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CONTROLLED MULTIPLIERS WITH TWO OPERATORS IN HILBERT C^*-MODULES

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Abstract. Improving and extending the notion of controlled frames, in this paper, we introduce controlled frames with two operators for Hilbert C^*-modules. This generalize controlled frames in Hilbert and Hilbert C^*-module. We show, in Hilbert C^*-module setting, controlled frame with two operators is classical frame. Also, we investigate controlled multiplier operators and their invertibility in Hilbert C^*-modules. We show that the inverse of controlled frame multiplier with two operators is controlled frame multiplier too.

Keywords: Frame, Controlled frame, Hilbert C^*-module, Multiplier operator.

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1. Introduction

Frames for Hilbert spaces were first introduced in 1952 by Duffin and Schaeffer [1] to study of nonharmonic Fourier series. They were reintroduced and developed in 1986 by Daubechies, Grossmann and Meyer [2], and popularized from then on. For basic results on frames, see [3].

Hilbert C^*-modules is a generalization of Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than in the field of complex numbers. Their structure was first used by Kaplansky [4] in 1952. Frank and Larson [5, 6] introduced frames in Hilbert C^*-modules and got a series of result for standard frames in finitely or countably generated Hilbert C^*-modules over unital C^*-algebras. Extending the results to Hilbert C^*-modules is not a routine generalization, as there are essential differences between Hilbert C^*-modules and Hilbert spaces. For example, any Hilbert space has a frame and any closed subspace in a Hilbert space has an orthogonal complement, but these fail in Hilbert C^*-module. We refer the readers to [7] and [8] for more details on Hilbert C^*-modules and to [9, 6, 10, 11, 12, 13] for a discussion of basic properties of frame in Hilbert C^*-modules and their generalizations.

Balazs and et al.[14] introduced controlled frames in Hilbert space to improve the numerical efficiency of iterative algorithms for inverting the frame operator on abstract Hilbert

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spaces, however they are used earlier in [15] for spherical wavelets. Recently controlled frames with two operators were introduced in [16] by the idea of controlled g-frames in [17]. Also the authors defined and investigated controlled frames in Hilbert C^*-modules [18].

In this paper, we extend controlled frames with two operators for Hilbert C^*-modules. This generalize controlled frames in Hilbert and Hilbert C^*-module. Similar to Hilbert spaces, we show controlled frames with two operators in Hilbert C^*-modules are classical frame. Then we investigate controlled multipliers in Hilbert C^*-modules. We show the inverse of controlled frame multiplier is controlled frame multiplier too. The presented results can be relevance for wider audience in the areas of wave-packet analysis [19, 20] and [21].

The paper is organized as follows. In Section 2, we review the concept Hilbert C^*-modules, frames and multiplier operators in Hilbert C^*-modules. Also the analysis, synthesis, frame operator and dual frames are reviewed. In Section 3, we define controlled frames with two operators in Hilbert C^*- modules and characterize them. In Section 4, we investigate controlled multiplier operators with two controller operators in Hilbert C^*-modules and verify their invertibility.

2. Preliminaries

In this section, we review some basic notations and definitions.

Hilbert C^*-modules form a wide category between Hilbert spaces and Banach spaces. Hilbert C^*-modules are generalizations of Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than in the field of complex numbers and define as follows:

Let A be a C^*-algebra with involution $*$. An inner product A-module (or pre Hilbert A-module) is a complex linear space $\mathcal{H}$ which is a left A-module with an inner product map $\langle\cdot,\cdot\rangle : \mathcal{H}\times\mathcal{H}\to A$ which satisfies the following properties:

(1) $\langle \alpha f+\beta g, h\rangle = \alpha\langle f,h\rangle + \beta\langle g,h\rangle$ for all $f,g,h\in\mathcal{H}$ and $\alpha,\beta\in\mathbb{C}$;
(2) $\langle af, g\rangle = a\langle f,g\rangle$ for all $f,g\in\mathcal{H}$ and $a\in A$;
(3) $\langle f,g\rangle = \langle g,f\rangle^*$ for all $f,g\in\mathcal{H}$;
(4) $\langle f,f\rangle \geq 0$ for all $f\in\mathcal{H}$ and $\langle f,f\rangle = 0$ iff $f=0$.

For $f\in\mathcal{H}$, we define a norm on $\mathcal{H}$ by $\|f\|_{\mathcal{H}} = \|\langle f,f\rangle\|_A^{1/2}$. If $\mathcal{H}$ is complete with this norm, it is called a (left) Hilbert C^*-module over A or a (left) Hilbert A-module.

An element a of a C^*-algebra A is positive if $a^*=a$ and its spectrum is a subset of positive real numbers. In this case, we write $a\geq 0$. It is easy to see that $\langle f,f\rangle\geq 0$ for every $f\in\mathcal{H}$, hence we define $|f| = \langle f,f\rangle^{1/2}$. If $a,b\in A$ and $0\leq a\leq b$, then $\|a\|\leq\|b\|$. Thus norm preserves order for positive members in C^*-algebras.

