

THE FEKETE-SZEGÖ PROBLEMS FOR A SUBCLASS OF M-FOLD SYMMETRIC BI-UNIVALENT FUNCTIONS

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ABSTRACT. In this paper, we investigate a new subclass $\mathcal{P}_{\Sigma_m}^{h,p}(\lambda, \gamma)$ of m -fold symmetric bi-univalent functions. Moreover, for functions of this subclass, we obtain the coefficient estimates of the Taylor-Maclaurin coefficients $|a_{m+1}|, |a_{2m+1}|$ and Fekete-Szegő problems. The coefficients estimates presented in this paper would generalize and improve those in related works of several earlier authors

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1. INTRODUCTION

Let \mathcal{A} denote the class of functions f which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, with in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1)$$

we let \mathcal{S} to denote the class of functions $f \in \mathcal{A}$ which are univalent in \mathbb{U} (see details [3, 5]).

Every function $f \in \mathcal{S}$ has an inverse f^{-1} , which is defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f), \quad r_0(f) \geq \frac{1}{4} \right).$$

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In fact, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in \mathbb{U} , if both f and f^{-1} are univalent in \mathbb{U} (see [17]). We denote $\sigma_{\mathcal{B}}$ the class of bi-univalent functions in \mathbb{U} given by (1).

Lewin [9] investigated the class $\sigma_{\mathcal{B}}$ of bi-univalent functions and showed that $|a_2| < 1.51$ for the Taylor-Maclaurin coefficient $|a_2|$ of functions belonging to $\sigma_{\mathcal{B}}$. Subsequently, Brannan et al. [2] conjectured that $|a_2| \leq \sqrt{2}$. However, finding upper bounds of the Taylor-Maclaurin coefficients $|a_n| (n \in \mathbb{N} - \{2, 3\})$ for each $f \in \sigma_{\mathcal{B}}$ is coefficient estimate problem and still an open problem.

For a brief history and interesting examples of functions in the class $\sigma_{\mathcal{B}}$, refer to the papers by Sirvastava et al. [13, 14, 23, 24].

For each function $f \in \mathcal{S}$ function

$$h(z) = \sqrt[m]{f(z^m)} \tag{3}$$

is univalent and maps unit disk \mathbb{U} into a region with m -fold symmetry. A function f is said to be m -fold symmetric (see [8, 10]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1}z^{mk+1} \quad (z \in \mathbb{U}, m \in \mathbb{N}). \tag{4}$$

We denote by \mathcal{S}_m the class of m -fold symmetric univalent functions in \mathbb{U} , which are normalized by the series expansion (4). In fact, the functions in class \mathcal{S} are one-fold symmetric.

In [18] Srivastava et al. defined m -fold symmetric bi-univalent functions analogues to the concept of m -fold symmetric univalent functions. They gave some important results, such as each function $f \in \sigma_{\mathcal{B}}$ generates an m -fold symmetric bi-univalent function for each $m \in \mathbb{N}$. Furthermore, for the normalized form of f given by (4), they obtained the series expansion for f^{-1} as follows:

$$f^{-1}(w) = w - a_{m+1}w^{m+1} + [(m + 1)a_{m+1}^2 - a_{2m+1}]w^{2m+1} - \left[\frac{1}{2}(m + 1)(3m + 2)a_{m+1}^3 - (3m + 2)a_{m+1}a_{2m+1} + a_{3m+1}\right]w^{3m+1} + \dots \tag{5}$$

We denote by Σ_m the class of m -fold symmetric bi-univalent functions in \mathbb{U} . For $m = 1$, formula (5) coincides with formula (2) of the class $\sigma_{\mathcal{B}}$. Some examples of m -fold symmetric bi-univalent functions are given as follows:

$$\left(\frac{z^m}{1 - z^m}\right)^{\frac{1}{m}}, \left[\frac{1}{2} \log\left(\frac{1 + z^m}{1 - z^m}\right)\right]^{\frac{1}{m}} \text{ and } [-\log(1 - z^m)]^{\frac{1}{m}}$$

with the corresponding inverse functions

$$\left(\frac{w^m}{1 + w^m}\right)^{\frac{1}{m}}, \left(\frac{e^{2w^m} - 1}{e^{2w^m} + 1}\right)^{\frac{1}{m}} \text{ and } \left(\frac{e^{w^m} - 1}{e^{w^m}}\right)^{\frac{1}{m}},$$

respectively.

