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ON FUZZY TOPOLOGIES GENERATED BY FUZZY RELATIONS

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Abstract. In this paper, we have introduced a fuzzy topology generated by a fuzzy relation as a generalization of the corresponding concept given by Smithson and obtained sufficient conditions under which this fuzzy topology becomes fuzzy T_0, fuzzy T_1 and fuzzy T_2. We have also introduced 'finite intersection property (F.I.P.)' for fuzzy topological spaces and shown that in a fuzzy topological space (X, τ), this property is equivalent to the Lowen's fuzzy compactness. We have also obtained a sufficient condition under which a fuzzy topology generated by a fuzzy relation becomes fuzzy compact.

Keywords: Fuzzy topology, fuzzy relation, separation axioms, fuzzy compactness.

AMS Subject Classification: 54A40.

1. Introduction

Smithson[11] initiated the study of topologies induced by binary relations. Since then many researchers have been working in this area(cf.[6],[4],[1] etc.).

Various types of binary relations have been used to define induced topologies. Dallen and Wattel[3] had obtained a characterization of orderable topologies. Campión, Candeal and Induráin[2] introduced and studied preorderable topologies. Knoblauch[6] introduced topologies induced by a relation $\mathcal{R}$ of a general kind, which are generated by the set of all upper and lower contours of $\mathcal{R}$ taken as a subbase for open sets. He obtained a characterization of a topology on a set X which is induced by a relation of general kind.

Smithson[11] defined a topology on X, induced by a relation $\mathcal{R}$, taking the set $\mathcal{S} = D(\mathcal{A}) \cup I(\mathcal{A}) \cup \{X\} \cup \{\phi\}$ as a subbase for closed sets in X, where $\mathcal{A}$ is a collection of antisets in X, with respect to $\mathcal{R}$ ($A \subseteq X$ is called an antiset if no two distinct elements of A are $\mathcal{R}$-related) and $D(\mathcal{A})$, $I(\mathcal{A})$ are defined as follows:

$$D(\mathcal{A}) = \{\mathcal{R}A : A \in \mathcal{A}\}, \quad I(\mathcal{A}) = \{A\mathcal{R} : A \in \mathcal{A}\}$$

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where,

$$\mathcal{R}A = \{x : (x, a) \in \mathcal{R}, \text{ for some } a \in A\},$$
$$A\mathcal{R} = \{x : (a, x) \in \mathcal{R}, \text{ for some } a \in A\}.$$

He obtained sufficient conditions under which the topology induced by a binary relation(in his sense) satisfies separation axioms, compactness and connectedness.

In this paper, we have introduced a fuzzy topology generated by a fuzzy relation, as a generalization of the corresponding concept given by Smithson[11]. In Section 2, we have obtained sufficient conditions under which this generated fuzzy topology will satisfy separation axioms, fuzzy T_0, fuzzy T_1 and fuzzy T_2. In Section 3, we have introduced 'finite intersection property(F.I.P.)' in a fuzzy topological space and then obtained a characterization of (Lowen's) fuzzy compactness in terms of F.I.P. Using this result, we have obtained a sufficient condition under which a fuzzy topology generated by a fuzzy relation, becomes fuzzy compact.

2. Preliminaries

Definition 2.1. [14] *A fuzzy set in X is a function $f : X \to I$, where I is the closed unit interval $[0, 1]$. Now we define some basic fuzzy set operations as follows:*

Let f and g be two fuzzy sets in X. Then

(1) $f = g$ *if* $f(x) = g(x)$, $\forall x \in X$.
(2) $f \subseteq g$ *if* $f(x) \leq g(x)$, $\forall x \in X$.
(3) $(f \cup g)(x) = \max\{f(x), g(x)\}, \forall x \in X$.
(4) $(f \cap g)(x) = \min\{f(x), g(x)\}$, $\forall x \in X$.
(5) $f^c(x) = 1 - f(x)$, $\forall x \in X$ (*here f^c denotes the complement of f*).

A constant fuzzy set in X taking value $\alpha \in [0, 1]$ will be denoted by α_X.

Definition 2.2. [9] *Let Ω be an index set and $\{f_i : i \in \Omega\}$ be a family of fuzzy sets in X. Then their union $\bigcup_{i\in\Omega} f_i$ and intersection $\bigcap_{i\in\Omega} f_i$ are defined respectively as follows:*

(1) $(\bigcup_{i\in\Omega} f_i)(x) = \sup\{f_i(x) : i \in \Omega\}, \ \forall x \in X$.
(2) $(\bigcap_{i\in\Omega} f_i)(x) = \inf\{f_i(x) : i \in \Omega\}, \ \forall x \in X$.

Definition 2.3. [12] *A fuzzy point $x_\lambda (0 < \lambda < 1)$ in X is a fuzzy set in X such that*

$$x_\lambda(x') = \begin{cases} \lambda, & \text{if } x' = x \\ 0, & \text{otherwise.} \end{cases}$$

Here x and λ are respectively called the support and value of x_λ. A fuzzy point x_λ is said to belong to a fuzzy set f if $\lambda < f(x)$ and two fuzzy points x_r and y_s in X are said to be distinct if $x \neq y$.

