The proof of (3) is established using mathematical induction.

Take $n=1$,

$$\int_a^b P_1^1(x,t)df'(t)$$
$$= \frac{1}{2}\left[\int_a^x (t-a)^2df'(t)+\int_x^b (t-b)^2df'(t)\right]$$
$$= -(b-a)f(x)-(b-a)\left(x-\frac{a+b}{2}\right)+\int_a^b f(t)dt.$$

The identity (3) is provided for $n=1$.

Assume that (3) is true for n. We will show that (3) is true for $n+1$.

$$\begin{aligned}
&\int_a^b P_{n+1}^1(x,t)df^{(n+1)}(t) \\
&= \frac{1}{(n+2)!}\left[\int_a^x (t-a)^{n+2}df^{(n+1)}(t)+\int_x^b (t-b)^{n+2}df^{(n+1)}(t)\right] \\
&= \frac{1}{(n+1)!}\left[(x-a)^{n+2}-(x-b)^{n+2}\right]f^{(n+1)}(x) \\
&\quad -\frac{1}{(n+2)!}\left[\int_a^x (t-a)^{n+1}df^{(n)}(t)+\int_x^b (t-b)^{n+1}df^{(n)}(t)\right] \\
&= \frac{(-1)^{n+1}}{(n+2)!}\left[(-1)^{n+1}(x-a)^{n+2}+(b-x)^{n+2}\right]f^{(n+1)}(x) \\
&\quad -(-1)^n\sum_{k=0}^{n}\frac{1}{(k+1)!}\left[(-1)^k(x-a)^{k+1}+(b-x)^{k+1}\right]f^{(k)}(x)-(-1)^{n+1}\int_a^b f(t)dt \\
&= (-1)^{n+1}\sum_{k=0}^{n}\frac{1}{(k+1)!}\left[(-1)^k(x-a)^{k+1}+(b-x)^{k+1}\right]f^{(k)}(x)+(-1)^{n+2}\int_a^b f(t)dt.
\end{aligned}$$

This completes the proof. □

Lemma 2.2. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ is a continuous function of bounded variation on $[a,b]$, then the following identity holds*

$$\begin{aligned}
&\int_a^b P_n^2(x,t)df^{(n)}(t) \qquad (4)\\
&= \sum_{k=0}^{n}\frac{(-1)^{n+k}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}\left(\frac{a+x}{2}\right)\right. \\
&\quad +\left\{\left(x-\frac{3a+b}{4}\right)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}(x) \\
&\quad +(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-\left(x-\frac{3a+b}{4}\right)^{k+1}\right\}f^{(k)}(a+b-x) \\
&\quad \left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-(x-a)^{k+1}\right\}f^{(k)}\left(\frac{a+2b-x}{2}\right)\right] \\
&\quad +(-1)^{n+1}\int_a^b f(t)dt,
\end{aligned}$$

where the mapping $P_n^2(x,t)$ is defined by

$$P_n^2(x,t)=\begin{cases}\frac{1}{(n+1)!}(t-a)^{n+1}, & t\in\left(a,\frac{a+x}{2}\right]\\ \frac{1}{(n+1)!}\left(t-\frac{3a+b}{4}\right)^{n+1}, & t\in\left(\frac{a+x}{2},x\right]\\ \frac{1}{(n+1)!}\left(t-\frac{a+b}{2}\right)^{n+1}, & t\in(x,a+b-x]\\ \frac{1}{(n+1)!}\left(t-\frac{a+3b}{4}\right)^{n+1}, & t\in\left(a+b-x,\frac{a+2b-x}{2}\right]\\ \frac{1}{(n+1)!}(t-b)^{n+1}, & t\in\left(\frac{a+2b-x}{2},b\right]\end{cases}$$

for all $x\in\left[a,\frac{a+b}{2}\right]$.

Proof. We prove the Lemma using mathematical induction. Take $n=1$,

$$\begin{aligned}&\int_a^b P_1^2(x,t)df'(t)\\ =&\int_a^b f(t)dt-\frac{b-a}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right.\\ &+\left(x-\frac{5a+3b}{8}\right)\left\{f'(a+b-x)-f'(x)\right\}\\ &\left.+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)\left\{f'\left(\frac{a+2b-x}{2}\right)-f'\left(\frac{a+x}{2}\right)\right\}\right].\end{aligned}$$

The identity (4) is provided for $n=1$.

