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THE MESHLESS METHODS FOR NUMERICAL SOLUTION OF THE NONLINEAR KLEIN-GORDON EQUATION

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Abstract. In this paper, we develop the numerical solution of nonlinear Klein-Gordon equation (NKGE) using the meshless methods. The finite difference scheme and the radial basis functions (RBFs) collocation methods are used to discretize time derivative and spatial derivatives, respectively. Numerical results are given to confirm the accuracy and efficiency of the presented schemes.

Keywords: Klein-Gordon equation; RBF; RBF-PS; Collocation method

AMS Subject Classification: 65M06, 65M12, 65M70, 65M20.

1. Introduction

Nonlinear phenomena has many application such as solid state physics, plasma physics, fluid dynamics, mathematical biology and chemical kinetics. This kind of phenomena can be prescribed by partial differential equations. For solving such problems, a wide class of analytical solution methods and numerical solution methods have used. The nolinear Klein-Gordon equation (NKGE) is one of the most fundamental nonlinear equation.

In this paper, we consider NKGE in the following form:

$$\frac{\partial^2 u(x,y,t)}{\partial t^2} - \frac{\partial^2 u(x,y,t)}{\partial x^2} - \frac{\partial^2 u(x,y,t)}{\partial y^2} + \alpha u(x,y,t) + \beta u^3(x,y,t) = f(x,y,t), \qquad (1)$$
$$(x,y) \in \Omega \subset \mathbb{R}^2, \quad 0 < t \leq T,$$

with initial conditions:

$$u(x,y,0) = g_1(x,y), \quad \frac{\partial u(x,y,t)}{\partial t}|_{t=0} = g_2(x,y), \quad x,y \in \Omega \cup \partial\Omega,$$

and boundry condition

$$u(x,y,t) = \psi(t), \quad x,y \in \partial\Omega, \quad t \in [0,T],$$

where α and β are constant.

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Many of scientific fields like solid state physics, the nonlinear optics and quantum field theory can be described by the NKGE [30]. As mentioned in [30], to investigate aforementioned kinds of equations, there exist several powerful methods such as the inverse scattering method, Bcklund transformation, the auxiliary equation method, the Wadati trace method, Jacobi elliptic functions, pseudo-spectral method, the tanhsech method, the Riccati equation expansion method, Hirota bilinear forms and the sine-cosine method [28, 27]. The NKGE has been recognized to demonstrate phenomena in the mathematical physics [30, 29, 8]. This equation has been extensively condensed matter physics that investigate the interaction of solitons in a collisionless plasma, the recurrence of initial states, and in examining the nonlinear wave equations [2, 7]. Many researchers proposed some numerical algorithms for the NKGE. The Cubic B-spline collocation approach for numerical solution of NKGE proposed by [23]. The authors of [25] solved the mention equation by using Tension spline approach. In [21], the B-spline collocation approach is presented for solution of the NKGE. The stability analysis of the finite difference schemes are discussed in [18]. In explicit finite difference procedures, we need large amount of CPU time, its limit the applicability of these methods [3]. In addition, the implicit finite difference schemes develop the solution of the problem only on mesh points. Also, the accuracy of these methods is reduced in non-smooth and non-regular domains. Recently, the spectral and pseudo-spectral methods have been given in [1]. In [6], a numerical scheme is presented to solve the one-dimensional NKGE using radial basis functions (RBFs). In [5], a fourth-order compact method is proposed for solving the one-dimensional NKGE. The authors [16] adopted two numerical techniques based on the finite difference and collocation methods for the solution of the NKGE. In [4], the dual reciprocity boundary integral equation and boundary element methods are applied to obtain approximate solutions of the NKGE. The semi-analytical technique has been developed in [26] to obtain numerical solution of the NKGE.