Definition 2.4. [7] *A fuzzy topological space is a pair (X, τ) consisting of a non empty set X and a family τ of fuzzy sets in X satisfying the following conditions:*

(1) $\alpha_X \in \tau, \forall \alpha \in [0, 1]$;
(2) *If $\{f_i : i \in \Omega\}$ is an arbitrary family of fuzzy sets in τ, then $\bigcup_{i\in\Omega} f_i \in \tau$.*
(3) *If $f, g \in \tau$, then $f \cap g \in \tau$.*

Then τ is called a fuzzy topology on X and the members of τ are called fuzzy open sets (or $\tau-$ fuzzy open sets). A fuzzy set f in X is called fuzzy closed if $f^c \in \tau$.

Definition 2.5. [14] *A fuzzy relation $\mathcal{R}$ on X is a fuzzy set in $X \times X$ i.e., $\mathcal{R}$ is a mapping from $X \times X$ to* $[0, 1]$.

Two elements x, y of X are said to be $\mathcal{R}$-related if $\mathcal{R}(x, y) > 0$.

Definition 2.6. [15] *A fuzzy relation $\mathcal{R}$ on X is said to be*

(1) *reflexive if $\mathcal{R}(x, x) = 1$, for each $x \in X$;*
(2) *antisymmetric if $\mathcal{R}(x, y) > 0$ and $\mathcal{R}(y, x) > 0$ implies that $x = y$;*
(3) *transitive if $\mathcal{R}(x, z) \geq \min\{\mathcal{R}(x, y), \mathcal{R}(y, z)\}$, for each $x, y, z \in X$;*

Definition 2.7. [15] *A fuzzy relation in X is said to be a fuzzy partial order if it is reflexive, antisymmetric and transitive.*

Definition 2.8. *Let X be a non empty set and $A \subseteq X$. Then A is called an $\mathcal{R}$-antiset of X if no two distinct elements of A are $\mathcal{R}$-related.*

Definition 2.9. *Let X be a non empty set, $\mathcal{R}$ be a fuzzy relation on X and $A \subseteq X$. Then fuzzy sets L_A and R_A on X, are defined as follows:*

$$L_A(y) = \sup_{a \in A} \mathcal{R}(y, a),$$
$$R_A(y) = \sup_{a \in A} \mathcal{R}(a, y),$$

for each $y \in X$.

Smithson[11] introduced and studied a topology on a set X, induced by a relation $\mathcal{R}$. Here we generalize this concept in the fuzzy setting.

Definition 2.10. *Let $\mathcal{R}$ be a fuzzy relation on X and $\mathcal{A}$ be a collection of $\mathcal{R}$-antisets of X. Then $\tau_{\mathcal{R},\mathcal{A}}$ is the fuzzy topology on X generated by taking*

$$\mathcal{S} = \{L_A\}_{A \in \mathcal{A}} \cup \{R_A\}_{A \in \mathcal{A}} \cup \{\alpha_X : \alpha \in [0, 1]\},$$

as a subbase for fuzzy closed sets in X i.e., every fuzzy closed set in X can be written as an intersection of finite unions of members of $\mathcal{S}$.

Example 2.1. *Let $\mathcal{R}$ be a fuzzy relation on $X = \{a, b\}$, which is given as follows:*

$\mathcal{R}$	a	b
a	0.1	0
b	0	0.3

and $\mathcal{A} = \{\{a\}, \{a, b\}\}$. Then the fuzzy topology $\tau_{\mathcal{R},\mathcal{A}}$ is generated by the following subbase $\mathcal{S}$ for the fuzzy closed sets in X:

$$\mathcal{S} = \{L_{\{a\}}, L_{\{a,b\}}, R_{\{a\}}, R_{\{a,b\}}\} \cup \{\alpha_X : \alpha \in [0, 1]\},$$

where $L_{\{a\}}, L_{\{a,b\}}, R_{\{a\}}, R_{\{a,b\}}$ are given by:

$$L_{\{a\}} = \frac{0.1}{a} + \frac{0}{b}, \quad L_{\{a,b\}} = \frac{0.1}{a} + \frac{0.3}{b}, \quad R_{\{a\}} = \frac{0.1}{a} + \frac{0}{b}, \quad R_{\{a,b\}} = \frac{0.1}{a} + \frac{0.3}{b}.$$

So, the collection $\mathcal{H}$ of fuzzy closed sets in X consists of arbitrary intersections of finite unions of members of $\mathcal{S}$ and hence $\tau_{\mathcal{R},\mathcal{A}}$ is the set obtained by taking complements of members of $\mathcal{H}$.

We mention here that in [10], for a given fuzzy relation $\mathcal{R}$ on X, the authors have introduced and studied the fuzzy topology τ on X generated by $\{L_{\{x\}}, R_{\{x\}}\}_{x \in X}$ considered as a subbase for fuzzy open sets in X, i.e, every member of τ is a union of finite intersections of members of the set $\{L_{\{x\}}, R_{\{x\}}\}_{x \in X}$. This is a generalization of the corresponding concept given in [6].