We call $Z(A)=\{a\in A : ab=ba, \forall b\in A\}$, the center of A. If $a\in Z(A)$, then $a^*\in Z(A)$, and if a is an invertible element of $Z(A)$, then $a^{-1}\in Z(A)$, also if a is a positive element of $Z(A)$, then $a^{1/2}\in Z(A)$. Let $Hom_A^*(\mathcal{H},\mathcal{K})$ denotes the set of all adjointable A-linear operators from $\mathcal{H}$ to $\mathcal{K}$ and $GL(\mathcal{H},\mathcal{K})$ as the set of all adjointable bounded linear operators with an adjointable bounded inverse, and similarly for $GL(\mathcal{H})$. If $T\in GL(\mathcal{H})$ is positive, i.e. $\langle Tf,f\rangle\geq 0$ for all $f\in\mathcal{H}$, then we denote that by $T\in GL^+(\mathcal{H})$. One of the standard references for Hilbert space and operator theory is [23].

Let

$$\ell^2(A) = \left\{ \{a_j\}\subseteq A : \sum_{j\in J} a_j^* a_j \ converges \ in \|.\| \right\}$$

with inner product

$$\langle \{a_j\}, \{b_j\} \rangle = \sum_{j\in J} a_j^* b_j, \quad \{a_j\}, \{b_j\} \in \ell^2(A)$$

and

$$\|\{a_j\}\| := \sqrt{\|\sum a_j^* a_j\|},$$

it was shown that [8], $\ell^2(A)$ is Hilbert A-modules.

Note that in Hilbert C^*-modules the Cauchy-Schwartz inequality is valid.

Let $f, g \in \mathcal{H}$, where $\mathcal{H}$ is a Hilbert C^*-module, then

$$\|\langle f, g\rangle\|^2 \le \|\langle f, f\rangle\| \times \|\langle g, g\rangle\|.$$

We are focusing in finitely and countably generated Hilbert C^*- modules over unital C^*-algebra A. A Hilbert A-module $\mathcal{H}$ is finitely generated if there exists a finite set $\{x_1, x_2, ..., x_n\} \subseteq \mathcal{H}$ such that every $x \in \mathcal{H}$ can be expressed as $x = \sum_{i=1}^n a_i x_i$, $a_i \in A$. A Hilbert A-module $\mathcal{H}$ is countably generated if there exits a countable set of generators.

The notion of (standard) frames in Hilbert C^*-modules is first defined by Frank and Larson [6]. Basic properties of frames in Hilbert C^*-modules are discussed in [22, 23, 24, 25].

Let $\mathcal{H}$ be a Hilbert C^*-module, and J a set which is finite or countable, a system $\{f_j\}_{j\in J} \subseteq \mathcal{H}$ is called a frame for $\mathcal{H}$ if there exist constants $C, D > 0$ such that

$$C\langle f, f\rangle \le \sum_{j\in J} \langle f, f_j\rangle\langle f_j, f\rangle \le D\langle f, f\rangle \tag{1}$$

for all $f \in \mathcal{H}$. The constants C and D are called the frame bounds. If $C = D$ it called a tight frame and in the case $C = D = 1$ it called Parseval frame. It is called a Bessel sequence if the second inequality in (1) holds.

Unlike Banach spaces, it is known that every finitely generated or countably generated Hilbert C^*-modules admits a frame [6] but this is not true for every Hilbert C^*-module [9] and [26].

The following characterization of frames in Hilbert C^*-modules, which was obtained independently in [27] and [28], enables us to verify whether a sequence is a frames in Hilbert C^*-modules in terms of norms. It also allows us to characterize frames in Hilbert C^*-modules from the operator theory point of view.

Theorem 2.1. *Let $\mathcal{H}$ be a finitely or countably generated Hilbert A-module over a unital C^*-algebra A and $\{f_j : j \in J\} \subset \mathcal{H}$ a sequence. Then $\{f_j : j \in J\}$ is a frame for $\mathcal{H}$ if and only if there exist constants $C, D > 0$ such that*

$$C\|f\|^2 \le \left\| \sum_{j\in J} \langle f, f_j\rangle\langle f_j, f\rangle \right\| \le D\|f\|^2, \;\; f \in \mathcal{H}.$$

Let $\{f_j : j \in J\}$ be a frame in Hilbert A-module $\mathcal{H}$ over a unital C^*-algebra A and $\{g_j : j \in J\}$ be a sequence of $\mathcal{H}$. Then $\{g_j\}_{j\in J}$ is called a dual sequence of $\{f_j\}_{j\in J}$ if

$$f = \sum_{j\in J} \langle f, g_j\rangle f_j$$

for all $f \in \mathcal{H}$. The sequences $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are called a dual frame pair when $\{g_j\}_{j\in J}$ is also a frame.

