TWMS J. App. and Eng. Math. V.11, N.4, 2021, pp. 1061-1071

FIXED POINT THEOREMS FOR MULTIVALUED WORDOWSKI TYPE CONTRACTIONS IN B-METRIC SPACES WITH AN APPLICATION TO INTEGRAL INCLUSIONS

HEDDI KADDOURI[1], SAID BELOUL[2], §

Abstract. The aim of this work is to give some fixed point results for set valued F-contractions combined with the concept of α_s-admissible in b-metric spaces. some consequences are established on b-metric spaces endowed with a partial ordering, graph. An example and an application to integral inclusions are given to demonstrate the usability of our results.

Keywords: b-metrics spaces, multivalued F-contraction, α-admissible, integral inclusion..

AMS Subject Classification: 47H10, 54H25

1. Introduction

The metric spaces were generalized to some types, and an important one of these generalizations so called b-metric spaces, which have been introduced by Bakhtin [7] and Czerwik [11]. Later, some fixed point results were obtained in such spaces, for single valued or set valued mappings, for instance see [1, 6, 10, 12]. On other hand Wordowski [25] introduced a new contraction type called F-contraction (or Wordowski contraction), which considered as a generalization of Banach contraction and in this way many works were done.
The concept of α-admissible in the setting of metric spaces was introduced by Samet et al. [21], where they proved some fixed point theorems for $\alpha - \psi$-contractive mappings, some results were obtained via such concepts, see [4,13, 15, 19]. Later Ali et al. [2] introduced the concept of α_s-admissible in the setting of b-metric spaces.

In this paper, we present an existence theorem of multivalued fixed point in b-metric space, using F-contractions concept combined with the notion of α_s-admissible. As consequences, we present an existence of multivalued fixed point theorem in ordered b-metric spaces and another theorem in b-metric spaces endowed with graph. We give also an example and an application to the existence of the solution for a Fredholm integral inclusion.

[1] Department of Mathematics, University of El Oued, P.O.Box 789, El Oued, 39000, Algeria.
e-mail: kaddouriheddi11@gmail.com; ORCID: https://orcid.org/0000-0003-2344-9102.
e-mail: beloulsaid@gmail.com; ORCID: https://orcid.org/0000-0002-2814-2161.

§ Manuscript received: October 12, 2019; accepted: January 20, 2020.

2. **Preliminaries**

Before going towards our findings, we need the following definitions and notions.

Definition 2.1 (2)**.** *Let X be a nonempty set. A mapping $d : X \times X \to [0, \infty)$ is said to be a b-metric on X if for all $x, y, z \in X$, we have a real number $s \geq 1$ such that:*

$(b_1) :$ *$d(x, y) = 0$ if and only if $x = y$;*
$(b_2) :$ *$d(x, y) = d(y, x)$;*
$(b_3) :$ *$d(x, z) \leq s\,[d(x, y) + d(y, z)]$.*

Then the triplet (X, d, s) is said to be a b-metric space.
Note that every metric space is a b-metric but the converse is not always true.

Example 2.1. *Let $X = [0, \infty)$ and $d : X \times X \to [0; \infty)$, $d(x, y) = |x - y|^2$ for each $x, y \in X$. Clearly, $(X, d, 2)$ is a b-metric space, but not a metric space.*

Let (X, d, s) be a b-metric space. The closed and bounded sets in X are defined in a similar manner as for a metric space. We denote by $CB(X)$ the family of all bounded and closed subsets of X.
Let $x \in X$ and $A \subset X$, $D(x, A) = \inf\{d(x, a), a \in A\}$. For $A, B \in CB(X)$, the function $H : CB(X) \times CB(X) \to [0; \infty)$

$$H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\},$$

is said to be a Hausdorff b-metric [11] induced by the b-metric d.
Also, denote the family of nonempty and closed subsets of X by $CL(X)$, the function $H : CL(X) \times CL(X) \to [0; \infty]$, given by

$$H(A, B) = \begin{cases} \max\left\{\sup\limits_{x \in A} d(x, B),\ \sup\limits_{y \in B} d(y, A)\right\} & \text{if the maximum exists;} \\ \infty, & \text{otherwise,} \end{cases}$$

is said to be a generalized Hausdorff b-metric induced by b-metric d.