Definition 2.11. *Let (X, τ) be a fuzzy topological space. Then (X, τ) is said to be*

(1) *fuzzy T_0 if for each $x, y \in X$ such that $x \neq y$, there exists a fuzzy closed set U in X such that $U(x) \neq U(y)$;*
(2) *fuzzy T_1 if for each $x, y \in X$ such that $x \neq y$, there exist two fuzzy closed sets U, V such that $U(x) = 1$, $U(y) = 0$, $V(x) = 0$, $V(y) = 1$;*
(3) *fuzzy T_2 if for each pair of distinct fuzzy points x_r, y_s in X, there exist two fuzzy closed sets U, V such that $r > U(x), s > V(y)$ and $U \cup V = 1_X$.*

We remark here that (1), (2), (3) given above are equivalent to the definitions of fuzzy T_0, fuzzy T_1 and fuzzy T_2 given in [8], [13], [12], respectively.

Definition 2.12. [7] *Let (X, τ) be a fuzzy topological space. Then a fuzzy set f in X is said to be fuzzy compact if for any $\beta \subseteq \tau$ such that $\bigcup\limits_{\mu \in \beta} \mu \supseteq f$ and for $\epsilon > 0$, there exists a finite subfamily $\beta_0 \subseteq \beta$ such that $\bigcup\limits_{\mu \in \beta_0} \mu \supseteq f - \epsilon$.*

A fuzzy topological space (X, τ) is said to be fuzzy compact if each constant fuzzy set in X is fuzzy compact.

Proposition 2.1. [13] *Let (X, τ) be a fuzzy topological space. Then the following statements are equivalent:*

(1) *(X, τ) is fuzzy T_1.*
(2) *$\{x\}$ is fuzzy closed, $\forall x \in X$.*

3. Separation axioms

In this section, we prove some results in the fuzzy setting, which are counterparts of the corresponding results given in [11]. In our discussion here, we shall assume that $\mathcal{R}$ is a fuzzy relation on X and $\mathcal{A}$ is a collection of $\mathcal{R}$-antisets.

Proposition 3.1. *Let for each $x, y \in X$ such that $x \neq y$, there exist $z \in X$ such that $\mathcal{R}(x, z) \neq \mathcal{R}(y, z)$ (or $\mathcal{R}(z, x) \neq \mathcal{R}(z, y)$)and $\mathcal{A}$ contains singletons. Then $(X, \tau_{\mathcal{R},\mathcal{A}})$ is fuzzy T_0.*

Proof. Let $x, y \in X$ such that $x \neq y$. Then by our assumption, there exists $z \in X$ such that $\mathcal{R}(x, z) \neq \mathcal{R}(y, z)$ which implies that $L_{\{z\}}(x) \neq L_{\{z\}}(y)$. Since $L_{\{z\}}$ is fuzzy closed in $\tau_{\mathcal{R},\mathcal{A}}$ and is such that $L_{\{z\}}(x) \neq L_{\{z\}}(y)$, so $(X, \tau_{\mathcal{R},\mathcal{A}})$ is fuzzy T_0.

Similarly, we can proceed for the case when $\mathcal{R}(z, x) \neq \mathcal{R}(z, y)$. □

Example 3.1. *Let $\mathcal{R}$ be a fuzzy relation on $X = \{a, b\}$, which is given as follows:*

$\mathcal{R}$	a	b
a	*0.7*	*0.3*
b	*0.6*	*0.5*

and $\mathcal{A} = \{\{a\}, \{b\}\}$. Then the fuzzy topology $\tau_{\mathcal{R},\mathcal{A}}$ is generated by the following subbase $\mathcal{S}$ for the fuzzy closed sets in X:

$$\mathcal{S} = \{L_{\{a\}}, L_{\{b\}}, R_{\{a\}}, R_{\{b\}}\} \cup \{\alpha_X : \alpha \in [0, 1]\},$$

where $L_{\{a\}}, L_{\{b\}}, R_{\{a\}}, R_{\{b\}}$ are given by:

$$L_{\{a\}} = \frac{0.7}{a} + \frac{0.6}{b}, \quad L_{\{b\}} = \frac{0.3}{a} + \frac{0.5}{b}, \quad R_{\{a\}} = \frac{0.7}{a} + \frac{0.3}{b}, \quad R_{\{b\}} = \frac{0.6}{a} + \frac{0.5}{b}.$$

Note that $(X, \tau_{\mathcal{R},\mathcal{A}})$ is fuzzy T_0 since for $a, b \in X$, $a \neq b$, there exists a fuzzy closed set $U = L_{\{a\}}$ in X such that $U(a) \neq U(b)$.

Proposition 3.2. *Let for each $x \in X$ and $y \in X \setminus \{x\}$, there exists $z \in X$ such that $\mathcal{R}(x,z) = 1$ (or $\mathcal{R}(z,x) = 1$) and $\mathcal{R}(y,z) = 0$ (or $\mathcal{R}(z,y) = 0$). Then $\mathcal{A}$ contains singletons implies that $(X, \tau_{\mathcal{R},\mathcal{A}})$ is fuzzy T_1.*

Proof. In view of Proposition 2.1, we show that $\{x\}$ is fuzzy closed. Let $y_r \in X \setminus \{x\}$. Then $x \neq y$. So according to our assumption, there exists $z \in X$ such that $\mathcal{R}(x,z) = 1$ and $\mathcal{R}(y,z) = 0$, which implies that $L_{\{z\}}(x) = 1$ and $L_{\{z\}}(y) = 0$. Therefore, there exists $L^c_{\{z\}} \in \tau_{\mathcal{R},\mathcal{A}}$ such that $y_r \in L^c_{\{z\}} \subseteq X \setminus \{x\}$, which implies that $\{x\}$ is fuzzy closed.
The other case can be handled similarly. □