In fact that this widely-cited work by Srivastava et al. [18] actually revived the study of m -fold symmetric bi-univalent functions in recent years and that it led to a flood of papers on the subject by (for example) Srivastava et al. [15, 16, 18, 19, 20], and others [6, 7, 11, 12, 21, 22].

The object of the present paper is to introduce new subclass $\mathcal{P}_{\Sigma_m}^{h,p}(\lambda, \gamma)$ of Σ_m and obtain estimates on initial coefficients $|a_{m+1}|$, $|a_{2m+1}|$ for functions in subclass and improve some recent works of many authors.

2. SUBCLASS $\mathcal{P}_{\Sigma_m}^{h,p}(\lambda, \gamma)$

In this section, we introduce the general subclass $\mathcal{P}_{\Sigma_m}^{h,p}(\lambda, \gamma)$.

Definition 2.1. Let the functions $h, p : \mathbb{U} \rightarrow \mathbb{C}$ be so constrained that

$$\min\{\operatorname{Re}((h(z)), \operatorname{Re}(p(z)))\} > 0 \quad (z \in \mathbb{U}) \text{ and } h(0) = p(0) = 1.$$

A function $f \in \Sigma_m$ given by (4) is said to be in the subclass $\mathcal{P}_{\Sigma_m}^{h,p}(\lambda, \gamma)$, if the following conditions are satisfied:

$$1 + \frac{1}{\gamma} \left(\frac{zf'(z)}{f(z)} + \lambda \frac{z^2 f''(z)}{f(z)} - 1 \right) \in h(\mathbb{U}) \quad (0 \leq \lambda \leq 1, \gamma \in \mathbb{C} - \{0\}, z \in \mathbb{U}) \quad (6)$$

and

$$1 + \frac{1}{\gamma} \left(\frac{wg'(w)}{g(w)} + \lambda \frac{w^2 g''(w)}{g(w)} - 1 \right) \in p(\mathbb{U}) \quad (0 < \lambda \leq 1, \gamma \in \mathbb{C} - \{0\}, w \in \mathbb{U}), \quad (7)$$

where g is the extension of f^{-1} to \mathbb{U} .

Remark 2.1. There are many selections of the functions $h(z)$ and $p(z)$ which would provide interesting classes of m -fold symmetric bi-univalent functions Σ_m . For example, if we let

$$h(z) = p(z) = \left(\frac{1+z^m}{1-z^m} \right)^\alpha = 1 + 2\alpha z^m + 2\alpha^2 z^{2m} + \dots \quad (0 < \alpha \leq 1),$$

it is easy to verify that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2.1.

If $f \in \mathcal{P}_{\Sigma_m}^{h,p}(\lambda, \gamma)$, then

$$\left| \arg \left\{ 1 + \frac{1}{\gamma} \left(\frac{zf'(z)}{f(z)} + \lambda \frac{z^2 f''(z)}{f(z)} - 1 \right) \right\} \right| < \frac{\alpha\pi}{2}$$

and

$$\left| \arg \left\{ 1 + \frac{1}{\gamma} \left(\frac{wg'(w)}{g(w)} + \lambda \frac{w^2 g''(w)}{g(w)} - 1 \right) \right\} \right| < \frac{\alpha\pi}{2}.$$

Therefore, for $h(z) = p(z) = \left(\frac{1+z^m}{1-z^m} \right)^\alpha$, $\gamma = 1$ and $\lambda = 0$, the subclass $\mathcal{P}_{\Sigma_m}^{h,p}(\lambda, \gamma)$ reduces to the subclass $\mathcal{S}_{\Sigma_m}^\alpha$ which was considered by Altinkaya and Yalcin [1].

If we let

$$h(z) = p(z) = \frac{1 + (1-2\beta)z^m}{1-z^m} = 1 + 2(1-\beta)z^m + 2(1-\beta)z^{2m} + \dots \quad (0 \leq \beta < 1),$$

it is easy to verify that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2.1.