Assume that (3) is true for n. We will show that (3) is true for $n+1$.

$$
\begin{aligned}
&\int_a^b P_n^2(x,t)df^{(n)}(t) \\
= \; & \frac{(-1)^{2n+2}}{(k+1)!}\left[\frac{1}{2^{n+2}}\left\{(x-a)^{n+2}-\left(x-\frac{a+b}{2}\right)^{n+2}\right\}f^{(n+1)}\left(\frac{a+x}{2}\right)\right. \\
&+\left\{\left(x-\frac{3a+b}{4}\right)^{n+2}-\left(x-\frac{a+b}{2}\right)^{n+2}\right\}f^{(n+1)}(x) \\
&+(-1)^{n+2}\left\{\left(x-\frac{a+b}{2}\right)^{n+2}-\left(x-\frac{3a+b}{4}\right)^{n+2}\right\}f^{(n+1)}(a+b-x) \\
&\left.+\left(\frac{-1}{2}\right)^{n+2}\left\{\left(x-\frac{a+b}{2}\right)^{n+2}-(x-a)^{n+2}\right\}f^{(n+1)}\left(\frac{a+2b-x}{2}\right)\right] \\
&-\sum_{k=0}^{n}\frac{(-1)^{n+k}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}\left(\frac{a+x}{2}\right)\right. \\
&+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}(x) \\
&+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-\left(x-\frac{3a+b}{4}\right)^{k+1}\right\}f^{(k)}(a+b-x) \\
&\left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-(x-a)^{k+1}\right\}f^{(k)}\left(\frac{a+2b-x}{2}\right)\right] \\
&-(-1)^{n+1}\int_a^b f(t)dt \\
= \; & \sum_{k=0}^{n+1}\frac{(-1)^{n+k+1}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}\left(\frac{a+x}{2}\right)\right. \\
&+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}(x) \\
&+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-\left(x-\frac{3a+b}{4}\right)^{k+1}\right\}f^{(k)}(a+b-x) \\
&\left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-(x-a)^{k+1}\right\}f^{(k)}\left(\frac{a+2b-x}{2}\right)\right] \\
&+(-1)^{n+2}\int_a^b f(t)dt.
\end{aligned}
$$

This completes the proof. $\square$

Now using above identities, we state and prove the following theorems.

Theorem 2.1. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ is a continuous function of bounded variation on $[a,b]$. Then, for all $x\in[a,b]$, we have the inequality*

$$\left|(-1)^n\sum_{k=0}^{n}\frac{1}{(k+1)!}\left[(-1)^k(x-a)^{k+1}+(b-x)^{k+1}\right]f^{(k)}(x)+(-1)^{n+1}\int_a^b f(t)dt\right| \tag{5}$$
$$\leq \frac{1}{(n+1)!}\max\left\{(x-a)^{n+1},(b-x)^{n+1}\right\}\bigvee_a^b(f^{(n)}).$$

where $\bigvee_a^b(f^{(n)})$ denotes the total variation of $f^{(n)}$ on $[a,b]$.

Proof. It is well known that if $g,f:[a,b]\to\mathbb{R}$ are such that g is continuous on $[a,b]$ and f is of bounded variation on $[a,b]$, then $\int_a^b g(t)df(t)$ exists and

$$\left|\int_a^b g(t)df(t)\right|\leq\sup_{t\in[a,b]}|g(t)|\bigvee_a^b(f). \tag{6}$$