Corollary 3.1. *Let $\mathcal{R}$ be a fuzzy relation on X which is reflexive and antisymmetric. Then $\mathcal{A}$ contains singletons implies that $(X, \tau_{\mathcal{R},\mathcal{A}})$ is fuzzy T_1.*

Proof. Let $x, y \in X$ such that $x \neq y$. Then by the antisymmetry of $\mathcal{R}$, either $\mathcal{R}(x,y) = 0$ or $\mathcal{R}(y,x) = 0$. Also by the reflexivity of $\mathcal{R}$, $\mathcal{R}(x,x) = 1$, for each $x \in X$. In both the cases, if we set $z = x$, then by Proposition 3.2, $(X, \tau_{\mathcal{R},\mathcal{A}})$ is fuzzy T_1. □

Definition 3.1. *A collection $\mathcal{A}$ of $\mathcal{R}$-antisets is called separating if for each $x \in X$ and $y \in X \setminus \{x\}$, there exists $A \in \mathcal{A}$ such that $L_A(x) = 1$ and $L_A(y) = 0$ or $R_A(x) = 1$ and $R_A(y) = 0$.*

Proposition 3.3. *If $\mathcal{A}$ is separating, then $(X, \tau_{\mathcal{R},\mathcal{A}})$ is fuzzy T_1.*

Proof. Let $x, y \in X$ such that $x \neq y$. Then $y \in X \setminus \{x\}$ and so by our assumption, there exists $A \in \mathcal{A}$ such that $L_A(x) = 1$ and $L_A(y) = 0$ or $R_A(x) = 1$ and $R_A(y) = 0$. Similarly, for $x \in X \setminus \{y\}$, there exists $A' \in \mathcal{A}$ such that $L_{A'}(y) = 1$ and $L_{A'}(x) = 0$ or $R_{A'}(y) = 1$ and $R_{A'}(x) = 0$. Since $L_A, L_{A'}, R_A, R_{A'}$ are fuzzy closed sets in X, therefore $(X, \tau_{\mathcal{R},\mathcal{A}})$ is fuzzy T_1. □

Example 3.2. *Let $\mathcal{R}$ be a fuzzy relation on $X = \{a, b\}$, which is given as follows:*

$\mathcal{R}$	a	b	c
a	1	0.3	0.4
b	0	1	0
c	0	0.5	1

and $\mathcal{A} = \{\{a\}, \{b\}, \{c\}\}$. Then the fuzzy topology $\tau_{\mathcal{R},\mathcal{A}}$ is generated by the following subbase $\mathcal{S}$ for the fuzzy closed sets in X:

$$\mathcal{S} = \{L_{\{a\}}, L_{\{b\}}, L_{\{c\}}, R_{\{a\}}, R_{\{b\}}, R_{\{c\}}\} \cup \{\alpha_X : \alpha \in [0,1]\},$$

where $L_{\{a\}}, L_{\{b\}}, L_{\{c\}}, R_{\{a\}}, R_{\{b\}}, R_{\{c\}}$ are given by:

$$L_{\{a\}} = \frac{1}{a} + \frac{0}{b} + \frac{0}{c}, \quad L_{\{b\}} = \frac{0.3}{a} + \frac{1}{b} + \frac{0.5}{c}, \quad L_{\{c\}} = \frac{0.4}{a} + \frac{0}{b} + \frac{1}{c},$$
$$R_{\{a\}} = \frac{1}{a} + \frac{0.3}{b} + \frac{0.4}{c}, \quad R_{\{b\}} = \frac{0}{a} + \frac{1}{b} + \frac{0}{c}, \quad R_{\{c\}} = \frac{0}{a} + \frac{0.5}{b} + \frac{1}{c}.$$

Since, $L_{\{a\}} = \{a\}, R_{\{b\}} = \{b\}$ and $L_{\{c\}} \cap R_{\{c\}} = \{c\}$ are fuzzy closed sets in X, so in view of the Proposition 2.1, $(X, \tau_{\mathcal{R},\mathcal{A}})$ is fuzzy T_1.

Definition 3.2. *A collection $\mathcal{A}$ of $\mathcal{R}$-antisets completely separates points of X if for each $x, y \in X$ such that $x \neq y$, there exist $B_1, B_2, ..., B_k$, where for each i, $B_i = L_A$ or R_A, $A \in \mathcal{A}$ such that $1_X = \bigcup_{i=1}^{k} B_i$ and $B_i(x) > 0$ implies that $B_i(y) = 0$.*

Theorem 3.1. *If $\mathcal{A}$ completely separates points of X, then $(X, \tau_{\mathcal{R},\mathcal{A}})$ is fuzzy T_2.*