A meshless method is a method used to establish system of algebraic equations for the whole problem domain without the use of a predefined mesh for the domain discretization. In recent years , researchers have used meshless techniques for avoiding the mesh generation. The main advantage of mesh free method is the use a set of scattered nodes instead of meshing the domain of the problem. Meshless schemes such as the element free Galerkin method, the reproducing kernel particle, the local point interpolation,etc have been discussed [17]. In the past two decades, the RBFs interpolation methods are theoretically spectrally accurate for the scattered data interpolation problem. The RBFs as a meshless method based on the collocation scheme have attracted the attention numerical solution of partial differential equations (PDEs) [15, 32]. The main advantage of numerical not need to evaluate any integral for the collocation technique and have the meshless property. Many researchers in engineering and science consider the development of the radial basis functions as a truly meshless method for approximating the solutions of PDEs. Kansa's method as the domain-type meshless method is obtained by directly collocating the RBFs [15]. This method was developed to approximate various ordinary and partial differential equations including the one-dimensional nonlinear Burgers equation [12] with shock wave, shallow water equations for tide and currents simulation [13] and free boundary problems [14, 19]. Hermite-type collocation method is the modified Kansa's method that implement for the solvability of the resultant collocation matrix [9]. The disadvantage of radial basis functions is a full resultant coefficient matrix that become due to ill-conditioning of the coefficient matrix For improving this ill-conditioning problem, a new category of compactly supported RBFs were constructed by [31]. The theoretical proof in using the RBFs for the numerical solutions of PDEs is presented in [10, 11]. The RBFs with those advanced approachs of domain decomposition, multigrid, Schwarz iterative schemes, and preconditioning in the FEM arrangement are described [14]. In [22, 24], the authors enhanced the eigenfunction expansion approach for evaluating Gaussian RBF interpolants by taking advantage of the orthogonality of the eigenfunctions which are based on Hermite polynomials and using their zeros for interpolating functions and collocating boundary value problems in one and two dimensional problems. In [20], the nonlinear Korteweg-de Vries-Benjamin-Bona-Mahony-Burgers is approximated by means of the meshless methods based the RBF.

In this paper, we use the meshless collocation method based on radial basis functions for solving NKGE. The struture of this current research work is as follows. In Section 2, the aforesaid equation is discretized in temporal direction. The proposed numerical schemes for the two dimensional NKGE based on RBFs meshless method are described in Section 3. In Section 4, we report the numerical results of solving two test problems with the proposed methods. Finally a brief conclusion is given Section 5.

2. Time derivative approximation

For descretization of time direction, we need some preliliminary. Let us consider

$$t_k = k\delta t,\ \ k = 0, 1, \ldots, M,$$

where $\delta t = \frac{T}{M}$ is the step size of time direction. Then, the second-order derivative of the time direction is approximated by using central finite difference formula. Now, we consider Eq. (1) in point (x, y, t_n)

$$\frac{u^{n-1} - 2u^n + u^{n+1}}{\delta t^2} - \frac{\partial^2 u^{n+1}(x,y)}{\partial x^2} - \frac{\partial^2 u^{n+1}(x,y)}{\partial y^2} + \alpha u^{n+1}(x,y) + \beta(u^n(x,y))^3 = f^{n+1}(x,y), \tag{2}$$

where $u^{n+1}(x, y) = u(x, y, t_{n+1}),\quad f^{n+1}(x, y) = f(x, y, t_{n+1})$.
Since u^3 is the first order continuous derivative, one obtains

$$u^3(x, y, t_{n+1}) = u^3(x, y, t_n) + O(\delta t).$$

By simplifying (2) leads to

$$u^{n+1} - \delta t^2 \frac{\partial^2 u^{n+1}}{\partial x^2} - \delta t^2 \frac{\partial^2 u^{n+1}}{\partial y^2} + \delta t^2 \alpha u^{n+1} = 2u^n - u^{n-1} - \delta t^2 \beta (u^n)^3 + \delta t^2 f^{n+1}, \qquad n \geq 0. \tag{3}$$

Now, by inserting $n = 0$ in (3), we have

$$u^1 - \delta t^2 \frac{\partial^2 u^1}{\partial x^2} - \delta t^2 \frac{\partial^2 u^1}{\partial y^2} + \delta t^2 \alpha u^1 = 2u^0 - u^{-1} - \delta t^2 \beta (u^0)^3 + \delta t^2 f^1, \tag{4}$$

and by using relation

$$\frac{u^1 - u^{-1}}{2\delta t} = \frac{\partial u(x, y, 0)}{\partial t} = g_2(x, y),$$

we conclude that

$$u^{-1} = u^1 - 2\delta t g_2(x, y).$$

By substituting this relation in Eq. (4), we obtain

$$(2 + \delta t^2 \alpha) u^1 - \delta t^2 \frac{\partial^2 u^1}{\partial x^2} - \delta t^2 \frac{\partial^2 u^1}{\partial y^2} = 2u^0 + 2\delta t g_2 - \delta t^2 \beta (u^0)^3 + \delta t^2 f^1, \qquad n = 0. \tag{5}$$

After simplification, Eq. (3) can be rewritten for $n \geq 1$

$$(\delta t^2 \alpha + 1) u^{n+1} - \delta t^2 \frac{\partial^2 u^{n+1}}{\partial x^2} - \delta t^2 \frac{\partial^2 u^1}{\partial y^2} = 2u^n - u^{n-1} - \delta t^2 \beta (u^n)^3 + \delta t^2 f^{n+1},\ n \geq 1. \tag{6}$$

3. Description of numerical methods

In this section, we implement the meshless methods in the solution of the time NKGE. The aim of section to demonstrate that the meshless methods based on the MQ-RBFs using collocation method and RBF-Pseudospectral (RBF-PS) method.