In Lemma 2.1, by using (6), we get

$$\begin{aligned}
&\left|\int_a^b P_n^1(x,t)df^{(n)}(t)\right|\\
\leq\ &\frac{1}{(n+1)!}\left[\left|\int_a^x(t-a)^{n+1}df^{(n)}(t)\right|+\left|\int_x^b(t-b)^{n+1}df^{(n)}(t)\right|\right]\\
\leq\ &\frac{1}{(n+1)!}\left[\sup_{t\in[a,x]}|t-a|^{n+1}\bigvee_a^x(f^{(n)})+\sup_{t\in[a,b]}|t-b|^{n+1}\bigvee_x^b(f^{(n)})\right]\\
=\ &\frac{1}{(n+1)!}\left[(x-a)^{n+1}\bigvee_a^x(f^{(n)})+(b-x)^{n+1}\bigvee_x^b(f^{(n)})\right]\\
\leq\ &\frac{1}{(n+1)!}\max\left\{(x-a)^{n+1},(b-x)^{n+1}\right\}\bigvee_a^b(f^{(n)}).
\end{aligned}$$

This completes the proof. $\square$

Remark 2.1. *If we choose $n=0$ in Theorem 2.1, the inequality (5) reduces the inequality (2).*

Corollary 2.1. *Under assumption of Theorem 2.1 with $n=1$, we obtain the inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-f(x)-\left(\frac{a+b}{2}-x\right)f'(x)\right| \tag{7}$$
$$\leq\frac{1}{4}\left[\frac{1}{(b-a)}\left[\frac{1}{2}(b-a)^2+2\left(x-\frac{a+b}{2}\right)^2\right]+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f').$$

Remark 2.2. *If we choose $x = \frac{a+b}{2}$ in (7), then we have the inequality*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{b-a}{8}\bigvee_a^b (f'),$$

which was given by Liu in [18].

Theorem 2.2. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ is a continuous function of bounded variation on $[a,b]$, then the following identity holds*

$$\begin{aligned}
&\left|\sum_{k=0}^{n} \frac{(-1)^{n+k}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1} - \left(x-\frac{a+b}{2}\right)^{k+1}\right\} f^{(k)}\left(\frac{a+x}{2}\right)\right.\right. \\
&+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1} - \left(x-\frac{a+b}{2}\right)^{k+1}\right\} f^{(k)}(x) \\
&+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1} - \left(x-\frac{3a+b}{4}\right)^{k+1}\right\} f^{(k)}(a+b-x) \\
&\left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1} - (x-a)^{k+1}\right\} f^{(k)}\left(\frac{a+2b-x}{2}\right)\right] \\
&\left.+(-1)^{n+1}\int_a^b f(t)dt\right| \\
\leq\ & \frac{1}{(n+1)!}\max\left\{\left|x-\frac{3a+b}{4}\right|^{n+1}, \left(\frac{a+b}{2}-x\right)^{n+1}, \frac{(x-a)^{n+1}}{2^{n+1}}\right\}\bigvee_a^b \left(f^{(n)}\right)
\end{aligned}$$

for all $x \in \left[a, \frac{a+b}{2}\right]$

Proof. In Lemma 2.2, by using (6), we get

$$\begin{aligned}
&\left|\int_a^b P_n^2(x,t)df^{(n)}(t)\right| \\
\leq\ & \frac{1}{(n+1)!}\left[\left|\int_a^{\frac{a+x}{2}} (t-a)^{n+1}\, df^{(n)}(t)\right| + \left|\int_{\frac{a+x}{2}}^{x} \left(t-\frac{3a+b}{4}\right)^{n+1} df^{(n)}(t)\right|\right. \\
&+\left|\int_x^{a+b-x} \left(t-\frac{a+b}{2}\right)^{n+1} df^{(n)}(t)\right| + \left|\int_{a+b-x}^{\frac{a+2b-x}{2}} \left(t-\frac{a+3b}{4}\right)^{n+1} df^{(n)}(t)\right| \\
&\left.+\left|\int_{\frac{a+2b-x}{2}}^{b} (t-b)^{n+1}\, df^{(n)}(t)\right|\right]
\end{aligned}$$