For the frame $\{f_j : j \in J\}$ in Hilbert A-module $\mathcal{H}$ over a unital C^*-algebra A, the operator S defined by

$$Sf = \sum_{j\in J}\langle f, f_j\rangle f_j, \;\; f \in \mathcal{H}$$

is called the frame operator. It was proved that [5], S is invertible, positive, adjointable and self-adjoint. Since

$$\langle Sf, f\rangle = \langle \sum_{j\in J}\langle f, f_j\rangle f_j, f\rangle = \sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle, \quad f \in \mathcal{H}$$

it follows that

$$C\langle f, f\rangle \le \langle Sf, f\rangle \le D\langle f, f\rangle, \quad f \in \mathcal{H}$$

and the following reconstruction formula holds

$$f = SS^{-1}f = S^{-1}Sf = \sum_{j\in J}\langle S^{-1}f, f_j\rangle f_j = \sum_{j\in J}\langle f, S^{-1}f_j\rangle f_j$$

for all $f \in \mathcal{H}$.
Let $\tilde{f}_j = S^{-1}f_j$, then

$$f = \sum_{j\in J}\langle f, \tilde{f}_j\rangle f_j = \sum_{j\in J}\langle f, f_j\rangle \tilde{f}_j,$$

for any $f \in \mathcal{H}$. The sequence $\{\tilde{f}_j : j \in J\}$ is also a frame for $\mathcal{H}$ which is called the canonical dual frame of $\{f_j : j \in J\}$.

In [29], R. Schatten provided a detailed study of ideals of compact operators using their singular decomposition. He investigated the operators of the form $\sum_j \lambda_j \varphi_j \otimes \psi_j$ where (φ_j) and (ψ_j) are orthonormal families. In [5], the orthonormal families were replaced with Bessel and frame sequences to define Bessel and frame multipliers in Hilbert space.

Basic properties and some applications of this operator for Bessel sequences, frames and Riesz basis have been proved by Peter Balazs in his Ph.D habilation [30] and related paper [31]. Recently, the concept of multipliers extended and introduced for continuous frames [32], fusion frames [33], p-Bessel sequences [34], generalized frames [35] and Hilbert C^*-module [36].

Definition 2.1. *Let A be a unital C^*-algebra, J be a finite or countable index set and $\{f_j : j \in J\}$ and $\{g_j : j \in J\}$ be Hilbert C^*-modules Bessel sequences for $\mathcal{H}$. For $m \in \ell^\infty(A)$ with $m_j \in Z(A)$, for each $j \in J$, the operator $M_{m,\{f_j\},\{g_j\}} : \mathcal{H} \to \mathcal{H}$ defined by*

$$M_{m,\{f_j\},\{g_j\}}f := \sum_{j\in J} m_j\langle f, f_j\rangle g_j, \quad f \in \mathcal{H}$$

is called the multiplier operator of $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$. The sequence $m = \{m_j\}$ called the symbol of $M_{m,\{f_j\},\{g_j\}}$.

The symbol m has an important role in the studying of multiplier operators. In this paper m is always a sequence $m = \{m_j\}_{j\in J} \in \ell^\infty(A)$ with $m_j \in Z(A)$, for each $j \in J$.

We need the following lemma and theorem of [35] to prove our results.

Lemma 2.2. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert C^*-module over A and let $T : \mathcal{H} \to \mathcal{K}$ be a linear map. Then the following conditions are equivalent:*

1. *the operator T is bounded and A-linear;*
2. *there exists a constant $M > 0$ such that the inequality $\langle Tf, Tf\rangle \le M\langle f, f\rangle$ holds in A for all $f \in \mathcal{H}$.*

Theorem 2.3. *Let $T : \mathcal{H} \to \mathcal{H}$ be a linear operator. Then the following conditions are equivalent:*

(1) *There exist $m > 0$ and $M < \infty$ such that $mI \leq T \leq MI$;*
(2) *$T \in GL^+(\mathcal{H})$.*

3. Controlled frames with two operators in Hilbert C^*-modules

In this section, we define and characterize controlled frames with two operators in Hilbert C^*-modules. Then we show every controlled frames with invertible bounded operators in Hilbert C^*-module are classical frames in Hilbert C^*-modules.

Definition 3.1. *Let $\mathcal{H}$ be a Hilbert C^*-module and $T, T' \in GL(\mathcal{H})$. Let $\mathcal{F} = \{f_j : j \in J\}$ be a sequence in Hilbert C^*-module $\mathcal{H}$. The sequence $\mathcal{F}$ is called a controlled frame by T and T' or (T, T')-controlled frame if there exists two constants $0 < C, D < \infty$ such that*

$$C\langle f, f\rangle \leq \sum_{j\in J} \langle f, Tf_j\rangle\langle T'f_j, f\rangle \leq D\langle f, f\rangle,$$

for all $f \in \mathcal{H}$. We call $\mathcal{F}$ a Parseval (T, T')-controlled frame if $C = D = 1$. If only the right inequality holds, then we call $\mathcal{F}$ a (T, T')-controlled Bessel sequence.