Proof. Let x_r and y_s be two distinct fuzzy points in X. Then $x \neq y$. Since $\mathcal{A}$ completely separates points of X, so there exist $B_1, B_2, ..., B_k$, where for each i, $B_i = L_A$ or R_A, $A \in \mathcal{A}$ such that

$$1_X = \bigcup_{i=1}^{k} B_i \tag{1}$$

and $B_i(x) > 0$ implies that $B_i(y) = 0$. Since $1_X = \bigcup\limits_{i=1}^{k} B_i$, so for $x, y \in X$, there exist i_1 and i_2 such that $B_{i_1}(x) = 1$ and $B_{i_2}(y) = 1$. Next, since $B_{i_1}(x) = 1 > 0$, so $B_{i_1}(y) = 0$. Let $X_1 = \cup\{B_i | B_i(y) = 0\}$ and $X_2 = \cup\{B_i | B_i(y) > 0\}$. Here $X_1(y) = 0, X_2(x) = 0$ and in view of (1), we have $X_1 \cup X_2 = 1_X$. Note that X_1 and X_2 are fuzzy closed sets in X such that $r > X_2(x)$, $s > X_1(y)$ and $X_1 \cup X_2 = 1_X$. Therefore, $(X, \tau_{\mathcal{R},\mathcal{A}})$ is fuzzy T_2. □

Example 3.3. *Let $\mathcal{R}$ be a fuzzy relation on $X = \{a, b, c\}$, which is given as follows:*

$\mathcal{R}$	a	b	c
a	1	0	0.3
b	0.7	1	0
c	0	0.8	1

and $\mathcal{A} = \{\{a\}, \{b\}, \{c\}\}$. Then the fuzzy topology $\tau_{\mathcal{R},\mathcal{A}}$ is generated by the following subbase $\mathcal{S}$ for the fuzzy closed sets in X:

$$\mathcal{S} = \{L_{\{a\}}, L_{\{b\}}, L_{\{c\}}, R_{\{a\}}, R_{\{b\}}, R_{\{c\}}\} \cup \{\alpha_X : \alpha \in [0, 1]\},$$

where $L_{\{a\}}, L_{\{b\}}, L_{\{c\}}, R_{\{a\}}, R_{\{b\}}, R_{\{c\}}$ are given by:

$$L_{\{a\}} = \frac{1}{a} + \frac{0.7}{b} + \frac{0}{c}, \quad L_{\{b\}} = \frac{0}{a} + \frac{1}{b} + \frac{0.8}{c}, \quad L_{\{c\}} = \frac{0.3}{a} + \frac{0}{b} + \frac{1}{c},$$
$$R_{\{a\}} = \frac{1}{a} + \frac{0}{b} + \frac{0.3}{c}, \quad R_{\{b\}} = \frac{0.7}{a} + \frac{1}{b} + \frac{0}{c}, \quad R_{\{c\}} = \frac{0}{a} + \frac{0.8}{b} + \frac{1}{c}.$$

Since for the fuzzy points $x_r, y_s \in X$, there exist two fuzzy closed sets $U = L_{\{b\}} \cup R_{\{c\}}$ and $V = L_{\{c\}} \cup R_{\{a\}}$ in X such that $r > U(x)$, $s > V(y)$ and $U \cup V = 1_X$, for the fuzzy points $y_r, z_s \in X$, there exist two fuzzy closed sets $U = L_{\{c\}} \cup R_{\{a\}}$ and $V = L_{\{a\}} \cup R_{\{b\}}$ in X such that $r > U(y)$, $s > V(z)$ and $U \cup V = 1_X$ and for the fuzzy points $z_r, x_s \in X$, there exist two fuzzy closed sets $U = L_{\{a\}} \cup R_{\{b\}}$ and $V = L_{\{b\}} \cup R_{\{c\}}$ in X such that $r > U(z)$, $s > V(x)$ and $U \cup V = 1_X$, so $(X, \tau_{\mathcal{R},\mathcal{A}})$ is fuzzy T_2.

4. Finite intersection property and fuzzy compactness

In this section, we introduce 'finite intersection property' in fuzzy topological spaces and find a sufficient condition under which a fuzzy topology generated by a fuzzy relation, becomes fuzzy compact.

Definition 4.1. *A family $\mathcal{F}$ of fuzzy sets is said to satisfy the finite intersection property(F.I.P.) if for every $\alpha \in (0, 1]$, there exists ϵ, $0 < \epsilon < \alpha$ such that for every finite subfamily $F_1, F_2, ..., F_n$ of $\mathcal{F}$, there exists $x \in X$ such that $(\bigcap\limits_{i=1}^{n} F_i)(x) > 1 - \alpha + \epsilon$.*

Theorem 4.1. *Let (X, τ) be a fuzzy topological space. Then the following statements are equivalent:*

(1) *If $\mathcal{F}$ is a family of fuzzy closed sets satisfying finite intersection property (F.I.P.), then for each $\alpha \in (0, 1]$, there exists $y \in X$ such that $(\bigcap\limits_{F \in \mathcal{F}} F)(y) > 1 - \alpha$.*

(2) *(X, τ) is fuzzy compact.*

Proof. (1) $\Rightarrow$ (2) Let $\mathcal{G}$ be a family of fuzzy open sets in X such that

$$\begin{aligned} & \alpha_X \subseteq \bigcup_{G_i \in \mathcal{G}} G_i, \alpha \in (0,1] \\ \Rightarrow \quad & \bigcap_{G_i \in \mathcal{G}} G_i^c \subseteq (1-\alpha)_X \\ \Rightarrow \quad & \nexists y \in X \text{ such that } (\bigcap_{G_i \in \mathcal{G}} G_i^c)(y) > 1-\alpha. \end{aligned}$$