3.1. **The derivation of RBF meshless method.** In this study, Ω is an arbitrary domain in $\mathbb{R}^k$. Let us consider that $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$; $N \geq 2$, then the approximate solution of $u(\mathbf{x}_i, t_n)$ is in the following form:

$$u(\mathbf{x}_i, t_n) = \sum_{j=1}^{N} {\lambda_j}^n \varphi(r_{ij}), \tag{7}$$

where

$$\varphi(r_{ij}) = \sqrt{(\mathbf{x}_i - \mathbf{x}_j)^2 + c^2} = \sqrt{r^2 + c^2}.$$

According to Kansa's method, we assume that $\{\mathbf{x}_i\}_{i=1}^{N}$ be N collocation points in Ω in which $\{\mathbf{x}_i\}_{i=1}^{N_I}$ are interior points and $\{\mathbf{x}_i\}_{i=N_I+1}^{N}$ are boundary points. For each point $\mathbf{x}_i$, let us define

$$\varphi_j(\mathbf{x}) = \sqrt{(\mathbf{x} - \mathbf{x}_j)^2 + c^2}.$$

Now, by substituting (7) into (5) and (6) in matrix form, we obtain

$$A\lambda^{n+1} = B^{n+1}, \qquad n = 0, 1, \ldots, M-1,$$

where

$$A = a_{ij} = \begin{cases} \ell(\varphi(r_{i,j})) & 1 \leq i \leq N_I,\ 1 \leq j \leq N, \\ \varphi(r_{i,j}) & N_I + 1 \leq i \leq N,\ 1 \leq j \leq N, \end{cases} \tag{8}$$

with

$$\ell(\varphi(r_{ij})) = \begin{cases} (2 + \delta t^2 \alpha)\varphi_j(\mathbf{x}) - \delta t^2 \Delta\varphi_j(\mathbf{x})|_{\mathbf{x}=\mathbf{x}_i} \ n = 0, \\ (1 + \delta t^2 \alpha)\varphi_j(\mathbf{x}) - \delta t^2 \Delta\varphi_j(\mathbf{x})|_{\mathbf{x}=\mathbf{x}_i} \ n \geq 1, \end{cases}$$

and

$$\lambda^{n+1} = (\lambda_1^{n+1}, \lambda_2^{n+1}, \ldots, \lambda_N^{n+1})^T,\ B^{n+1} = (b_1^{n+1}, b_2^{n+1}, \ldots, b_N^{n+1})^T,$$

in which

$$b_i^1 = 2g_1(\mathbf{x}) + 2\delta t g_2(\mathbf{x}) - \delta t^2 \beta (g_1(\mathbf{x}))^3 + \delta t^2 f^1, \qquad 1 \leq i \leq N_I + 1,\ n = 0,$$

$$b_i^{n+1} = 2\sum_{j=1}^{N} \lambda_j^n \varphi(r_{ij}) - \sum_{j=1}^{N} \lambda_j^{n-1} \varphi(r_{ij}) - \delta t^2 \beta (\sum_{j=1}^{N} \lambda_j^n \varphi(r_{ij}))^3 + \delta t^2 f^{n+1},$$

$$1 \leq i \leq N_I\ , 0 \leq n \leq M-1,$$

and

$$b_i^{n+1} = \psi(x_i, y_i, t_{n+1}), \qquad 0 \leq n \leq M-1,\ N_I + 1 \leq i \leq N.$$

3.2. **The implementation of RBF-PS meshless method.** First of all, we review the properties of differentiation matrices (DM). Let us assume that $\phi_j,\ j = 1, 2, \ldots, N$, be an arbitrary linearly independent set of smooth functions that will apply as the basis for our investigation space and $\mathbf{x} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$ be a set of distinct points in $\Omega \subseteq \mathbb{R}^s$. We suppose that the approximate expansion is as follows:

$$u^h(\mathbf{x}) = \sum_{j=1}^{N} \lambda_j \phi_j(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}, \tag{9}$$

where $h = h_{\mathbf{x},\Omega} := \sup_{\mathbf{x}\in\Omega} \min_{1\le j\le N} \|\mathbf{x} - \mathbf{x}_j\|_2$. Collocating Eq (9) at the grid points $\mathbf{x}_i$, we get

$$u^h(\mathbf{x}_i) = \sum_{j=1}^{N} \lambda_j \phi_j(\mathbf{x}_i), \quad i = 1, 2, \ldots, N, \tag{10}$$

then we gain the following matrix-vector form:

$$\mathbf{u} = \mathbf{A}\lambda, \tag{11}$$

where

$$\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_N]^T,$$

and $\mathbf{A}$ is the evaluation matrix with entries $A_{ij} = \phi_j(\mathbf{x}_i)$

$$\mathbf{u} = [u^h(\mathbf{x}_1), u^h(\mathbf{x}_2), \ldots, u^h(\mathbf{x}_N)]^T.$$