$$\leq \frac{1}{(n+1)!}\left[\sup_{t\in\left[a,\frac{a+x}{2}\right]}|t-a|^{n+1}\bigvee_{a}^{\frac{a+x}{2}}(f^{(n)})+\sup_{t\in\left[\frac{a+x}{2},x\right]}\left|t-\frac{3a+b}{4}\right|^{n+1}\bigvee_{\frac{a+x}{2}}^{x}(f^{(n)})\right.$$
$$+\sup_{t\in[x,a+b-x]}\left|t-\frac{a+b}{2}\right|^{n+1}\bigvee_{x}^{a+b-x}(f^{(n)})+\sup_{t\in\left[a+b-x,\frac{a+2b-x}{2}\right]}\left|t-\frac{a+3b}{4}\right|^{n+1}\bigvee_{a+b-x}^{\frac{a+2b-x}{2}}(f^{(n)})$$
$$\left.+\sup_{t\in\left[\frac{a+2b-x}{2},b\right]}|t-b|^{n+1}\bigvee_{\frac{a+2b-x}{2}}^{b}(f^{(n)})\right]$$
$$= \frac{1}{(n+1)!}\left[\frac{(x-a)^{n+1}}{2^{n+1}}\bigvee_{a}^{\frac{a+x}{2}}(f^{(n)})+\max\left\{\left|x-\frac{3a+b}{4}\right|^{n+1},\frac{1}{2^{n+1}}\left(\frac{a+b}{2}-x\right)^{n+1}\right\}\bigvee_{\frac{a+x}{2}}^{x}(f^{(n)})\right.$$
$$+\left(\frac{a+b}{2}-x\right)^{n+1}\bigvee_{x}^{a+b-x}(f^{(n)})+\max\left\{\left|x-\frac{3a+b}{4}\right|^{n+1},\frac{1}{2^{n+1}}\left(\frac{a+b}{2}-x\right)^{n+1}\right\}\bigvee_{a+b-x}^{\frac{a+2b-x}{2}}(f^{(n)})$$
$$\left.+\frac{(x-a)^{n+1}}{2^{n+1}}\bigvee_{\frac{a+2b-x}{2}}^{b}(f^{(n)})\right]$$
$$\leq \frac{1}{(n+1)!}\max\left\{\left|x-\frac{3a+b}{4}\right|^{n+1},\left(\frac{a+b}{2}-x\right)^{n+1},\frac{(x-a)^{n+1}}{2^{n+1}}\right\}\bigvee_{a}^{b}\left(f^{(n)}\right).$$

This completes the proof. □

Remark 2.3. *If we choose $n=0$ in Theorem 2.2, we get the result proved by Budak and Sarikaya in* [**?**].

Remark 2.4. *If we choose $n=1$ in Theorem 2.2, we get the result proved by Budak et al. in* [11].

3. Derivation of Numerical Quadrature Rules

We propose some new quadrature rules involving *nth*-derivatives of the function f. In fact, the following new quadrature rules can be obtained while investigating error bounds using theorem 6.

$$Q_{n,1}(f) := \int_a^b f(t)dt$$
$$\approx \sum_{k=0}^{n}\frac{1}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right],$$

$$Q_{n,2}(f) := \int_a^b f(t)dt$$
$$= \sum_{k=0}^{n} \frac{(-1)^k}{(k+1)!} \left[\frac{\left(1+(-1)^k\right)}{4^{k+1}2^{k+1}} (b-a)^{k+1} \left\{ f^{(k)}\left(\frac{7a+b}{8}\right) + f^{(k)}\left(\frac{a+7b}{8}\right) \right\} \right.$$
$$\left. + \frac{1}{4^{k+1}} (b-a)^{k+1} \left\{ (-1)^k f^{(k)}\left(\frac{3a+b}{4}\right) + f^{(k)}\left(\frac{a+3b}{4}\right) \right\} \right],$$

$$Q_{n,3}(f) := \int_a^b f(t)dt$$
$$= \sum_{k=0}^{n} \frac{(-1)^k}{(k+1)!} \left[\left\{ f^{(k)}\left(\frac{3a+b}{4}\right) + \left(1+(-1)^k\right) f^{(k)}\left(\frac{a+b}{2}\right) + (-1)^k f^{(k)}\left(\frac{a+3b}{4}\right) \right\} \right.$$
$$\left. \times \frac{1}{4^{k+1}} (b-a)^{k+1} \right]$$