The proof of the following lemma is straightforward.

Lemma 3.1. *Let $\mathcal{H}$ be a Hilbert C^*-module and $T \in GL(\mathcal{H})$. The Bessel sequence $\mathcal{F} = \{f_j : j \in J\}$ in $\mathcal{H}$ is (T, T)-controlled Bessel sequence(or (T, T)-controlled frame) if and only if there exists constant $D < \infty$ (and $C > 0$) such that*

$$\sum_{j\in J} |\langle f, Tf_j|^2 \leq D|f|^2, \ \forall f \in \mathcal{H}$$

$$(or\ C|f|^2 \leq \sum_{j\in J} |\langle f, Tf_j|^2 \leq D|f|^2, \ \forall f \in \mathcal{H}).$$

We call the (T, T)-controlled Bessel sequence and (T, T)-controlled frame, T^2-controlled Bessel sequence and T^2-controlled frame with bounds C, D.

Let $\mathcal{F} = \{f_j : j \in J\}$ be a Bessel sequence of elements in Hilbert C^*-module $\mathcal{H}$. We define a linear operator $U_{T\mathcal{F}} : \mathcal{H} \to \ell^2(A)$ as follows:

$$U_{T\mathcal{F}}f = \{\langle f, Tf_j\rangle\}_{j\in J},$$

for all $f \in \mathcal{H}$. If $\mathcal{F}$ is also a (T, T')-controlled frame for $\mathcal{H}$, then it is a bounded linear operator and this is called the analysis operator of (T, T')-controlled frame. The adjoint operator $U^*_{T\mathcal{F}} : \ell^2(A) \to \mathcal{H}$ which is called the synthesis operator of (T, T')-controlled frame and is defined as follows:

$$U^*_{T\mathcal{F}}(\{a_j\}_{j\in J}) = \sum_{j\in J} a_j Tf_j,$$

for all $\{a_j\}_{j\in J} \in \ell^2(A)$.

Controlled frame operator $S_{TT'}$ on Hilbert C^*-module $\mathcal{H}$ for (T, T')-controlled frame $\mathcal{F}$ is defined by

$$S_{TT'}f := U^*_{T'\mathcal{F}}U_{T\mathcal{F}}(f) = \sum_{j\in J} \langle f, Tf_j\rangle T'f_j,$$

for all $f \in \mathcal{H}$.

It is easy to see that $S_{TT'}$ is well defined and

$$CId_{\mathcal{H}} \leq S_{TT'} \leq DId_{\mathcal{H}}.$$

Hence $S_{TT'}$ is a bounded, invertible, self-adjoint and positive linear operator. Therefore we have $S_{TT'} = S^*_{TT'} = S_{T'T}$.

Now, by using the following lemma in [27], we give a characterization of controlled frames.

Lemma 3.2. [27] *Let A be a C^*-algebra, U and V two Hilbert A-modules, and $T \in End^*_A(U,V)$. Then the following statements are equivalent:*

1. *T is surjective;*
2. *T^* is bounded below with respect to norm, that is, there is $m > 0$ such that $\|T^*f\| \geq m\|f\|$ for all $f \in U$;*
3. *T^* is bounded below with respect to the inner product, that is, there is $m' > 0$ such that $\langle T^*f, T^*f\rangle \geq m'\langle f, f\rangle$ for all $f \in U$.*

Theorem 3.3. *Let $\mathcal{H}$ be a Hilbert C^*-module, $T, T' \in GL(\mathcal{H})$ and $\mathcal{F} = \{f_j\}_{j\in J}$ be a sequence in $\mathcal{H}$. Then $\mathcal{F}$ is a (T,T')-controlled frame for $\mathcal{H}$ if and only if there exist constants $C, D > 0$ such that*

$$C\|f\|^2 \leq \left\| \sum_{j\in J} \langle f, Tf_j\rangle\langle T'f_j, f\rangle \right\| \leq D\|f\|^2, \quad f \in \mathcal{H}. \tag{2}$$

Proof. Let the sequence $\mathcal{F} = \{f_j\}_{j\in J}$ be (T,T')-controlled frame in Hilbert C^*-module $\mathcal{H}$. By the definition of (T,T')-controlled frame inequality (2) holds.

Now suppose that the inequality (2) holds. Since (T,T')-controlled frame operator $S_{TT'}$ is positive, self-adjoint and invertible, we have

$$\langle S^{\frac{1}{2}}_{TT'}f, S^{\frac{1}{2}}_{TT'}f\rangle = \langle S_{TT'}f, f\rangle = \sum_{j\in J} \langle f, Tf_j\rangle\langle T'f_j, f\rangle.$$

So we have

$$\sqrt{C}\|f\| \leq \|S^{\frac{1}{2}}_{TT'}f\| \leq \sqrt{D}\|f\|$$

for any $f \in \mathcal{H}$. According to Lemma 3.2 and Lemma 2.3 there are constants $m, M > 0$ such that

$$m\langle f, f\rangle \leq \sum_{j\in J} \langle f, Tf_j\rangle\langle T'f_j, f\rangle \leq M\langle f, f\rangle,$$

which implies that $\mathcal{F} = \{f_j : j \in J\}$ is (T,T')-controlled frame in Hilbert C^*-module $\mathcal{H}$. □

The following proposition shows every (T,T')-controlled is a frame in Hilbert C^*-modules. Also it gives a condition that every classical frame is a (T,T')-controlled frame.

