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EXISTENCE AND UNIQUENESS RESULTS FOR AN INITIAL-BOUNDARY VALUE PROBLEM OF PARABOLIC OPERATOR-DIFFERENTIAL EQUATIONS IN A WEIGHT SPACE

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Abstract. In this article, through the coefficients of the second order parabolic operator-differential equation, the regular solvability problem with initial-boundary conditions is proved in a weight space. The association between the lower bound of the spectrum of the self-adjoint operator in the main part of the differential equation and the weight exponent is clearly provided. A mixed problem of a partial differential equation is introduced as an applied result of this article.

Keywords: operator-differential equation, multiple characteristics, Hilbert space, regular solvability.

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1. Introduction

The studies of the last 50 years have enriched the theory of operator-differential equations with significant results. The theory of initial-value problems of operator- differential equations in Banach or Hilbert space enabled us to study both of ordinary and partial differential operators.

Nowadays, a large number of papers concerning the study of initial value problems of the operator-differential equations in Banach spaces have been published.

M. G. Gasymov analyzed both the solvability of operator-differential equations and the multiple completeness of some of the eigen and associated vectors of corresponding operator pencils. His works were the most valuable as they inspired many papers, including the present one and others to be mentioned below. Moreover, the interest of investigation of the solvability of operator-differential equations in Hilbert spaces with an exponential weight has been extensively studied. The main motivation for considering the solvability problem of a second order operator-differential equations was to provide an effective approach to solve dynamic problems of arches as well as rings and modeling the stability of the plates from plastic (see [1,2]).

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In a Hilbert space H with a countable dense subset, we have the following initial-boundary value problem

$$\prod_{k=1}^{2}\left(\frac{d}{dx}-\Psi_k A\right)u(x)+\sum_{j=1}^{2}A_j\frac{d^{2-j}u(t)}{dt^{2-j}}=f\left(t\right),\qquad t\in R_+=[0,+\infty), \tag{1}$$

$$u(0)=\frac{du(0)}{dt}=0, \tag{2}$$

where, $\Psi_1=\Psi_2=-1$, $f\left(t\right)$ and $u(t)$ are H- valued functions, the derivatives are understood here and below in the sense of distribution theory, A_j $(j=1,\ 2)$ are linear unbounded operators, and A is a self-adjoint positive definite operator in H. In this paper, we find the Green's function and formulate the solvability conditions for the problem (1), (2) in the case of

$$\sum_{j=1}^{2}A_j\frac{d^{2-j}u(t)}{dt^{2-j}}\neq 0.$$

For functions $\omega(t)$ defined in $R_+$with values in H, we introduce the following spaces with $e^{-\frac{\kappa}{2}t}$ weight , $-\infty<\kappa<\ +\infty$:

$$L_{2,\kappa}\left(R_+;H\right)=$$
$$=\left\{\omega(t):\|\omega(t)\|_{L_{2,\kappa}(R_+;H)}=\left(\int_0^{+\infty}\|\omega(t)\|_H^2\,e^{-\kappa t}\,dt\right)^{\frac{1}{2}}<+\infty\right\},$$

$$W_{2,\kappa}^2\left(R_+;H\right)=\left\{\omega(t):\|\omega(t)\|_{W_{2,\kappa}^2(R_+;H)}\right.$$
$$\left.=\left(\int_0^{+\infty}\left(\left\|\frac{d^2\omega(t)}{dt^2}\right\|_H^2+\left\|A^2\omega(t)\right\|_H^2\right)e^{-\kappa t}\,dt\right)^{\frac{1}{2}}<+\infty\right\},$$

$$W_{2,\kappa}^2\left(R_+;H;\{i\}\right)=\left\{u(t):u(t)\in W_{2,\kappa}^2\left(R_+;H\right),\frac{d^iu\left(0\right)}{dt^i}=0,(i=0,1)\right\}.$$

Simply, for $k\ =\ 0$, we have the spaces $L_{2,0}\left(R_+;H\right)$=$L_2(R_+;H)$ and $W_{2,0}^2\left(R_+;H\right)$= $W_2^2\left(R_+;H\right)$ (see [3]).

Definition 1.1. The function $u(t)\in W_{2,k}^2\left(R_+;H\right)$ which satisfies equation (1) and the initial condition (2) is called a regular solution of the problem (1),(2), for any $f(t)\in L_{2,k}(R+;\ H)$ where:

$$\lim_{t\to 0}\left\|A^{\frac{3}{2}-i}\frac{d^iu(t)}{dt^i}\right\|_H=0\qquad(i=0,1),$$

$$\|u\|_{W_{2,\kappa}^2(R_+;H)}\leq const\,\|f\|_{L_{2,\kappa}(R_+;H)}\,.$$

In this article, in the weighted space $W^2_{2,\kappa}(R_+;H)$ under algebraic condition on the operator coefficients of Eq. (1), the regular solution of the problem (1), (2) is obtained. Additionally, when establishing coefficient conditions, we indicate there relation between the weight exponent k and the lower boundary of the spectrum of the basic operator A. It should be noted that coefficient conditions are convenient in applications.