Performance of the efficient quadrature rules

Sr. No.	**Method**	$n:\quad Q_{n,1}(f)$	$n:\quad Q_{n,2}(f)$	$n:\quad Q_{n,3}(f)$	**Exact Value**
1.	$\int_0^1 f_1(x)dx$	**2:** 2.83333	**2:** 2.83333	**2:** 2.83333	2.83333
	Error:	0	0	0	
2.	$\int_0^1 f_2(x)dx$	**6:** 0.301168	**4:** 0.301169	**4:** 0.301168	0.301169
	Error:	5.5921×10^{-7}	7.20674×10^{-7}	1.07696×10^{-6}	
3.	$\int_0^1 f_3(x)dx$	**6:** 0.909332	**4:** 0.909329	**4:** 0.909333	0.909331
	Error:	1.22345×10^{-6}	1.66306×10^{-6}	2.48778×10^{-6}	
4.	$\int_0^1 f_4(x)dx$	**4:** 0.793022	**3:** 0.793023	**4:** 0.793031	0.793031
	Error:	8.63182×10^{-6}	8.08331×10^{-6}	1.03192×10^{-7}	
5.	$\int_0^1 f_5(x)dx$	**10:** 1.46266	**6:** 1.46265	**6:** 1.46265	1.46265
	Error:	5.8789×10^{-6}	1.54452×10^{-6}	2.29707×10^{-6}	
6.	$\int_0^1 f_6(x)dx$	**10:** 1.31384	**6:** 1.31383	**6:** 1.31383	1.31383
	Error	7.37624×10^{-6}	3.15394×10^{-7}	1.47843×10^{-7}	
7.	$\int_0^1 f_7(x)dx$	**5:**1.34147	**4:** 1.34147	**4:** 1.34147	1.34147
	Error:	2.46065×10^{-6}	1.26192×10^{-7}	1.88891×10^{-7}	
8.	$\int_0^1 f_8(x)dx$	**8:** 0.62977	**4:** 0.629773	**4:** 0.629762	0.629769
	Error:	1.18074×10^{-6}	4.86274×10^{-6}	6.3567×10^{-6}	

Table: 1

$$\begin{aligned}
f_1(x) &= x^2+x+2, \qquad f_2(x)=x\sin x, \\
f_3(x) &= e^x \sin x\ , \qquad f_4(x)=x^2+\sin x, \\
f_5(x) &= e^{x^2}, \qquad f_6(x)=e^x\cos\left(e^x-2x\right), \\
f_7(x) &= x+\cos x, \qquad f_8(x)=\log\left(x^2+2\right)\sin\left[\log\left(x^2+2\right)\right].
\end{aligned} \tag{8}$$

We conclude that all three quadrature rules show exact value of the integral of f_1 for $n=2$. For any polynomial of degree k, $n=k+1$ will give exact value of the integral f_1. Acceptable error estimates can be obtained for smaller values of n to save computational time.

In general $Q_{n,2}(f)$ gave better results as compared to other two quadrature rules for much smaller values of n. Therefore we can conclude that overall $Q_{n,2}(f)$ is computationally more efficient both in terms of error approximation, simplicity, and time. As a rough estimate we integrated $\log\left(x^2+2\right)\sin\left[\log\left(x^2+2\right)\right]$ using the built in algorithms of Mathematica 8.0 which took 26.30 seconds to give its approximate answer. To obtain similar approximation for the integral of f_8, $Q_{n,2}(f)$ took less than a second.

Based on this analysis, we can conjecture that $Q_{n,2}(f)$ is the most efficient quadrature rule. It should be noted that if desired the value of n can be adjusted to improve the error bounds or decrease computational time.

References

[1] Alomari, M. W., (2011), A generalization of weighted companion of Ostrowski integral inequality for mappings of bounded variation, RGMIA Research Report Collection, 14, Article 87, 11 pp.

[2] Alomari, M. W. and Latif, M.A.,(2011), Weighted companion for the Ostrowski and the generalized trapezoid inequalities for mappings of bounded variation, RGMIA Research Report Collection, 14, Article 92, 10 pp.

[3] Alomari, M.W. and Dragomir, S.S.,(2013), Mercer-Trapezoid rule for the Riemann–Stieltjes integral with applications Journal of Advances in Mathematics, 2 (2), pp. 67–85.

[4] Alshanti, W. G. and Qayyum, A.,(2017), A note on new Ostrowski type inequalities using a generalized kernel, Bulletin of Mathematical Analysis and Applications, 9(1), pp.1-18.