Lemma 2.1 (2)**.** *Let (X, d, s) be a b-metric space. For any $A, B \in CB(X)$ and any $x, y \in X$ the following properties are satisfied.*

(1) : *$D(x, A) \leq d(x, a)$ for each $a \in A$;*
(2) : *$D(x, B) \leq H(A, B)$ for each $x \in X$;*
(3) : *$D(x, A) \leq s\,[d(x, y) + D(y, A)]$.*

Lemma 2.2. [**?**] *Let (X, d, s) be a b-metric space and $A, B \in CL(X)$ with $H(A, B) > 0$. Then, for each $b \in B$, there exists $a = a(b) \in A$ such that*

$$d(a, b) \leq sH(A, B).$$

Lemma 2.3 (12)**.** *Let (X, d, s) be a b-metric space and $A, B \in CL(X)$. For each $\varepsilon > 0$ and all $b \in B$, there exists $a \in A$ such that*

$$d(a, b) \leq H(A, B) + \varepsilon.$$

Definition 2.2 (10)**.** *Let $s \geq 1$ be a real number. We denote by $\mathcal{F}_s$ the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ with the following properties:*

$(F_1) :$ *F is strictly increasing;*
$(F_2) :$ *For each sequence $\{\alpha_n\} \subset \mathbb{R}^+$, $\lim\limits_{n\to\infty} \alpha_n = 0$ if and only if $\lim\limits_{n\to\infty} F(\alpha_n) = -\infty$;*
$(F_3) :$ *There exists $k \in (0, 1)$ such that $\lim\limits_{\alpha\to 0^+} (\alpha)^k F(\alpha) = 0$;*

(F_4) : *For each sequence* $\{\alpha_n\} \subset \mathbb{R}^+$ *such that* $\tau + F(s\alpha_n) \leq F(\alpha_{n-1})$ *for* $n \in \mathbb{N}$ *and some* $\tau > 0$, *then* $\tau + F(s^n\alpha_n) \leq F(s^{n-1}\alpha_{n-1})$.

Example 2.2. (1) : *Let* $F : (0,\infty) \to \mathbb{R}$ *be defined by* $F(t) = t + \ln t$. *Clearly,* $F \in \mathcal{F}_s$.
(2) : *Let* $F : (0,\infty) \to \mathbb{R}$ *be defined by* $F(t) = \ln t$. $F \in \mathcal{F}_s$

Definition 2.3 (24). *Let* X *be a nonempty set and let* $T : X \to X$, $\alpha : X \times X \to [0,\infty)$ *be two mappings.For a giver real number* $s \geq 1$, T *is weak* α*-admissible of type* S *if for* $x \in X$ *and* $\alpha(x,Tx) \geq s$, *then* $\alpha(Tx,TTx) \geq s$.

Definition 2.4 (2). *Let* (X,d,s) *be a b-metric space and* $\alpha\colon X \times X \to [0,+\infty)$ *be a given function. A mapping* $T\colon X \to CL(X)$ *is an*
(1) α_s*-admissible, if for each* $x \in X$ *and* $y \in Tx$ *with* $\alpha(x,y) \geq s^2$, *we have* $\alpha(y,z) \geq s^2$ *for each* $z \in Ty$.
(2) α_s^**-admissible, if for* $x,y \in X$ *with* $\alpha(x,y) \geq s$ *we have* $\alpha^*(Tx,Ty) \geq s^2$, *where* $\alpha^*(Tx,Ty) = \inf\{\alpha(a,b) :\ a \in Tx, b \in Ty\}$.