Therefore in view of (1), $\mathcal{F} = \{G_i^c : G_i \in \mathcal{G}\}$ does not satisfy F.I.P., so for each ϵ such that $0 < \epsilon < \alpha$, there exist $G_1^c, G_2^c, ..., G_n^c \in \mathcal{F}$ such that

$$\begin{aligned} & (\bigcap_{i=1}^{n} G_i^c)(x) \leq 1-\alpha+\epsilon, \quad \text{for each } x \in X \\ \Rightarrow \quad & \bigcap_{i=1}^{n} G_i^c \subseteq (1-\alpha+\epsilon)_X \\ \Rightarrow \quad & (\alpha-\epsilon)_X \subseteq \bigcup_{i=1}^{n} G_i \\ \Rightarrow \quad & \alpha_X \text{ is fuzzy compact.} \\ \Rightarrow \quad & (X,\tau) \text{ is fuzzy compact.} \end{aligned}$$

(2) $\Rightarrow$ (1) Conversely, assume that (X,τ) is fuzzy compact i.e., each α_X, $\alpha \in [0,1]$, is fuzzy compact. Let $\mathcal{F}$ be a family of fuzzy closed sets satisfying F.I.P. We have to show that for each $\alpha \in (0,1]$, there exists $x \in X$ such that

$$(\bigcap_{F \in \mathcal{F}} F)(x) > 1-\alpha.$$

Assume the contrary, i.e, for some $\alpha \in (0,1]$,

$$\begin{aligned} & (\bigcap_{F \in \mathcal{F}} F)(x) \leq 1-\alpha, \quad \forall x \in X \\ \Rightarrow \quad & \bigcap_{F \in \mathcal{F}} F \subseteq (1-\alpha)_X \\ \Rightarrow \quad & \alpha_X \subseteq \bigcup_{F \in \mathcal{F}} F^c \end{aligned}$$

This implies that $\{F^c : F \in \mathcal{F}\}$ is an open cover of α_X. Since α_X is fuzzy compact, so for each ϵ such that $0 < \epsilon < \alpha$, there exist $F_1^c, F_2^c, ..., F_n^c \in \mathcal{F}$ such that

$$\begin{aligned} & (\alpha-\epsilon)_X \subseteq \bigcup_{i=1}^{n} F_i^c \\ \Rightarrow \quad & \bigcap_{i=1}^{n} F_i \subseteq (1-\alpha+\epsilon)_X \\ \Rightarrow \quad & \nexists \text{ any } x \text{ such that } (\bigcap_{i=1}^{n} F_i)(x) > 1-\alpha+\epsilon, \end{aligned}$$

implying that $\mathcal{F}$ does not satisfy F.I.P., which is a contradiction. $\square$

Definition 4.2. [5] *Let $\mathcal{R}$ be a fuzzy partial ordering on X and $A \subseteq X$. Then the fuzzy upper bound for A is the fuzzy set denoted by $U(\mathcal{R}, A)$ and defined by*

$$U(\mathcal{R}, A) = \bigcap_{x \in A} R_x, \text{ where } R_x(y) = \mathcal{R}(x, y), \text{ for each } y \in X.$$

Definition 4.3. *A subset A of X is said to be α-level bounded above, $\alpha \in (0, 1]$, if for every ϵ, $0 < \epsilon < \alpha$, there exists $y \in X$ such that*

$$U(\mathcal{R}, A)(y) > 1 - \alpha + \epsilon.$$

We say that A is fuzzy bounded above if it is α-level bounded above, for each $\alpha \in (0, 1]$.

Definition 4.4. *A subset A of X is said to have α-level least upper bound, $\alpha \in (0, 1]$, if for every ϵ, $0 < \epsilon < \alpha$, there exists $z \in X$ such that $U(\mathcal{R}, A)(z) > 1 - \alpha + \epsilon$ and $\mathcal{R}(z, y) > 1 - \alpha + \epsilon$, $\forall y$ such that $\mathcal{R}(x, y) > 1 - \alpha$, for each $x \in A$.*

Definition 4.5. *A set X is said to be α-level complete if every $A \subseteq X$ which is α-level bounded above has an α-level least upper bound.*

We say that X is fuzzy complete if it is $\alpha-$level complete for every $\alpha \in (0, 1]$.