Proposition 3.4. *Let $\mathcal{H}$ be a Hilbert C^*-module, $T, T' \in GL(\mathcal{H})$ and $\mathcal{F} = \{f_j : j \in J\}$ a sequence in $\mathcal{H}$. Then the following statements hold:*

1. *If $\mathcal{F}$ is a (T,T')-controlled frame for $\mathcal{H}$, then $\mathcal{F}$ is a frame for $\mathcal{H}$.*
2. *If $\mathcal{F}$ is a frame for $\mathcal{H}$ and $T'S_{\mathcal{F}}T^*$ is a positive operator, then $\mathcal{F}$ is a (T,T')-controlled frame for $\mathcal{H}$.*

Proof. (1) Let f be an arbitrary element of Hilbert C^*-module $\mathcal{H}$. The operator

$$Sf := (T')^{-1}S_{TT'}(T^*)^{-1}(f) = \sum_{j\in J} \langle f, f_j\rangle f_j$$

is well defined, bounded and invertible. Since

$$\langle Sf, f\rangle = \sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle$$

the operator S is a positive linear operator. Also we have

$$\|S^{-1}\| = \|T^* S_{TT'}^{-1} T'\| \leq \|T^*\|\|S_{TT'}^{-1}\|\|T'\| \leq \frac{1}{C}\|T\|\|T'\|.$$

So $S \in GL^+(\mathcal{H})$. Therefore, by Theorem 3.9 in [18] there exist $m > 0$ and $M < \infty$ such that $mI \leq S \leq MI$. Therefore $\mathcal{F}$ is a frame for $\mathcal{H}$.

(2) Since $T' S_{\mathcal{F}} T^*$ is a positive operator so $S_{TT'} := T' S_F T^* \in GL^+(\mathcal{H})$ and again by Theorem 3.9 in [18] there exist $m > 0$ and $M < \infty$ such that $mI \leq S_{TT'} \leq MI$. Therefore $\mathcal{F}$ is a (T, T')-controlled frame for $\mathcal{H}$.

□

The following proposition shows that any frame is a T^2-controlled frame and versa.

Proposition 3.5. *Let $\mathcal{H}$ be a Hilbert C^*-module, $T \in GL(\mathcal{H})$ be self-adjoint and $\mathcal{F} = \{f_j : j \in J\}$ a sequence in $\mathcal{H}$. The sequence $\mathcal{F}$ is a frame if and only if $\mathcal{F}$ is a T^2-controlled frame.*

Proof. Let the sequence $\mathcal{F}$ be a frame in Hilbert C^*-module $\mathcal{H}$ with bounds C', D'. Then by Theorem 2.1

$$C'\|f\|^2 \leq \left\|\sum_{j\in J}\langle f, f_j\rangle\langle f_j, f\rangle\right\| \leq D'\|f\|^2,$$

for all $f \in \mathcal{H}$. So, we have

$$\left\|\sum_{j\in J}\langle Tf, f_j\rangle\langle f_j, Tf\rangle\right\| \leq D'\|Tf\|^2 \leq D'\|T\|^2\|f\|^2,$$

for all $f \in \mathcal{H}$.

For lower bound,

$$C'\|f\|^2 = C'\|T^{-1}Tf\|^2 \leq C'\|T^{-1}\|^2\|Tf\|^2 \leq \|T^{-1}\|^2\left\|\sum_{j\in J}\langle f, Tf_j\rangle\langle Tf_j, f\rangle\right\|,$$

for all $f \in \mathcal{H}$. Therefore $\mathcal{F}$ is a T^2-controlled frame with bounds $C'\|T^{-1}\|^{-2}$ and $D'\|T\|^2$.

The converse is valid by Proposition 3.4 (1).

□

The idea of the following proposition is from Proposition 3.6. in [17].

Proposition 3.6. *Let $\mathcal{H}$ be a Hilbert C^*-module, $\mathcal{F} = \{f_j : j \in J\}$ a frame in $\mathcal{H}$ and $T, T' \in G^+L(\mathcal{H})$, which commute with each other and commute with $S_{\mathcal{F}}$. Then $\mathcal{F}$ is a (T, T')-controlled frame.*

Proof. Let $\mathcal{F}$ be a frame with bounds C, D. Since $T, T' \in GL^+(\mathcal{H})$ by Theorem 2.4 there exist $m, m' > 0$ and $M, M' < \infty$ such that

$$mI \leq T \leq MI, \ m'I \leq T' \leq M'I.$$

Then

$$m'CI \leq T' S_{\mathcal{F}} \leq M'DI,$$

because T' commute with $S_{\mathcal{F}}$. Again T commute with $T'S_{\mathcal{F}}$ and then

$$mm'CI \leq S_{TT'} \leq MM'DI.$$

□

The following theorem characterize all operators $T, T' \in G^+L(\mathcal{H})$ which can generate Parseval controlled frames from a usual frame. This is a generalization of Theorem 3.1 in [16] for Hilbert C^*-modules.