Throughout this paper, we will denote by Φ the set of all continuous functions ψ : $[0,+\infty) \to [0,\infty)$ satisfying :

(1) : ψ is nondecreasing;
(2) : $\psi(t) = 0$ if and only if $t = 0$;
(3) : $\sum_{n=1}^{\infty} s^n\psi^n(t) < \infty$, for all $t \in [0;+\infty)$.

Clearly, if $\psi \in \Phi$, then $\psi(t) \leq t$, for all $t \in [0;+\infty)$.

3. Main results

Theorem 3.1. *Let* (X,d,s) *be a complete b-metric space,* $\alpha : X \times X \to [0,\infty)$ *a function. Let* $T : X \to CB(X)$ *be a multi-valued mapping such that*

$$\tau + F(s^3H(Tx,Ty)) \leq F(\psi(M_s(x,y))), \tag{1}$$

for all $x,y \in X$, *with* $\alpha(x,y) \geq s^2$ *and* $H(Tx,Ty) > 0$, *where* $F \in \mathcal{F}_s$, $\psi \in \Phi$ *and*

$$M_s(x,y) = \max\left\{d(x,y), D(x,Tx), D(y,Ty), \frac{D(x,Ty) + D(y,Tx)}{2s}\right\}. \tag{2}$$

Suppose that the following conditions hold:

(i) *T is α_s admissible;*
(ii) *There exist $x_0 \in X$, and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq s^2$;*
(iii) *For every sequence $\{x_n\}$ in X converges to x in X and $\alpha(x_n,x_{n+1}) \geq s^2$, for all $n \in N$, then $\alpha(x_n,x) \geq s^2$, for all $n \in \mathbb{N}$.*

Then T has a fixed point.

Proof. By the hypothesis (ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq s^2$. If $x_0 = x_1$, so $x_0 \in Tx_0$ and x_1 is a fixed point of T, which completes the proof. Suppose $x_0 \neq x_1$ and $x_0 \notin Tx_0$, so $H(Tx_0,Tx_1) > 0$.

By Lemma 2.2, there exists $x_2 \in Tx_1$ such that

$$d(x_1,x_2) \leq s^2H(Tx_0,Tx_1),$$

which implies

$$sd(x_1,x_2) \leq s^3H(Tx_0,Tx_1).$$

F is strictly increasing and $\psi(t) \leq t$ for all $t \geq 0$, we get

$$\begin{aligned} F(sd(x_1,x_2)) &\leq F(s^3H(Tx_0,Tx_1)) \\ &\leq F\big(\psi(M_s(x_0,x_1))\big) - \tau \\ &\leq F(M_s(x_0,x_1)) - \tau. \end{aligned}$$

This yields,

$$F(sd(x_1,x_2)) \leq F(M_s(x_0,x_1)) - \tau, \tag{3}$$

where

$$\begin{aligned} M_s(x_0,x_1) &= \max\left\{d(x_0,x_1), D(x_0,Tx_0), D(x_1,Tx_1), \frac{D(x_0,Tx_1)+D(x_1,Tx_0)}{2s}\right\} \\ &\leq \max\left\{d(x_0,x_1), d(x_0,x_1), D(x_1,Tx_1), \frac{D(x_0,Tx_1)+d(x_1,x_1)}{2s}\right\} \\ &\leq \max\left\{d(x_0,x_1), D(x_1,Tx_1), \frac{D(x_0,Tx_1)}{2s}\right\}. \end{aligned}$$

Since

$$\begin{aligned} \frac{D(x_0,Tx_1)}{2s} &\leq \frac{s\,[d(x_0,x_1)+D(x_1,Tx_1)]}{2s} \\ &\leq \frac{[d(x_0,x_1)+D(x_1,Tx_1)]}{2} \\ &\leq \max\{d(x_0,x_1), D(x_1,Tx_1)\}, \end{aligned}$$

we get

$$M_s(x_0,x_1) \leq \max\{d(x_0,x_1), D(x_1,Tx_1)\}.$$

If $\max\{d(x_0,x_1), D(x_1,Tx_1)\} = D(x_1,Tx_1)$, then

$$\begin{aligned} F(D(x_1,Tx_1) &< F(s^3H(Tx_0,Tx_1) \\ &\leq F(\psi(D(x_1,Tx_1))) - \tau \\ &\leq F(D(x_1,Tx_1)) - \tau \\ &< F(D(x_1,Tx_1)). \end{aligned}$$