Proposition 4.1. *If X is α-level bounded above and $A \subseteq X$, then A is also α-level bounded above.*

Proof. Since X is α-level bounded above, so for every ϵ, $0 < \epsilon < \alpha$, there exists $y \in X$ such that

$$U(\mathcal{R}, X)(y) > 1 - \alpha + \epsilon.$$

We have to show that for every ϵ, $0 < \epsilon < \alpha$, there exists $y \in X$ such that $U(\mathcal{R}, A)(y) > 1 - \alpha + \epsilon$. Assume the contrary that for each $y \in X$,

$$\begin{aligned}
& U(\mathcal{R}, A)(y) \leq 1 - \alpha + \epsilon \\
\Rightarrow\ & \inf_{a \in A} \mathcal{R}(a, y) \leq 1 - \alpha + \epsilon \\
\Rightarrow\ & U(\mathcal{R}, X)(y) = \inf_{x \in X} \mathcal{R}(x, y) \leq \inf_{a \in A} \mathcal{R}(a, y) \leq 1 - \alpha + \epsilon, \text{ for each } y \in Y
\end{aligned}$$

which is a contradiction to the fact that X is α-level bounded above. □

Theorem 4.2. *Let (X, τ) be a fuzzy topological space. Then the following statements are equivalent:*

(1) *If $\mathcal{F}$ is a family of subbasic fuzzy closed sets satisfying finite intersection property (F.I.P.), then for each $\alpha \in (0, 1]$, there exists $y \in X$ such that $(\bigcap_{F \in \mathcal{F}} F)(y) > 1 - \alpha$.*

(2) *For any subbase $\mathcal{S}$ of τ, if $\mathcal{G} \subseteq \mathcal{S}$ such that $\alpha_X \subseteq \bigcup_{f \in \mathcal{G}} f$, then for each ϵ, $0 < \epsilon < \alpha$, there exists a finite subset $\mathcal{G}_0$ of $\mathcal{G}$ such that $(\alpha - \epsilon)_X \subseteq \bigcup_{f \in \mathcal{G}_0} f$.*

(3) *(X, τ) is fuzzy compact.*

Proof. We prove (1) $\Leftrightarrow$ (2). Let $\mathcal{S}$ be a subbase for τ and $\mathcal{G} \subseteq \mathcal{S}$ such that

$$\begin{aligned}
& \alpha_X \subseteq \bigcup_{f \in \mathcal{G}} f \\
\Rightarrow\ & \bigcap_{f \in \mathcal{G}} f^c \subseteq (1 - \alpha)_X \\
\Rightarrow\ & \nexists\, y \in X \text{ such that } (\bigcap_{f \in \mathcal{G}} f^c)(y) > 1 - \alpha.
\end{aligned}$$

Therefore in view of (1), $\mathcal{F} = \{f^c : f \in \mathcal{G}\}$ does not satisfy F.I.P., so for each ϵ such that $0 < \epsilon < \alpha$, there exist $f_1^c, f_2^c, ..., f_n^c \in \mathcal{G}$ such that

$$\begin{aligned} & (\bigcap_{i=1}^{n} f_i^c)(x) \leq 1 - \alpha + \epsilon, \quad \text{for each } x \in X \\ \Rightarrow\ & \bigcap_{i=1}^{n} f_i^c \subseteq (1 - \alpha + \epsilon)_X \\ \Rightarrow\ & (\alpha - \epsilon)_X \subseteq \bigcup_{i=1}^{n} f_i. \end{aligned}$$

Conversely, assume that $\mathcal{F}$ be a family of subbasic fuzzy closed sets satisfying F.I.P. We have to show that for each $\alpha \in (0, 1]$, there exists $x \in X$ such that

$$(\bigcap_{F \in \mathcal{F}} F)(x) > 1 - \alpha.$$

Assume the contrary i.e, for some $\alpha \in (0, 1]$,

$$\begin{aligned} & (\bigcap_{F \in \mathcal{F}} F)(x) \leq 1 - \alpha, \quad \forall x \in X \\ \Rightarrow\ & \bigcap_{F \in \mathcal{F}} F \subseteq (1 - \alpha)_X \\ \Rightarrow\ & \alpha_X \subseteq \bigcup_{F \in \mathcal{F}} F^c. \end{aligned}$$

So according to our assumption, for each ϵ such that $0 < \epsilon < \alpha$, there exist $F_1^c, F_2^c, ..., F_n^c \in \mathcal{F}$ such that

$$\begin{aligned} & (\alpha - \epsilon)_X \subseteq \bigcup_{i=1}^{n} F_i^c \\ \Rightarrow\ & \bigcap_{i=1}^{n} F_i \subseteq (1 - \alpha + \epsilon)_X \\ \Rightarrow\ & \nexists \text{ any } x \text{ such that } (\bigcap_{i=1}^{n} F_i)(x) > 1 - \alpha + \epsilon, \end{aligned}$$

implying that $\mathcal{F}$ does not satisfy F.I.P., which is a contradiction.
(2) $\Leftrightarrow$ (3) has already been proved in [7]. $\square$

Theorem 4.3. *Let $\mathcal{R}$ be a fuzzy partial order. If X is fuzzy complete, fuzzy bounded above and $\mathcal{A}$ contains singletons, then $(X, \tau_{\mathcal{R},\mathcal{A}})$ is fuzzy compact.*

Proof. Let $\mathcal{F}$ be a family of subbasic fuzzy closed sets of X satisfying F.I.P. First, we show that there does not exist any δ_X, where $\delta \in (0, 1)$, belonging to $\mathcal{F}$. Since if we assume that $\mathcal{F}$ contains some δ_X, where $\delta \in (0, 1)$, then

$$\bigcap_{F \in \mathcal{F}_1} F \subseteq \delta_X,$$

where $\mathcal{F}_1$ is a finite subfamily of $\mathcal{F}$ containing δ_X, this implies that

$$\bigcap_{F\in\mathcal{F}_1} F \subseteq (1-\alpha_1)_X \subseteq (1-\alpha_1+\epsilon)_X, \quad \text{for}\,\alpha_1 \in (0,1)\,\text{such that}\ \delta = (1-\alpha_1)\ \text{and}$$
$$\text{for each}\,\epsilon,\ 0<\epsilon<\alpha_1$$

$\Rightarrow$ $\mathcal{F}$ does not satisfy F.I.P., which is a contradiction.