Theorem 3.7. *Let $\mathcal{H}$ be a Hilbert C^*-module, $\mathcal{F} = \{f_j : j \in J\}$ a frame in $\mathcal{H}$ and $T, T' \in G^+L(\mathcal{H})$. Then $\mathcal{F}$ is a Parseval (T, T')-controlled frame if and only if $T = WS_{\mathcal{F}}^{-q}$ and $T' = W'S_{\mathcal{F}}^{-p}$, where W, W' are two operators on $\mathcal{H}$ such that $W'W^* = Id_{\mathcal{H}}$ and p, q are real numbers such that $p + q = 1$.*

Proof. Let $\mathcal{F}$ be a Parseval (T, T')-controlled frame for $\mathcal{H}$. So $S_{TT'} = Id_{\mathcal{H}}$. Therefore, for each pairs of real numbers p, q such that $p + q = 1$, we have

$$Id_{\mathcal{H}} = S_{T,T'} = T'S_{\mathcal{F}}T^* = T'S_{\mathcal{F}}^p S_{\mathcal{F}}^q T^*.$$

We define $W' := T'S_{\mathcal{F}}^p$ and $W := TS_{\mathcal{F}}^q$. So

$$W'W^* = T'S_{\mathcal{F}}^p S_{\mathcal{F}}^q T^* = T'S_F T^* = S_{TT'} = Id_{\mathcal{H}}.$$

Conversely, let W, W' be two operators on $\mathcal{H}$ such that $W'W^* = Id_{\mathcal{H}}$. We define $T := WS_{\mathcal{F}}^{-q}$ and $T' = W'S_{\mathcal{F}}^{-p}$ where p, q are real numbers and $p + q = 1$. So

$$f = T'S_{\mathcal{F}}T^*(f) = \sum_{j\in J} \langle f, Tf_j\rangle T'f_j, \ \ \forall f \in \mathcal{H}.$$

Therefore, $\mathcal{F}$ is a Parseval (T, T')-controlled frame. □

Corollary 3.8. *Let $\mathcal{H}$ be a Hilbert C^*-module, $\mathcal{F} = \{f_j : j \in J\}$ a frame in $\mathcal{H}$. Then $T\mathcal{F} = \{Tf_j : j \in J\}$ is a Parseval frame for $\mathcal{H}$ if and only if $T = WS_{\mathcal{F}}^{-\frac{1}{2}}$ where W is an operator on $\mathcal{H}$ such that $WW^* = Id_{\mathcal{H}}$.*

Proof. Let $W' = W$ and $p = q = \frac{1}{2}$ in theorem 3.8. □

4. Multipliers of Controlled Frames in Hilbert C^*-modules

In this section, we introduce controlled multipliers with two operators in Hilbert C^*-module. We show the inverse of controlled frame multiplier with two operators is a controlled frame multiplier too.

First we start by the following lemma. This is a generalization of Proposition 3.1 in [36] to controlled multipliers with two operators.

Lemma 4.1. *Let $\mathcal{H}$ be a Hilbert C^*-module, $T, T' \in GL(\mathcal{H})$ and $\mathcal{F} = \{f_j : j \in J\}$, $\mathcal{G} = \{g_j : j \in J\}$ be T^2 and T^2-controlled Bessel sequences for $\mathcal{H}$, respectively. Let $m \in \ell^\infty(A)$. The operator*

$$M_{m,T\mathcal{F},T'\mathcal{G}} : \mathcal{H} \to \mathcal{H}$$

defined

$$M_{m,T\mathcal{F},T'\mathcal{G}}f := \sum_{j\in J} m_j \langle f, T'g_j\rangle Tf_j$$

is a well defined bounded operator.

Proof. Let $\mathcal{F} = \{f_j : j \in J\}$ and $\mathcal{G} = \{g_j : j \in J\}$ be T^2 and T'^2-controlled Bessel sequences for $\mathcal{H}$ with bounds D and D', respectively.