We can obtain the derivative of u^h by differentiating the basis function in (9)

$$\frac{\partial u^h(\mathbf{x})}{\partial x} = \sum_{j=1}^{N} \lambda_j \frac{\partial \phi_j(\mathbf{x})}{\partial x}, \tag{12}$$

$$\frac{\partial u^h(\mathbf{x})}{\partial y} = \sum_{j=1}^{N} \lambda_j \frac{\partial \phi_j(\mathbf{x})}{\partial y}. \tag{13}$$

Now, collocating Eqs. (12) and (13) at the grid points $\mathbf{x}_i$ in the form matrix, we get

$$\mathbf{u}_x = \mathbf{A}_x\lambda, \qquad \mathbf{u}_y = \mathbf{A}_y\lambda, \tag{14}$$

where the entries of matrices $\mathbf{A}_x$ and $\mathbf{A}_y$ are given by $\frac{\partial \phi_j(\mathbf{x})}{\partial x}$ and $\frac{\partial \phi_j(\mathbf{x})}{\partial y}$, respectively. we require to ensure invertibility of the evaluation of matrix $\mathbf{A}$ for obtaining the differentiation matrix $\mathbf{D}$. This relies on both the basis functions selected and the location of the grid points $\mathbf{x_i}$. According to Bochner's theorem, the invertibility of the matrix $\mathbf{A}$ for any set of distinct grid points $\mathbf{x_i}$ is insured by using the positive definite radial basis functions. Now using (11)

$$\mathbf{u} = \mathbf{A}\lambda \implies \lambda = \mathbf{A}^{-1}\mathbf{u}.$$

Considering Eq.(14) and the above result, we have

$$\mathbf{u}_x = \mathbf{A}_x\mathbf{A}^{-1}\mathbf{u},$$

and

$$\mathbf{u}_y = \mathbf{A}_y\mathbf{A}^{-1}\mathbf{u}.$$

Now, the approximate solution can be rewritten by considering Eqs. (6) and (7)

$$u^{n+1}(\mathbf{x}_i) = \sum_{j=1}^{N} \lambda_j \Phi\left(\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + c^2}\right) = \sum_{j=1}^{N} \lambda_j \Phi(r_{ij}), \quad i = 1, 2, \ldots, N. \tag{15}$$

Then, the matrix-vector from Eq (15) is follows as:

$$\mathbf{u}^{n+1} = \mathbf{A}\Lambda, \tag{16}$$

where

$$\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)^T, \ \mathbf{u}^{n+1} = (u_1^{n+1}, u_2^{n+1}, \ldots, u_N^{n+1})^T,$$

and the elements of matrix $\mathbf{A}$ are $A_{ij} = \varphi(r_{ij})$. The following matrix-vector form is achieved by differentiating Eq. (15) with respect to c and evaluating it at the grid points (x_i, y_i) :

$$\mathbf{u}_{xx}^{n+1} = \mathbf{A}_{xx}\Lambda, \tag{17}$$

where

$$\mathbf{u}_{xx}^{n+1} = \left(\frac{\partial^2 u_1^{n+1}}{\partial x^2}, \frac{\partial^2 u_2^{n+1}}{\partial x^2}, \ldots, \frac{\partial^2 u_N^{n+1}}{\partial x^2}\right)^T,$$

and elements of matrix $\mathbf{A}_{xx}$ are $a_{xx,ij} = \frac{\partial^2 \varphi(\|\mathbf{x}_i - \mathbf{x}_j\|)}{\partial x^2}$. Now, from Eq. (16) we have:

$$\Lambda = \mathbf{A}^{-1}\mathbf{u}^{n+1},$$

and (17) yields

$$\mathbf{u}_{xx}^{n+1} = \mathbf{A}_{xx}\mathbf{A}^{-1}\mathbf{u}^{n+1}. \tag{18}$$