2. Example

Now we introduce the following problem of a partial differential equation on the strip $R_+\times$ $[0,\ \pi]$,

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)^2 u(t,x)+p(x)\frac{\partial^3 u(t,x)}{\partial x^2\partial t}+q(x)\frac{\partial^4 u(t,x)}{\partial x^4}=f(t,x). \tag{3}$$

Where, $f(t,x)\in L_{2,k}(R_+;L_2[0,\pi])$ and $p(x)$, $q(x)$ are bounded functions on the interval $[0,\ \pi]$. This problem can be reduced to the problem (1), (2) which means that we search for a solution of Eq. (3) that additionally satisfies the conditions

$$u(0,x)=\frac{\partial u(0,x)}{\partial t}=0, \tag{4}$$

$$\frac{\partial^{2s}u(t,0)}{\partial x^{2s}}=\frac{\partial^{2s}u(t,\pi)}{\partial x^{2s}}=0 \quad (s=0,1). \tag{5}$$

Here, $H=L_2[0,\ \pi]$, $A_1=\ p(x)\dfrac{\partial^2}{\partial x^2}$, and $A_2=q(x)\dfrac{\partial^4}{\partial x^4}$.

The operator A is defined in $L_2\ [0,\ \pi]$ by the equality $Au=-\dfrac{d^2u}{dx^2}$ with conditions $u(0)=u(\pi)=\ 0$.

On the other hand, some issues of mathematical physics lead to problems for equations of the form (1) (see [4,5,6]). Note that, setting $A_j=0$ for $j\ =\ 1,\ 2$ and defining the operator A as the Laplacian $-\Delta$, $x\ \in R_n$, we see that this equation characterizes, for example, diffusion or heat conduction in solids.

Once again, this confirms the usefulness of methods of operator-differential equations as applied to numerous natural science problems.

3. Main results

Let P is an operator from the space $W^2_{2,\kappa}$(R$_+$;H; {i}) to L$_{2,k}$(R+; H) defined as

$$Pu(t)=\left(\frac{d}{dt}+A\right)^2 u(t)+\sum_{j=1}^{2}A_j\frac{d^{2-j}u(t)}{dt^{2-j}}\ ,\quad u(t)\in W^2_{2,\kappa}(R_+;H;\{i\}).$$

Lemma 3.1. *Let the operators A_jA^{-j} $(j=1,\ 2,)$ be bounded in H. Then the operator P_2 from $W^2_{2,k}(R_+,H,\{i\})$ to $L_{2,k}(R_+;H)$ defined as*

$$P_2u(t)=\sum_{j=1}^{2}A_j\frac{d^{2-j}u(t)}{dt^{2-j}},\quad u(t)\in W^2_{2,\kappa}(R_+;H;\{i\}),$$

is also bounded.

Proof:
From the theorem of intermediate derivatives (see [7,8]), we get

$$\begin{aligned}\|P_2u\|_{L_{2,\kappa}(R_+;H)} &= \left\| \sum_{j=1}^{2} A_j \frac{d^{2-j}u}{dt^{2-j}} \right\|_{L_{2,\kappa}(R_+;H)} \\ &\le \sum_{j=1}^{2} \left\|A_j A^{-j}\right\| \left\| A^j \frac{d^{2-j}u}{dt^{2-j}} \right\|_{L_{2,\kappa}(R_+;H)} \\ &\le const\, \|u\|_{W^2_{2,\kappa}(R_+,H)},\end{aligned}$$

$$P_2 : W^2_{2,\kappa}\left(R_+; H; \{i\}\right) \to L_{2,\kappa}\left(R_+; H\right).$$

Hence, P_2 is bounded.
Let P denote an operator from the space $W^2_{2,\kappa}\left(R_+; H; \{i\}\right)$ to $L_{2,\kappa}\left(R_+; H\right)$ defined as

$$Pu\left(t\right) = P_1u\left(t\right) + P_2u\left(t\right), \;\; u\left(t\right) \in W^2_{2,\kappa}\left(R_+; H; \{i\}\right).$$

Theorem 3.2. *Suppose that A is a self-adjoint positive definite operator with a lower spectral boundary λ_0 ($A = A{*}\geq\lambda_0 E$ ($\lambda_0 > 0$), E is the identity operator) and $k > -2\lambda_0$. Then the operator $P_1 : W^2_{2,\kappa}$ ($R+$; H; $\{i\}$) $\to L_{2,k}\left(R_+; H\right)$ is an isomorphism (see [9,10]) and the solution of the equation*

$$\begin{aligned}&Pu\left(t\right) = f\left(t\right), \\ &u\left(t\right) \in W^2_2\left(R_+; H\right), \\ &f\left(t\right) \in L_2\left(R_+; H\right),\end{aligned} \tag{6}$$

is given by the formula

$$u\left(t\right) = \int_0^{+\infty} G\left(t-s\right)\, h\left(s\right)\, ds = P^{-1}h(t).$$