For any $f, g \in \mathcal{H}$ and finite subset $I \subset J$,

$$\begin{aligned}\left\|\sum_{i\in I} m_i\langle g, T'g_i\rangle Tf_i\right\| &= sup_{f\in\mathcal{H},\|f\|=1}\left\|\sum_{i\in I} m_i\langle g, T'g_i\rangle\langle Tf_i, f\rangle\right\| \\ &\leq sup_{f\in\mathcal{H},\|f\|=1}\left\|\left(\sum_{i\in I}|m_i|^2|\langle g, T'g_i\rangle|^2\right)^{\frac{1}{2}}\left(\sum_{i\in I}|\langle Tf_i, f\rangle|^2\right)^{\frac{1}{2}}\right\| \\ &\leq sup_{f\in\mathcal{H},\|f\|=1}\|m\|_\infty\left\|\left(\sum_{i\in I}|\langle g, T'g_i\rangle|^2\right)^{\frac{1}{2}}\left(\sum_{i\in I}|\langle Tf_i, f\rangle|^2\right)^{\frac{1}{2}}\right\| \\ &\leq \|m\|_\infty\sqrt{DD'}\|f\|\end{aligned}$$

This show that $M_{m,T\mathcal{F},T'\mathcal{G}}$ is well defined and

$$\|M_{m,T\mathcal{F},T'\mathcal{G}}\| \leq \|m\|_\infty\sqrt{DD'}.$$

□

Now we can define controlled multipliers with two operators in Hilbert C^*-modules as follows:

Definition 4.1. *Let $\mathcal{H}$ be a Hilbert C^*-module, $T, T' \in GL(\mathcal{H})$ and $\mathcal{F} = \{f_j : j \in J\}$, $\mathcal{G} = \{g_j : j \in J\}$ be T^2 and T'^2-controlled Bessel sequences for $\mathcal{H}$, respectively. Let $m \in \ell^\infty(A)$. The operator*

$$M_{m,T\mathcal{F},T'\mathcal{G}} : \mathcal{H} \to \mathcal{H}$$

defined by

$$M_{m,T\mathcal{F},T'\mathcal{G}}f := \sum_{j\in J} m_j\langle f, T'g_j\rangle Tf_j$$

is called the (T, T')-controlled multiplier operator with symbol m.

For frames we will call the resulting (T, T')-controlled Bessel multiplier a (T, T')-controlled frame multiplier.

Let us denote $M_{m,T\mathcal{F}} = M_{m,T\mathcal{F},T\mathcal{F}}$.

Consider the diagonal operator

$$D_m : \ell^2(A) \to \ell^2(A)$$

corresponding to sequence $m = \{m_j\} \in \ell^p$ for $p > 0$ which is defined by

$$D_m\{a_j\}_{j\in J} := \{m_j a_j\}_{j\in J}, \ \{a_j\}_{j\in J} \in \ell^2,$$

so definition of a controlled Bessel multiplier can also be expressed in the following way:

$$M_{m,T\mathcal{F},T'\mathcal{G}} = U^*_{T\mathcal{F}} D_m U_{T'\mathcal{G}}.$$

We have the following lemma from [36].

Lemma 4.2. *Let $m = \{m_j\} \in \ell^\infty(Z(A))$. Then the operator D_m is well defined, adjointable with $D^*_m = D_{m^*}$ and $\|D_m\| \leq \|m\|_\infty$, where $m^* = \{m^*_j\}_{j\in J}$.*

The following theorem is a generalization of Theorem 6.1 in [31] and Theorem 3.3 in [36] to controlled multipliers with two operators in Hilbert C^*-modules.

Theorem 4.3. *Let $\mathcal{H}$ be a Hilbert C^*-module and $T, T' \in GL(\mathcal{H})$. Also, let $M = M_{m,T\mathcal{F},T'\mathcal{G}}$ be a (T,T')-controlled Bessel multiplier for T^2-controlled Bessel sequences $\mathcal{F} = \{f_j\}_{j\in J}$ and T'^2-controlled Bessel sequences $\mathcal{G} = \{g_j\}_{j\in J}$ with the bounds D and D'. Then*

(1) *If $m \in \ell^\infty$, M is a well-defined bounded operator with $\|M\|_{Op} \leq \sqrt{DD'}\|m\|_\infty$. Furthermore the sum $\sum_{j\in J} m_j\langle g, g_j\rangle f_j$ converges unconditionally for all $g \in \mathcal{H}$;*
(2) *$(M_{m,T_{\mathcal{F}},T'_{\mathcal{G}}})^* = M_{m,T'_{\mathcal{G}},T_{\mathcal{F}}}$. Therefore even if m is self-adjoint and $\{f_j\}_{j\in J} = \{g_j\}_{j\in J}$, the multiplier operator M is not self-adjoint;*
(3) *If $m \in c_0$, then the multiplier operator M is compact operator;*
(4) *If $m \in \ell^2$, then the multiplier operator M is a Hilbert-Schmidt operator with $\|M\|_{\mathcal{H}S} \leq \sqrt{DD'}\|m\|_2$.*

Proof. (1)

$$\|M\|_{Op} = \|U^*_{T\mathcal{F}} D_m U_{T'\mathcal{G}}\|_{Op} \leq \|U^*_{T\mathcal{F}}\|_{Op}\|D_m\|_\infty\|U_{T'\mathcal{G}}\|_{Op} \leq \sqrt{DD'}\|m\|_\infty$$

As $\{f_j\}_{j\in J}$ is a T^2-controlled Bessel sequence, $\sum C_j T f_j$ convergence unconditionally for all $\{c_j\}_{j\in J} \in \ell^2(A)$, in particular for $\{m_j.\langle g, Tg_j\rangle\}_{j\in J}$.