Similarly for y direction, we can obtain:

$$\mathbf{u}_{yy}^{n+1} = \mathbf{A}_{yy}\mathbf{A}^{-1}\mathbf{u}^{n+1}, \tag{19}$$

where the elements of matrix $\mathbf{A}_{yy}$ are given by

$$a_{yy,ij} = \frac{\partial^2 \varphi(\|\mathbf{x}_i - \mathbf{x}_j\|)}{\partial y^2}.$$

Now, by substituting Eqs. (18) and (19) in the Eqs. (5) and (6), we obtain

$$(2 + \delta t^2 \alpha)\mathbf{u}^1 - \delta t^2 \mathbf{A}_{xx}\mathbf{A}^{-1}u^1 - \delta t^2 \mathbf{A}_{yy}\mathbf{A}^{-1}\mathbf{u}^1 = 2\mathbf{u}^0 + 2\delta t g_2 - \delta t^2 \beta (u^0)^3 + \delta t^2 f^1, \quad n = 0,$$

and

$$(\delta t^2 \alpha + 1)\mathbf{u}^{n+1} - \delta t^2 \mathbf{A}_{xx}\mathbf{A}^{-1}\mathbf{u}^{n+1} - \delta t^2 \mathbf{A}_{yy}\mathbf{A}^{-1}\mathbf{u}^{n+1} = 2\mathbf{u}^n - \mathbf{u}^{n-1} - \delta t^2 \beta (u^n)^3 + \delta t^2 f^{n+1},$$

for $n \geq 1$. The above mentioned relations can be restated in the following form:

$$\begin{aligned} \mathbf{D}_1\mathbf{u}^1 &= 2\mathbf{u}^0 + 2\delta t g_2 - \delta t^2 \beta (u^0)^3 + \delta t^2 f^1, && n = 0, \\ \mathbf{D}_2\mathbf{u}^{n+1} &= -2\mathbf{u}^n + \mathbf{u}^{n-1} - \delta t^2 \beta (u^n)^3 + \delta t^2 f^{n+1}, && n \geq 1, \end{aligned}$$

where

$$\begin{aligned} \mathbf{D}_1 &= (2 + \delta t^2 \alpha)\mathbf{I} - \delta t^2 \mathbf{A}_{xx}\mathbf{A}^{-1} - \delta t^2 \mathbf{A}_{yy}\mathbf{A}^{-1}, \\ \mathbf{D}_2 &= (\delta t^2 \alpha + 1)\mathbf{I} - \delta t^2 \mathbf{A}_{xx}\mathbf{A}^{-1} - \delta t^2 \mathbf{A}_{yy}\mathbf{A}^{-1}, \end{aligned}$$

where $\mathbf{I}$ is the identity matrix.

By solving this nonlinear system we can obtain the numerical solution at each time levels.

4. Numerical Illustration

This section presents the above-mentioned numerical schemes for the NKGE. In this paper we compute the following maximum error norm:

$$L_\infty = \max_{1\leq j\leq N-1} |u(\mathbf{x}_j, T) - U(\mathbf{x}_j, T)|.$$

The computational order of the methods is evaluated in this paper by:

$$C - \text{order} = \frac{log(\frac{E_1}{E_2})}{log(\frac{\delta t_1}{\delta t_2})}.$$

Example 4.1. *Let us consider the following NKGE:*

$$\frac{\partial^2 u(x,y,t)}{\partial t^2} - \frac{\partial^2 u(x,y,t)}{\partial x^2} - \frac{\partial^2 u(x,y,t)}{\partial y^2} + u(x,y,t) + u^3(x,y,t) = \cos^3(x+y+t) + 2\cos(x+y+t),$$
$$(x,y) \in \Omega, \quad 0 \leq t \leq T,$$

with initial conditions:

$$u(x,y,0) = \cos(x+y), \quad \frac{\partial u(x,y,t)}{\partial t}|_{t=0} = -\sin(x+y).$$

The exact solution of this example is $u(x,y,t) = \cos(x+y+t)$.
We solve this problem with the presented methods with several values of h and δt on $\Omega = [-1,1]\times[-1,1]$ at final time T. Table 1 lists the errors and computational orders obtained for the present methods with $h = \frac{1}{10}$ on Ω at $T = 1$ for Example 4.1. Table 2 reports the errors and the condition number obtained for the present methods $\delta t = \frac{1}{100}$ on Ω at $T = 1$ for Example 4.1. In this table the error obtained by RBF collocation technique is close to the error obtained by RBF-PS collocation method but the coefficient matrix RBF-PS collocation technique is more well-posed than the coefficient matrix of the RBF collocation method. For example the condition number of the coefficient matrices of MQ-RBF and RBF-PS collocation schemes for $h = \dfrac{1}{25}$ is $2.340e + 08$ and 5.31, respectively. Figure 1 depicts the graphs of the approximate solution and resulting error with $h = \frac{1}{10}, \delta t = \frac{1}{100}$ on Ω at $T = 1, 3$. Figure 2 shows the error of RBF meshless method on the square domain $[0,1]^2$ using irregular distribution of collocation points.