Where

$$G\left(t-s\right) = k_3 \begin{cases} e^{-k_2(t-s)}, & t-s > 0 \\ e^{-k_1(t-s)}, & t-s < 0, \end{cases}$$

$$\begin{aligned}k_1 &= \frac{b}{2} - \frac{\sqrt{b^2-4c}}{2}, \\ k_2 &= \frac{b}{2} + \frac{\sqrt{b^2-4c}}{2}, \\ k_3 &= -\frac{1}{\sqrt{b^2-4c}}, \\ b &= 2A + A_1, \\ c &= A^2 + A_2.\end{aligned}$$

Corollary 3.3. *For $\kappa > -2\lambda_0$, the norms $\|Pu\|_{L_{2,\kappa}(R_+;H)}$ and $\|u\|_{W^2_{2,\kappa}(R_+;H)}$ are equivalent in the space $W^2_{2,\kappa}\left(R_+; H; \{i\}\right)$, and for $k = -2\lambda_0$, the operator P is not invertible.*

For the norms of the operators of intermediate derivatives, we have the following estimation:

$$A^j \frac{d^{2-j}}{dt^{2-j}} : W_{2,\kappa}^2 (R_+; H; \{i\}) \to L_{2,\kappa} (R_+; H), \quad j = 1, 2.$$

Since these operators are continuous, by Corollary 3.3, their norms can be estimated in terms of$\|Pu\|_{L_{2,\kappa}(R_+;H)}$.

Theorem 3.4. *Let $-2\lambda_0 < \kappa < 2\lambda_0$. Then, for any function $u(t) \in W_{2,\kappa}^2 (R_+; H; \{i\})$ (see [11,12,13])*

$$\left\| A^j \frac{d^{2-j} u}{dt^{2-j}} \right\|_{L_{2,\kappa}(R_+;H)} \le c_j (\kappa) \|P_1 u\|_{L_{2,\kappa}(R_+;H)}, \quad j = 1, 2. \tag{7}$$

Where

$$c_1 (\kappa) = c_2 (\kappa) = \left(1 - \frac{|k|}{2\lambda_0}\right)^{-1}.$$

Proof. We denote $y(t) = \left(\frac{d}{dt} + A\right) u(t)$. Then from the Eq. (1) for $A_j = 0, j = 1, 2$ and boundary conditions (2) with respect to $y(t)$ we have the following problem:

$$\frac{dy(t)}{dt} + Ay(t) = f(t), t \in R_+, \tag{8}$$

$$y(0) = 0. \tag{9}$$

After substitution $w(t) = y(t) e^{-\frac{kt}{2}}$ from the problem (8), (9) we obtain

$$\left(\frac{d}{dt} + \frac{k}{2}\right) w(t) + Aw(t) = h(t), t \in R_+, \tag{10}$$

$$w(0) = 0. \tag{11}$$

Where

$$w(t) \in W_2^1 (R_+, H), h(t) = f(t) e^{-\frac{kt}{2}} \in L_{2,k} (R_+, H).$$

Multiplying both sides of Eq. (10) by Aw as a scalar product in the space $L_2 (R_+, H)$ we have

$$\left(\frac{d}{dt}, Aw\right)_{L_2(R_{+,H})} + \left(\frac{k}{2} w, Aw\right)_{L_2(R_{+,H})} + (Aw, Aw)_{L_2(R_{+,H})} = (h, Aw)_{L_2(R_{+,H})}.$$

Now, integrating by parts we obtain

$$Re\,(h, Aw)_{L_2(R_{+,H})} = \|Aw\|_{L_2(R_{+,H})}^2 + \frac{k}{2} \left\| A^{\frac{1}{2}} w \right\|_{L_2(R_{+,H})}^2,$$

$$Re\,(h, Aw)_{L_2(R_{+,H})} \geq \gamma\,(\lambda_0)\,\|Aw\|^2_{L_2(R_{+,H})},$$

$$\|h\|_{L_2(R_{+,H})}\,\|Aw\|_{L_2(R_{+,H})} \geq \gamma\,(\lambda_0)\,\|Aw\|^2_{L_2(R_{+,H})},$$

$$\|Aw\|_{L_2(R_{+,H})} \leq \gamma^{-1}\,(\lambda_0)\,\|h\|_{L_2(R_{+,H})}, \tag{12}$$

$$h\,(t) = f\,(t)\,e^{-\frac{kt}{2}},\, w\,(t) = y\,(t)\,e^{-\frac{kt}{2}},$$

$$\|Ay\|_{L_2(R_{+,H})} \leq \gamma^{-1}\,(\lambda_0)\,\|f\|_{L_2(R_{+,H})},$$

$$\|Ay\|^2_{L_2(R_{+,H})} \leq \gamma^{-2}\,(\lambda_0)\,\|f\|^2_{L_2(R_{+,H})}.$$