If $f \in \mathcal{P}_{\Sigma_m}^{h,p}(\lambda, \gamma)$, then

$$\operatorname{Re} \left\{ 1 + \frac{1}{\gamma} \left(\frac{zf'(z)}{f(z)} + \lambda \frac{z^2 f''(z)}{f(z)} - 1 \right) \right\} > \beta$$

and

$$\operatorname{Re} \left\{ 1 + \frac{1}{\gamma} \left(\frac{wg'(w)}{g(w)} + \lambda \frac{w^2g''(w)}{g(w)} - 1 \right) \right\} > \beta.$$

Therefore, for $h(z) = p(z) = \frac{1+(1-2\beta)z^m}{1-z^m}$, $\gamma = 1$ and $\lambda = 0$, the subclass $\mathcal{P}_{\Sigma_m}^{h,p}(\lambda, \gamma)$ reduces to the subclass $\mathcal{S}_{\Sigma_m}^\beta$ which was considered by Altinkaya and Yalcin [1].

Remark 2.2. For one-fold symmetric bi-univalent functions, we denote the subclass $\mathcal{P}_{\Sigma_1}^{h,p}(\lambda, \gamma) = \mathcal{P}_{\Sigma}^{h,p}(\lambda, \gamma)$. Special cases of this subclass illustrated below:

- By putting $h(z) = p(z) = \left(\frac{1+z^m}{1-z^m}\right)^\alpha$, $\gamma = 1$ and $\lambda = 0$, then the subclass $\mathcal{P}_{\Sigma}^{h,p}(\lambda, \gamma)$ reduces to the subclass $\mathcal{S}_{\Sigma}^*[\alpha]$ of strongly bi-starlike functions of order α ($0 < \alpha \leq 1$).
- By putting $h(z) = p(z) = \frac{1+(1-2\beta)z^m}{1-z^m}$, $\gamma = 1$ and $\lambda = 0$, then the subclass $\mathcal{P}_{\Sigma}^{h,p}(\lambda, \gamma)$ reduces to the subclass $\mathcal{S}_{\Sigma}^*(\beta)$ of bi-starlike functions of order β ($0 \leq \beta < 1$).

Theorem 2.1. Let $f(z)$ given by (4) be in subclass $\mathcal{P}_{\Sigma_m}^{h,p}(\lambda, \gamma)$ ($0 \leq \lambda < 1, \gamma \in \mathbb{C} - \{0\}$). Then

$$|a_{m+1}| \leq \min \left\{ \frac{|\gamma||h_m|}{m[(1 + \lambda(m + 1))]}, \sqrt{\frac{|\gamma|(|h_{2m}| + |p_{2m}|)}{2m^2[1 + 2\lambda(m + 1)]}} \right\}$$

and

$$|a_{2m+1}| \leq \min \left\{ \frac{|\gamma|(|h_{2m}| + |p_{2m}|)}{4m[1 + \lambda(2m + 1)]} + \frac{|\gamma|(m + 1)(|h_m|^2 + |p_m|^2)}{4m^2[1 + \lambda(m + 1)]^2}, \frac{|\gamma|[(2m + 1) + \lambda(m + 1)(4m + 1)]|h_{2m}| + |\gamma|[\lambda(m + 1) + 1]|p_{2m}|}{4m^2(1 + \lambda(2m + 1))(1 + 2\lambda(m + 1))} \right\}.$$

Proof. First of all, we write the argument inequalities in (6) and (7) in their equivalent forms as follows:

$$1 + \frac{1}{\gamma} \left(\frac{zf'(z)}{f(z)} + \lambda \frac{z^2f''(z)}{f(z)} - 1 \right) = h(z) \quad (z \in \mathbb{U}) \tag{8}$$

and

$$1 + \frac{1}{\gamma} \left(\frac{wg'(w)}{g(w)} + \lambda \frac{w^2g''(w)}{g(w)} - 1 \right) = p(w) \quad (w \in \mathbb{U}), \tag{9}$$

respectively, where functions $h(z)$ and $p(w)$ satisfy the conditions of Definition 2.1.

Furthermore, the functions $h(z)$ and $p(w)$ have the forms:

$$h(z) = 1 + h_m z^m + h_{2m} z^{2m} + h_{3m} z^{3m} + \dots \tag{10}$$

and

$$p(w) = 1 + p_m w^m + p_{2m} w^{2m} + p_{3m} w^{3m} + \dots, \tag{11}$$

respectively.