(2) $M_{m,T\mathcal{F},T'\mathcal{G}} = U^*_{T\mathcal{F}} D_m U_{T'\mathcal{G}}$, so with Lemma 4.3

$$(M_{m,T\mathcal{F},T'\mathcal{G}})^* = (U^*_{T\mathcal{F}} D_m U_{T'\mathcal{G}})^* = U^*_{T\mathcal{G}} D^*_m U_{T\mathcal{F}} = M_{m^*,T'\mathcal{G},T\mathcal{F}}.$$

(3) Let m_N be the finite sequences, then

$$\begin{aligned}\|M_{m_N} - M_m\|_{Op} &= \|U^*_{T\mathcal{F}} D_{m_N} U_{T'\mathcal{G}} - U^*_{T\mathcal{F}} D_m U_{T'\mathcal{G}}\|_{Op} = \|U^*_{T\mathcal{F}}(D_{m_N} - D_m)U_{T'\mathcal{G}}\|_{Op}\\ &\leq \|U^*_{T\mathcal{F}}\|_{Op}\|D_{m_N} - D_m\|_{Op}\|U_{T'\mathcal{G}}\|_{Op} \leq \sqrt{DD'}\epsilon.\end{aligned}$$

For every $\epsilon' = \frac{\epsilon}{\sqrt{DD'}}\|C\|_{Op}$, there is a N_ϵ such that $\|D_{m_N} - D_m\|_{Op} < \epsilon'$ and therefore $\|M_{m_N} - M_m\|_{Op} < \epsilon$ for all $N > N_\epsilon$. The operator M_{m_N} is a finite sum of rank one operators and so has finite rank. This means that M_m is a limit of finite-rank operators and therefore compact.

(4) The operator $D_m : \ell^2 \to \ell^2$ is in $\mathcal{H}S$ due with bound $\|D_m\|_{\mathcal{H}S} = \|m\|_2$. Using the properties of $\mathcal{H}S$ operators we get

$$\|U^*_{T\mathcal{F}} D_m U_{T'\mathcal{G}}\|_{\mathcal{H}S} \leq \|U_{T\mathcal{F}}\|_{Op}\|m\|_2\|U_{T'\mathcal{G}}\|_{Op} \leq \sqrt{DD'}\|m\|_2.$$

□

The following theorem shows that the inverse of a controlled multiplier operator is a controlled multiplier operator. This generalized Theorem 1.1. in [37] for controlled multipliers with two operators in Hilbert C^*-modules.

Theorem 4.4. *Let $\mathcal{H}$ be an Hilbert A-module, $T, T' \in GL(\mathcal{H})$ and $\mathcal{F}, \mathcal{G}$ be T^2-controlled and T'^2-controlled frames for $\mathcal{H}$. Also let the symbol $m = (m_j) \in \ell^\infty(Z(A))$ satisfy $0 < inf_n|m_n| \leq sup_n|m_n| < \infty$. Assume that the (T,T')-controlled frame operator $M_{m,T\mathcal{F},T'\mathcal{G}}$ is invertible. Then there exists a dual frame $\mathcal{F}^+$ of $\mathcal{F}$, so that for any T'^2-controlled dual frame $\mathcal{G}^d$ of $\mathcal{G}$*

$$M^{-1}_{m,T\mathcal{F},T'\mathcal{G}} = M_{m^{-1},T'\mathcal{G}^d,T\mathcal{F}^+}.$$

Proof. Denote $M := M_{m,T\mathcal{F},T'\mathcal{G}}$ and $T\mathcal{F}^+ = (M^{-1}(m_j T' g_j))_{j\in J}$. First observe that TF^+ is a dual frame of $T\mathcal{F}$. Therefore,

$$M^{-1}U^*_{T'\mathcal{G}} e_j = U^*_{T\mathcal{F}^+} D_{m^{-1}} e_j, \; j \in J.$$

Now the boundedness of the operators implies that $M^{-1}U^*_{T'\mathcal{G}} = U^*_{T\mathcal{F}^+}D_{m^{-1}}$ on $\ell^2(A)$. Using any dual frame $T'\mathcal{G}^d$ of $T'\mathcal{G}$ we get $M^{-1} = U^*_{T\mathcal{F}^+}D_{m^{-1}}U_{T'\mathcal{G}^d}$ on Hilbert A-module $\mathcal{H}$. Therefore

$$M^{-1}_{m,T\mathcal{F},T'\mathcal{G}} = M_{m^{-1},T'\mathcal{G}^d,T\mathcal{F}^+}.$$

□

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