Since

$$y\,(t) = \left(\frac{d}{dt} + A\right) u\,(t),$$

$$\left\|A\frac{du}{dt}\right\|^2_{L_2(R_{+,H})} + \left\|A^2u\right\|^2_{L_2(R_{+,H})} \leq \gamma^{-2}\,(\lambda_0)\,\|f\|^2_{L_2(R_{+,H})}. \tag{13}$$

As a result, from the inequality (13) the following estimation is obtained:

$$\left\|A^j\frac{d^{2-j}u}{dt^{2-j}}\right\|_{L_{2,\kappa}(R_+;H)} \leq c_j\,(\kappa)\,\|P_0u\|_{L_{2,\kappa}(R_+;H)}, \quad j = 1, 2.$$

Where

$$c_1\,(\kappa) = c_2\,(\kappa) = \left(1 - \frac{|k|}{2\lambda_0}\right)^{-1}.$$

Estimates (7) play a key role in finding coefficient conditions for the regular solvability of problem (1), (2).

Note that the norms of operators of intermediate derivatives in Sobolev-type spaces with no weight are calculated in detail in (see [14]). □

Theorem 3.5. *Let the operators A_jA^{-j} $(j = 1,\ 2)$ be bounded in H. Then P is a bounded operator from $W^2_{2,\kappa}\,(R_+; H; \{i\})$ to $L_{2,\kappa}\,(R_+; H)$.*

The boundedness of $P : W^2_{2,\kappa}\,(R_+; H; \{i\})$ to $L_{2,\kappa}\,(R_+; H)$ follows from the inequality

$$\begin{aligned}\|Pu\|_{L_{2,\kappa}(R_+;H)} &\leq \|P_1u\|_{L_{2,\kappa}(R_+;H)} + \|P_2u\|_{L_{2,\kappa}(R_+;H)} \\ &\leq \left\|\frac{d^2u}{dt^2}\right\|_{L_{2,\kappa}(R_+;H)} + 2\left\|A\frac{du}{dt}\right\|_{L_{2,\kappa}(R_+;H)} + \left\|A^2u\right\|_{L_{2,\kappa}(R_+;H)} + \\ &\qquad + \|P_2u\|_{L_{2,\kappa}(R_+;H)} \\ &\leq const\,\|u\|_{W^2_{2,\kappa}(R_+;H)}.\end{aligned}$$

Which is proved by applying the theorem on intermediate derivatives and the previous lemma.

Now exactly for the problem (1), (2) we determined the sufficient conditions for the regular solvability in terms of the operator coefficients of Eq. (1).

Theorem 3.6. *Let $A = A^* \geq \lambda_o E$ ($\lambda_o > 0$), $-2\lambda_o < k < 2\lambda_o$, and the operators $A_j A^{-j}$ ($j = 1, 2$) be bounded in H and satisfy the inequality*

$$\sum_{j=1}^{2} c_j(\kappa) \left\| A_j A^{-j} \right\|_{H \to H} < 1.$$

Where the numbers $c_j(k)$, ($j = 1, 2,$) are defined in Theorem 3.4 Then the initial–boundary value problem (1), (2) is regularly solvable.

Corollary 3.7. *Under the conditions of Theorem 3.6, the operator P is an isomorphism from $W_{2,\kappa}^2(R_+; H; \{i\})$ to $L_{2,\kappa}(R_+; H)$.*

Now we apply theorem 3.6 to the mixed problem (3)–(5) at $\lambda_o = 1$.

Applying Theorem 4, we see that, if $-2 < \kappa < 2$ and

$$\left(1 - \frac{|k|}{2}\right)^{-1} \left(\sup_{x \in [0,\pi]} |p(x)| + \sup_{x \in [0,\pi]} |q(x)| \right) < 1,$$

then the mixed problem (3)–(5) (see [15]) has a unique solution in

$$W_{t,x,2,k}^{2,4}(R_+; L_2[0,\pi]).$$

4. Conclusions

In summary, we formulated exact conditions on the regular solvability of problem (1), (2), expressed only by its operator coefficients, we estimated the norms of the intermediate derivatives of the operators participating in the main part of the given equation. In the case when in the perturbed part of the equation (1) there participate variable operator coefficients, i.e. A_j ($j = 1, 2$) for all t$\in R_+$, were investigated in a similar way.

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