Now, upon substituting from (10) and (11) into (8) and (9), respectively, and equating the coefficients, we get

$$\frac{m[1 + \lambda(m + 1)]a_{m+1}}{\gamma} = h_m, \quad (12)$$

$$\frac{2m[1 + \lambda(2m + 1)]}{\gamma} a_{2m+1} - \frac{m[1 + \lambda(m + 1)]}{\gamma} a_{m+1}^2 = h_{2m}, \quad (13)$$

$$-\frac{m[1 + \lambda(m + 1)]}{\gamma} a_{m+1} = p_m, \quad (14)$$

and

$$-\frac{2m[1 + \lambda(2m + 1)]}{\gamma} a_{2m+1} + \frac{m[(2m + 1) + \lambda(m + 1)(4m + 1)]}{\gamma} a_{m+1}^2 = p_{2m}. \quad (15)$$

From (12) and (14), we get

$$h_m = -p_m \quad (16)$$

and

$$a_{m+1}^2 = \frac{\gamma^2(h_m^2 + p_m^2)}{2m^2[1 + \lambda(m + 1)]^2}. \quad (17)$$

Adding (13) and (15), we get

$$a_{m+1}^2 = \frac{\gamma(h_{2m} + p_{2m})}{2m^2[1 + 2\lambda(m + 1)]}. \quad (18)$$

Therefore, we find from the equations (16), (17) and (18) that

$$|a_{m+1}| \leq \frac{|\gamma||h_m|}{m[(1 + \lambda(m + 1))]},$$

$$|a_{m+1}| \leq \sqrt{\frac{|\gamma|(|h_{2m}| + |p_{2m}|)}{2m^2[1 + 2\lambda(m + 1)]}}.$$

respectively. So we get the desired estimate on the coefficient $|a_{m+1}|$.

Next, in order to find the bound on the coefficient $|a_{2m+1}|$, we subtract (15) from (13), we get

$$a_{2m+1} = \frac{\gamma(h_{2m} - p_{2m})}{4m[1 + \lambda(2m + 1)]} + \frac{(m + 1)}{2} a_{m+1}^2. \quad (19)$$

Therefore, we find from (17) and (19) that

$$a_{2m+1} = \frac{\gamma(h_{2m} - p_{2m})}{4m[1 + \lambda(2m + 1)]} + \frac{\gamma^2(m + 1)(h_m^2 + p_m^2)}{4m^2[1 + \lambda(m + 1)]^2}. \quad (20)$$

Also, from (18) and (19), we have

$$a_{2m+1} = \frac{\gamma[(2m + 1) + \lambda(m + 1)(4m + 1)]h_{2m} + \gamma[\lambda(m + 1) + 1]p_{2m}}{4m^2(1 + \lambda(2m + 1))(1 + 2\lambda(m + 1))}. \quad (21)$$

So, from the equations (20) and (21), we get

$$|a_{2m+1}| \leq \frac{|\gamma|(|h_{2m}| + |p_{2m}|)}{4m[1 + \lambda(2m + 1)]} + \frac{|\gamma|(m + 1)(|h_m|^2 + |p_m|^2)}{4m^2[1 + \lambda(m + 1)]^2}$$

and

$$|a_{2m+1}| \leq \frac{|\gamma|[(2m + 1) + \lambda(m + 1)(4m + 1)]|h_{2m}| + |\gamma|[\lambda(m + 1) + 1]|p_{2m}|}{4m^2(1 + \lambda(2m + 1))(1 + 2\lambda(m + 1))}.$$

□

Theorem 2.2. Let $f(z)$ given by (4) be in subclass $\mathcal{P}_{\Sigma_m}^{h,p}(\lambda, \gamma)$ ($0 \leq \lambda < 1, \gamma \in \mathbb{C} - \{0\}$). Also let ρ be real number. Then

$$|a_{2m+1} - \rho a_{m+1}^2| \leq \begin{cases} \frac{|\gamma|}{4m(1+\lambda(2m+1))} \{(1 + T(\rho)) |h_{2m}| + (1 - T(\rho)) |p_{2m}|\}; & |T(\rho)| \leq 1 \\ \frac{|\gamma|}{4m(1+\lambda(2m+1))} \{|1 + T(\rho)| |h_{2m}| + |T(\rho) - 1| |p_{2m}|\}; & |T(\rho)| \geq 1. \end{cases}$$

where

$$T(\rho) = \frac{(m - 2\rho + 1)(1 + \lambda(2m + 1))}{m(1 + 2\lambda(m + 1))}.$$

Proof. From the equation (19), we get

$$a_{2m+1} - \rho a_{m+1}^2 = \frac{\gamma(h_{2m} - p_{2m})}{4m[1 + \lambda(2m + 1)]} + \frac{m - 2\rho + 1}{2} a_{m+1}^2. \tag{22}$$