Table 1. Errors and computational order obtained for present methods with $h = \frac{1}{10}$ on Ω at $T = 1$ for Example 4.1

δt	MQ-RBF($c = 0.65$)		MQ-RBF-PS ($c = 0.65$)	
	L_∞	C-order	L_∞	C-order
1/10	1.702×10^{-2}	–	1.964×10^{-2}	–
1/20	8.502×10^{-3}	1.0014	8.645×10^{-3}	1.1839
1/40	4.019×10^{-3}	1.0808	3.735×10^{-3}	1.2108
1/80	1.842×10^{-3}	1.1255	1.482×10^{-3}	1.3336
1/160	8.319×10^{-4}	1.1467	5.825×10^{-4}	1.3472
1/320	3.546×10^{-4}	1.2302	2.274×10^{-4}	1.3570
1/640	1.127×10^{-4}	1.6537	8.806×10^{-5}	1.3678
1/1280	2.909×10^{-5}	1.9536	3.406×10^{-5}	1.3704

Table 2. Errors and condition number obtained for present methods with $\delta t = 1/100$ on Ω at $T = 1$ for Example 4.1

h	MQ-RBF($c = 0.65$)		MQ-RBF-PS ($c = 0.65$)	
	L_∞	Cond (A)	L_∞	Cond(A)
1/5	8.268×10^{-4}	$2.393e + 15$	8.527×10^{-4}	1.37
1/10	8.135×10^{-4}	$3.593e + 08$	8.654×10^{-4}	1.62
1/15	7.934×10^{-4}	$5.270e + 05$	8.103×10^{-4}	2.22
1/20	8.081×10^{-4}	$1.863e + 87$	8.153×10^{-4}	3.46
1/25	7.936×10^{-4}	$2.340e + 08$	7.905×10^{-4}	5.31

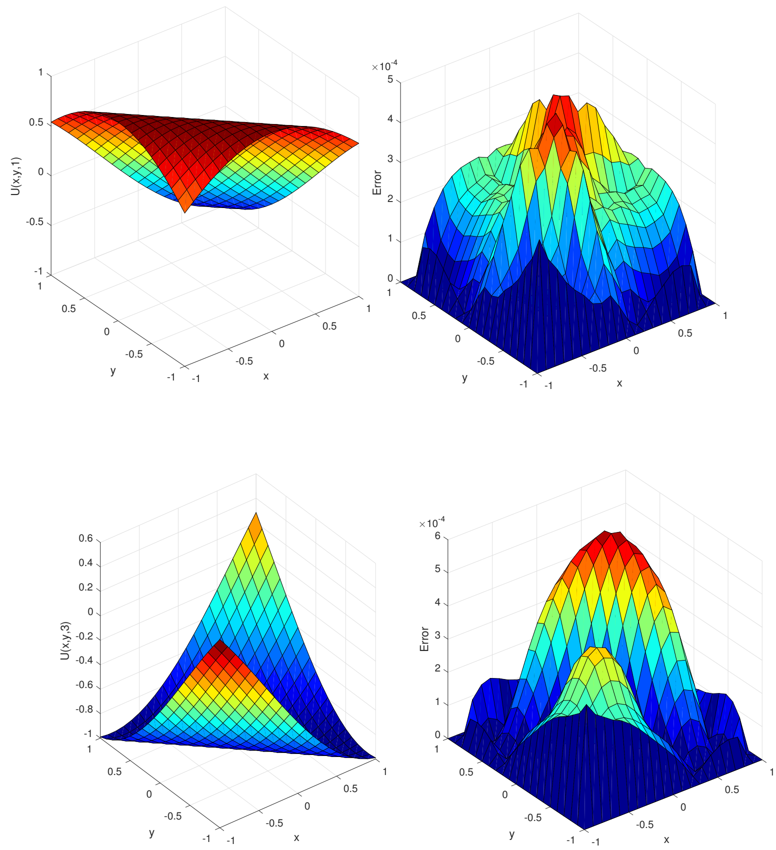

Figure 1. Graph of error in the Example 4.1 using the MQ-RBF method with $h = \frac{1}{10}, \delta t = \frac{1}{100}$ on Ω at $T = 1, 3$