Next, if $\mathcal{F}$ contains only 1_X, then the proof of the theorem is trivial and if there exists some member of the form $L_{\{x\}}$ or $R_{\{x\}}$ other than 1_X in $\mathcal{F}$, then 1_X plays no role in the intersection $\bigcap\limits_{F\in\mathcal{F}} F$, so it is sufficient to prove the theorem for $\mathcal{F}$ of the form $\{L_{\{x_\beta\}} : \beta\in\Omega_1\}\cup\{R_{\{x_\beta\}} : \beta\in\Omega_2\}$. Now, we have to show that for each $\alpha\in(0,1]$, there exists $z\in X$ such that $(\bigcap\limits_{F\in\mathcal{F}} F)(z) > 1-\alpha$.

Let $A=\{x_\beta : \beta\in\Omega_2\}\subseteq X$. Since for $\alpha\in(0,1]$, X is α-level bounded above, so by Proposition 4.1, A is also α-level bounded above. Therefore, by the fuzzy completeness of X, there exists an α-level least upper bound of A. So, for each ϵ such that $0<\epsilon<\alpha$, there exists an element $x_0\in X$ such that

$$\begin{aligned}
&U(\mathcal{R},A)(x_0) > 1-\alpha+\epsilon \text{ and } \mathcal{R}(x_0,y) > 1-\alpha+\epsilon,\ \forall\, y\in X\\
&\qquad \text{such that}\,\mathcal{R}(x,y) > 1-\alpha,\ \text{for each}\,x\in A \qquad (2)\\
&\Rightarrow\ \inf_{x\in A}\mathcal{R}(x,x_0) > 1-\alpha+\epsilon\\
&\Rightarrow\ \mathcal{R}(x_\beta,x_0) > 1-\alpha+\epsilon,\quad \text{for each}\,\beta\in\Omega_2\\
&\Rightarrow\ R_{\{x_\beta\}}(x_0) > 1-\alpha+\epsilon,\quad \text{for each}\,\beta\in\Omega_2. \qquad (3)
\end{aligned}$$

Since for $\beta\in\Omega_2$, $\mathcal{B}=\{R_{\{x_\beta\}}, L_{\{x_\gamma\}}\}$, where $\gamma\in\Omega_1$, is a finite subfamily of $\mathcal{F}$, so there exists ϵ_1, $0<\epsilon_1<\alpha$ and $y\in X$ such that

$$\begin{aligned}
&(R_{\{x_\beta\}}\cap L_{\{x_\gamma\}})(y) > 1-\alpha+\epsilon_1\\
\Rightarrow\ &\min\{R_{\{x_\beta\}}(y), L_{\{x_\gamma\}}(y)\} > 1-\alpha+\epsilon_1\\
\Rightarrow\ &\min\{\mathcal{R}(x_\beta,y),\mathcal{R}(y,x_\gamma)\} > 1-\alpha+\epsilon_1\\
\Rightarrow\ &\mathcal{R}(x_\beta,x_\gamma) > 1-\alpha+\epsilon_1 > 1-\alpha \quad (\text{Using transitivity of } \mathcal{R}).
\end{aligned}$$

Since the above inequality holds for each $\beta\in\Omega_2$, so we have

$$\begin{aligned}
\Rightarrow\ &\mathcal{R}(x_0,x_\gamma) > 1-\alpha+\epsilon \quad (\text{Using 2 and putting } y=x_\gamma)\\
\Rightarrow\ &L_{\{x_\gamma\}}(x_0) > 1-\alpha+\epsilon.
\end{aligned}$$

The above inequality holds for every $\gamma\in\Omega_1$. So

$$L_{\{x_\beta\}}(x_0) > 1-\alpha+\epsilon, \text{for each } \beta\in\Omega_1. \qquad (4)$$

From (3) and (4), we get

$$\begin{aligned}
&\inf_{F\in\mathcal{F}} F(x_0) \geq 1-\alpha+\epsilon > 1-\alpha\\
\Rightarrow\ &(\bigcap_{F\in\mathcal{F}} F)(x_0) > 1-\alpha. \qquad (5)
\end{aligned}$$

Therefore by Theorem 4.2, $(X,\tau_{\mathcal{R},\mathcal{A}})$ is fuzzy compact. $\square$

5. Conclusion

In this paper, we have introduced fuzzy topologies genearted by fuzzy relations, as a generalization of the corresponding concept given by Smithson[11]. We have then obtained sufficient conditions under which this generated fuzzy topology satisfies seperation axioms, fuzzy T_0, fuzzy T_1 and fuzzy T_2. Further, we have introduced 'finite intersection property' in fuzzy topological spaces and obtained a characterization of Lowen's fuzzy compactness in terms of this property. Using this result, we have obtained a sufficient condition under which a fuzzy topology genearted by a fuzzy relation, becomes fuzzy compact.

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