[5] Budak, H. and Sarıkaya, M.Z.,(2017), On generalization of Dragomir's inequalities, Turkish Journal of Analysis and Number Theory, 5(5), pp. 191-196.

[6] Budak, H. and Sarıkaya, M.Z.,(2016), New weighted Ostrowski type inequalities for mappings with first derivatives of bounded variation, TJMM, 8(1), pp. 21-27.

[7] Budak, H. and Sarıkaya, M.Z.,(2018), A new generalization of Ostrowski type inequality for mappings of bounded variation, Lobachevskii Journal of Mathematics, 39(9), pp. 1320–1326.

[8] Budak, H. and Sarıkaya, M.Z.,(2018), On generalization of weighted Ostrowski type inequalities for functions of bounded variation, Asian-European Journal of Mathematics, 11(4), 1850049 (11 pages).

[9] Budak, H. and Sarıkaya, M.Z.,(2016), A new Ostrowski type inequality for functions whose first derivatives are of bounded variation, Moroccan Journal of Pure and Applied Analysis, 2(1), pp. 1-11.

[10] Budak, H. and Sarıkaya, M.Z.,(2017) A companion of Ostrowski type inequalities for mappings of bounded variation and some applications,Transactions of A. Razmadze Mathematical Institute 171, pp. 136-143.

[11] Budak, H., Sarikaya M.Z., and Qayyum, A., (2017),Improvement in companion of Ostrowski type inequalities for mappings whose first derivatives are of bounded variation and application, Filomat, 31(16), pp. 5305–5314.

[12] Dragomir, S. S., (1999), The Ostrowski integral inequality for mappings of bounded variation, Bulletin of the Australian Mathematical Society, 60(1), pp. 495-508.

[13] Dragomir, S. S., (2001), On the Ostrowski's integral inequality for mappings with bounded variation and applications, Mathematical Inequalities & Applications, 4(1), pp.59–66.

[14] Dragomir, S. S., (2014), A companion of Ostrowski's inequality for functions of bounded variation and applications, International Journal of Nonlinear Analysis and Applications, 5(1), pp.89-97.

[15] Dragomir, S. S., (2015), Some perturbed Ostrowski type inequalities for functions of bounded variation, Asian-European J. Math.,8, 1550069, (14 pages).
[16] Dragomir, S. S., (2008), Approximating real functions which possess nth derivatives of bounded variation and applications, Computers and Mathematics with Applications, 56, pp 2268–2278.
[17] Liu, W. and Sun, Y., (2012),A Refinement of the Companion of Ostrowski inequality for functions of bounded variation and Applications, arXiv:1207.3861v1,
[18] Liu, Z., (2008),Some Ostrowski type inequalities, Mathematical and Computer Modelling, 48, pp. 949–960.
[19] Ostrowski, A. M., (1938), Über die absolutabweichung einer differentiebaren funktion von ihrem integralmitelwert, Comment. Math. Helv. 10, pp. 226-227.
[20] Qayyum, A., Shoaib, M., Faye, I., (2016), A companion of Ostrowski type integral inequality using a 5-step kernel with some applications, Filomat, 30(13), pp. 3601–3614.
[21] Qayyum, A., Shoaib, M., Faye, I., (2016), Companion of Ostrowski-type inequality based on 5-step quadratic kernel and applications, Journal of Nonlinear Sciences and Applications, 9, pp. 537-552.
[22] Qayyum, A., Shoaib, M., Faye, I., (2017), On new refinements and applications of efficient quadrature rules using n-times differentiable mappings, J. Computational Analysis and Applications, 3(4), pp. 723-739.

H. Budak for the photography and short autobiography, see TWMS J. App. and Eng. Math., V.9, N.4, 2019.

M. Z. Sarıkaya for the photography and short autobiography, see TWMS J. App. and Eng. Math., V.10, N.2, 2020.

Ather QAYYUM obtained his MS degree from B. Z. University, Pakistan. Currently, he has been a lecturer of mathematics in the University of Hail, KSA since 2008. He has 21 published research papers. His field of interest is Inequalities.