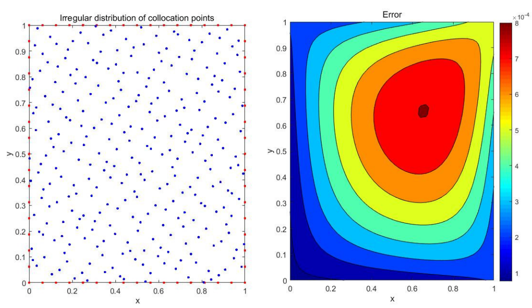

Figure 2. Plot of error (right panel) on the irregular distribution of collocation points (left panel) with $\delta t = 1/100, c = 0.5$ and $h = 1/10$ on the rectangular domain for Example 4.1

Example 4.2. *Now, we consider the following NKGE:*

$$\frac{\partial^2 u(x,y,t)}{\partial t^2} - \frac{\partial^2 u(x,y,t)}{\partial x^2} - \frac{\partial^2 u(x,y,t)}{\partial y^2} + u(x,y,t) + u^3(x,y,t) = \exp(3x+3y+3t),$$
$$(x,y) \in \Omega, \quad 0 \le t \le T,$$

with initial conditions:

$$u(x,y,0) = \exp(x+y), \qquad \frac{\partial u(x,y,t)}{\partial t}|_{t=0} = \exp(x+y).$$

This example has the exact solution $u(x,y,t) = \exp(x+y+t)$.
We solve this problem with the methods presented in this paper with several values of h and δt at the final times T on $\Omega = [-1,1] \times [-1,1]$. Table 3 displays the errors and computational orders obtained for the present methods with $h = \frac{1}{10}$ on Ω at $T = 1$ for Example 4.2. Table 4 shows the errors and the condition number obtained for the present methods $\delta t = 1/100$ on Ω at $T = 1$ for Example 4.2. In this table the error obtained by RBF collocation technique is close to the error obtained by RBF-PS collocation method but the coefficient matrix RBF-PS collocation technique is more well-posed than the coefficient matrix of the RBF collocation method. For example the condition number of the coefficient matrices of RBF and RBF-PS collocation schemes for $h = \frac{1}{25}$ is $2.902e + 19$ and 5.86, respectively. Figure 3 demonstrates the graphs of the approximate solution and resulting error with $h = \frac{1}{10}, \delta t = \frac{1}{100}$ on Ω at $T = 1, 3$. Figure 4 depicts the error of RBF meshless method on the on the circular domain using irregular distribution of collocation points.

TABLE 3. Errors and computational order obtained for present methods with $h = \frac{1}{10}$ on Ω at $T = 1$ for Example 4.2

δt	MQ-RBF($c = 0.088$)		MQ-RBF-PS ($c = 0.088$)	
	L_∞	C-order	L_∞	C-order
1/10	4.066×10^{-1}	–	4.103×10^{-1}	–
1/20	2.031×10^{-1}	1.0006	2.052×10^{-1}	0.9996
1/40	1.014×10^{-1}	1.0023	9.976×10^{-2}	1.0405
1/80	5.013×10^{-2}	1.0169	4.765×10^{-2}	1.0660
1/160	2.455×10^{-2}	1.0300	2.131×10^{-2}	1.1609
1/320	1.078×10^{-2}	1.1868	9.067×10^{-3}	1.2328
1/640	3.626×10^{-3}	1.5721	3.849×10^{-3}	1.2361
1/1280	1.285×10^{-4}	4.8185	1.605×10^{-4}	1.2619

TABLE 4. Errors and condition number obtained for present methods with $\delta t = 1/10$ on Ω at $T = 1$ for Example 4.2

h	MQ-RBF($c = 0.056$)		MQ-RBF-PS ($c = 0.056$)	
	L_∞	Cond (A)	L_∞	Cond(A)
1/5	2.007×10^{-4}	$1.241e + 16$	2.126×10^{-4}	1.32
1/10	7.918×10^{-3}	$4.551e + 14$	7.536×10^{-3}	4.91
1/15	4.218×10^{-2}	$1.024e + 13$	5.582×10^{-2}	3.46
1/20	5.532×10^{-2}	$1.479e + 13$	5.725×10^{-2}	4.12
1/25	3.342×10^{-2}	$2.902e + 19$	3.018×10^{-2}	5.86