From the equation (18) and (22), we have

$$a_{2m+1} - \rho a_{m+1}^2 = \frac{|\gamma|}{4m(1 + \lambda(2m + 1))} \left\{ \left[1 + \frac{(m - 2\rho + 1)(1 + \lambda(2m + 1))}{m(1 + 2\lambda(m + 1))} \right] h_{2m} + \left[\frac{(m - 2\rho + 1)(1 + \lambda(2m + 1))}{m(1 + 2\lambda(m + 1))} - 1 \right] p_{2m} \right\}.$$

Next, taking the absolute values we obtain

$$|a_{2m+1} - \rho a_{m+1}^2| \leq \frac{|\gamma|}{4m(1 + \lambda(2m + 1))} \left\{ \left| 1 + \frac{(m - 2\rho + 1)(1 + \lambda(2m + 1))}{m(1 + 2\lambda(m + 1))} \right| |h_{2m}| + \left| \frac{(m - 2\rho + 1)(1 + \lambda(2m + 1))}{m(1 + 2\lambda(m + 1))} - 1 \right| |p_{2m}| \right\}.$$

Then, we conclude that

$$|a_{2m+1} - \rho a_{m+1}^2| \leq \begin{cases} \frac{|\gamma|}{4m(1+\lambda(2m+1))} \{(1 + T(\rho)) |h_{2m}| + (1 - T(\rho)) |p_{2m}|\}; & |T(\rho)| \leq 1 \\ \frac{|\gamma|}{4m(1+\lambda(2m+1))} \{|1 + T(\rho)| |h_{2m}| + |T(\rho) - 1| |p_{2m}|\}; & |T(\rho)| \geq 1. \end{cases}$$

□

3. CONCLUSIONS

By putting

$$h(z) = p(z) = \left(\frac{1 + z^m}{1 - z^m} \right)^\alpha = 1 + 2\alpha z^m + 2\alpha^2 z^{2m} + \dots \quad (0 < \alpha \leq 1, z \in \mathbb{U}),$$

$\lambda = 0$ and $\gamma = 1$ in Theorems 2.1 and 2.2, we conclude the following results.

Corollary 3.1. Let f given by (4) be in subclass $\mathcal{S}_{\Sigma_m}^\alpha$ ($0 < \alpha \leq 1, m \in \mathbb{N}$). Then

$$|a_{m+1}| \leq \min \left\{ \frac{2\alpha}{m}, \frac{\sqrt{2}\alpha}{m} \right\} = \frac{\sqrt{2}\alpha}{m}$$

and

$$|a_{2m+1}| \leq \min \left\{ \frac{\alpha^2}{m} + \frac{2(m + 1)\alpha^2}{m^2}, \frac{(m + 1)\alpha^2}{m^2} \right\} = \frac{(m + 1)\alpha^2}{m^2}.$$

Corollary 3.2. Let f given by (4) be in subclass $\mathcal{S}_{\Sigma_m}^\alpha$ ($0 < \alpha \leq 1, m \in \mathbb{N}$). Also let ρ be real number. Then

$$|a_{2m+1} - \rho a_{m+1}^2| \leq \begin{cases} \frac{\alpha^2}{m}; & \frac{|m-2\rho+1|}{m} \leq 1 \\ \frac{|m-2\rho+1|\alpha}{m^2}; & \frac{|m-2\rho+1|}{m} \geq 1. \end{cases}$$

Remark 3.1. The bounds on $|a_{m+1}|$ and $|a_{2m+1}|$ given in Corollary 3.1 are better than those given in [1, Corollary 6]. Because

$$\frac{\sqrt{2}\alpha}{m} \leq \frac{2\alpha}{m\sqrt{\alpha+1}}$$

and

$$\frac{(m+1)\alpha^2}{m^2} \leq \frac{\alpha^2}{m} + \frac{2(m+1)\alpha^2}{m^2} \leq \frac{\alpha}{m} + \frac{2(m+1)\alpha^2}{m^2}.$$

By putting

$$h(z) = p(z) = \frac{1 + (1 - 2\beta)z^m}{1 - z^m} = 1 + 2(1 - \beta)z^m + 2(1 - \beta)z^{2m} + \dots \quad (0 \leq \beta < 1, z \in \mathbb{U}),$$

$\lambda = 0$ and $\gamma = 1$ in Theorems 2.1 and 2.2, we conclude the following results.