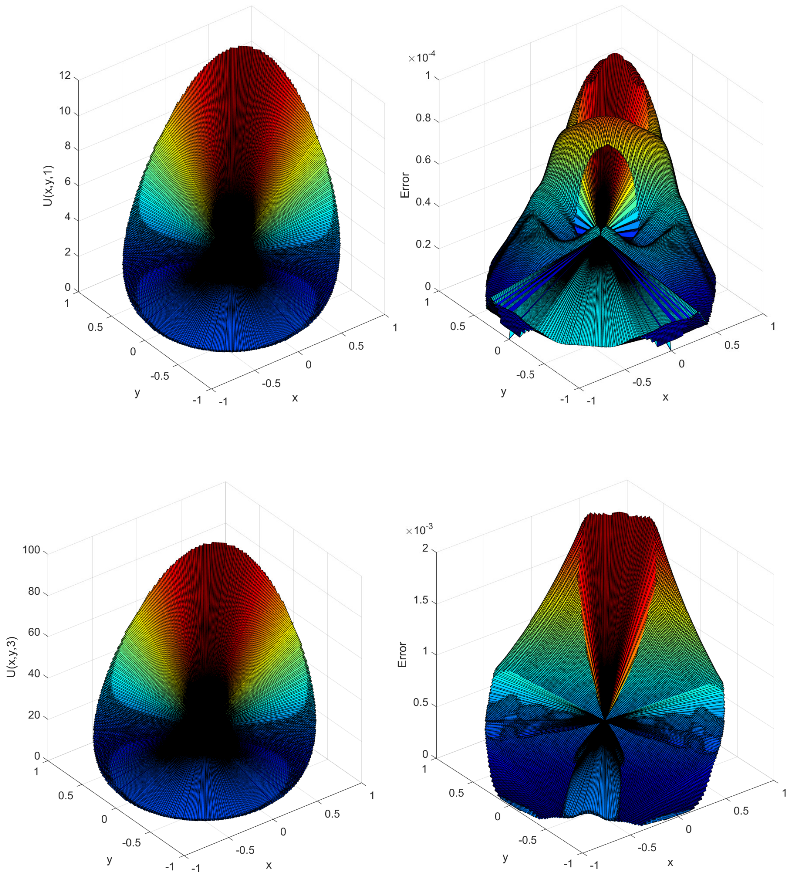

FIGURE 3. Graph of error in the Example 4.2 using the MQ-RBF method with $h = \frac{1}{10}, \delta t = \frac{1}{100}$ on Ω at $T = 1, 3$

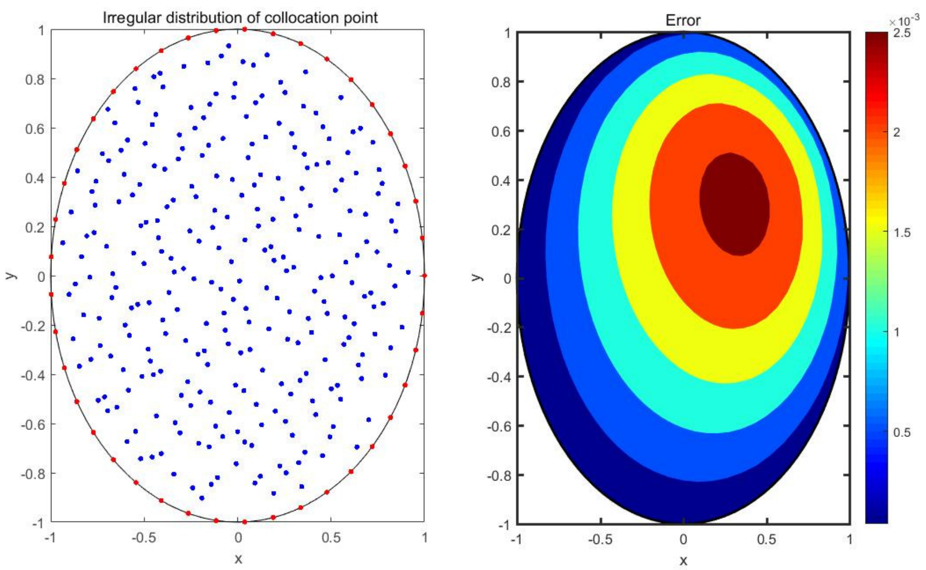

FIGURE 4. Plot of error (right panel) on the irregular distribution of collocation points (left panel) with $\delta t = 1/100, c = 0.5$ and $h = 1/10$ on the circular domain for Example 4.2

5. Conclusion

In this paper, we adopted the meshless methods for the solving of the NKGE. We showed that the meshless method is suitable for these problems. We conclude that the coefficient matrix RBF-PS collocation technique is more well-posed than the coefficient matrix of the MQ- RBF collocation method. Also, by considering ill-condition of coefficient matrix, we used the LU decomposition method for solving the linear system of algebraic equations which obtains from the process of collocating points. Comparing the numerical results with analytical solutions reveals the applicability and efficiency of the proposed approach.

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