Corollary 3.3. Let f given by (4) be in subclass $\mathcal{S}_{\Sigma_m}^\beta$ ($0 \leq \beta < 1, m \in \mathbb{N}$). Then

$$|a_{m+1}| \leq \begin{cases} \frac{\sqrt{2(1-\beta)}}{m}; & 0 \leq \beta \leq \frac{1}{2} \\ \frac{2(1-\beta)}{m}; & \frac{1}{2} \leq \beta < 1 \end{cases}$$

and

$$|a_{2m+1}| \leq \begin{cases} \frac{(m+1)(1-\beta)}{m^2}; & 0 \leq \beta \leq \frac{1+2m}{2(1+m)} \\ \frac{2(m+1)(1-\beta)^2}{m^2} + \frac{1-\beta}{m}; & \frac{1+2m}{2(1+m)} \leq \beta < 1. \end{cases}$$

Corollary 3.4. Let f given by (4) be in subclass $\mathcal{S}_{\Sigma_m}^\beta$ ($0 \leq \beta < 1, m \in \mathbb{N}$). Also let ρ be real number. Then

$$|a_{2m+1} - \rho a_{m+1}^2| \leq \begin{cases} \frac{(1-\beta)}{m}; & \frac{|m-2\rho+1|}{m} \leq 1 \\ \frac{(1-\beta)|m-2\rho+1|}{m^2}; & \frac{|m-2\rho+1|}{m} \geq 1. \end{cases}$$

Remark 3.2. The bounds on $|a_{m+1}|$ and $|a_{2m+1}|$ given in Corollary 3.3 are better than those given in [1, Corollary 7].

By setting $m = 1$ in Corollary 3.1, we conclude the following result.

Corollary 3.5. Let f given by (1) be in subclass $\mathcal{S}_{\Sigma}^*[\alpha]$ of strongly bi-starlike functions of order α ($0 < \alpha \leq 1$). Then

$$|a_2| \leq \min \{2\alpha, \sqrt{2}\alpha\} = \sqrt{2}\alpha$$

and

$$|a_3| \leq \min \{5\alpha^2, 2\alpha^2\} = 2\alpha^2.$$

Remark 3.3. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 3.5 are better than those given in [4, Corollary 2.5].

By setting $m = 1$ in Corollary 3.2, we conclude the following result.

Corollary 3.6. *Let f given by (1) be in subclass $\mathcal{S}_\Sigma^*[\alpha]$ of strongly bi-starlike functions of order $\alpha(0 < \alpha \leq 1)$. Also let ρ be real number. Then*

$$|a_3 - \rho a_2^2| \leq \begin{cases} \alpha^2 ; & |1 - \rho| \leq \frac{1}{2} \\ 2|1 - \rho|\alpha ; & |1 - \rho| \geq \frac{1}{2}. \end{cases}$$

By setting $m = 1$ in Corollary 3.3, we conclude the following result.

Corollary 3.7. *Let f given by (1) be in subclass $\mathcal{S}_\Sigma^*(\beta)$ of bi-starlike functions of order $\beta(0 \leq \beta < 1)$. Then*

$$|a_2| \leq \begin{cases} \sqrt{2(1 - \beta)} ; & 0 \leq \beta \leq \frac{1}{2} \\ 2(1 - \beta) ; & \frac{1}{2} \leq \beta < 1 \end{cases}$$

and

$$|a_3| \leq \begin{cases} 2(1 - \beta) ; & 0 \leq \beta \leq \frac{3}{4} \\ 4(1 - \beta)^2 + (1 - \beta) ; & \frac{3}{4} \leq \beta < 1. \end{cases}$$

Remark 3.4. *The bound on $|a_2|$ given in Corollary 3.7 is better than that given in [4, Corollary 3.5].*

By setting $m = 1$ in Corollary 3.4, we conclude the following result.

Corollary 3.8. *Let f given by (1) be in subclass $\mathcal{S}_\Sigma^*(\beta)$ of bi-starlike functions of order $\beta(0 \leq \beta < 1)$. Also let ρ be real number. Then*

$$|a_3 - \rho a_2^2| \leq \begin{cases} 1 - \beta ; & |1 - \rho| \leq \frac{1}{2} \\ 2(1 - \beta)|1 - \rho| ; & |1 - \rho| \geq \frac{1}{2}. \end{cases}$$

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