This yields, $F(D(x_1,Tx_1)) < F(D(x_1,Tx_1))$, From (F_1) we get $D(x_1,Tx_1) < D(x_1,Tx_1)$, which is a contradiction. Consequently, we obtain

$$F(sd(x_1,x_2)) \leq F(d(x_0,x_1)) - \tau. \tag{4}$$

Proceeding as before, assume that $x_1 \neq x_2$ and $x_1 \notin Tx_1$. Thus $d(x_2,Tx_2) > 0$, and $H(Tx_1,Tx_2) > 0$.
Since $\alpha(x_0,x_1) \geq s^2$, and T is α_s-admissible, we get $\alpha(x_1,x_2) \geq s^2$, for $x_2 \in Tx_1$. Also, by Lemma 2.2, there exists $x_3 \in Tx_2$ such that

$$d(x_2,x_3) \leq s^2H(Tx_1,Tx_2),$$

which implies

$$sd(x_2,x_3) \leq s^2H(Tx_1,Tx_2).$$

Since F is strictly increasing and $\psi(t) \leq t$ for all $t \geq 0$, we get

$$\begin{aligned} F(sd(x_2, x_3)) &\leq F(s^3 H(Tx_1, Tx_2)) \\ &\leq F\big(\psi(M_s(x_1, x_2))\big) - \tau \\ &\leq F(M_s(x_1, x_2)) - \tau, \end{aligned}$$

which gives

$$F(sd(x_2, x_3)) \leq F(M_s(x_1, x_2)) - \tau, \tag{5}$$

where

$$\begin{aligned} M_s(x_1, x_2) &= \max\left\{ d(x_1, x_2), D(x_1, Tx_1), D(x_2, Tx_2), \frac{D(x_1, Tx_2) + D(x_2, Tx_1)}{2s} \right\} \\ &\leq \max\left\{ d(x_1, x_2), D(x_2, Tx_2), \frac{D(x_1, Tx_2)}{2s} \right\}. \end{aligned}$$

Since

$$\frac{D(x_1, Tx_2)}{2s} \leq \max\left\{ d(x_1, x_2), D(x_2, Tx_2) \right\},$$

we get

$$M_s(x_1, x_2) \leq \max\left\{ d(x_1, x_2), D(x_2, Tx_2) \right\}.$$

If $\max\{d(x_1, x_2), D(x_2, Tx_2)\} = D(x_2, Tx_2)$, then

$$\begin{gathered} F(D(x_2, Tx_2) < F(s^3 H(Tx_1, Tx_2) \\ \leq F(\psi(D(x_2, Tx_2))) - \tau \\ \leq F(D(x_2, Tx_2)) - \tau < F(D(x_2, Tx_2)), \end{gathered}$$

which implies, $F(D(x_2, Tx_2)) < F(D(x_2, Tx_2))$. From (F_1) we get $D(x_2, Tx_2) < D(x_2, Tx_2)$, which is a contradiction. Consequently, we obtain

$$F(sd(x_2, x_3)) \leq F(d(x_1, x_2)) - \tau \tag{6}$$

By continuing in this manner, we can construct a sequence $\{x_n\} \subset X$ such that $x_n \neq x_{n+1} \in Tx_n, \alpha(x_n, x_{n+1}) \geq s^2$ and

$$F(sd(x_n, x_{n+1})) \leq F(d(x_{n-1}, x_n)) - \tau, \quad \text{for all } n \in \mathbb{N}. \tag{7}$$

Let $b_n := d(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$. Thus, from (F_4) and using (7), we get

$$F(s^n b_n) \leq F(b_{n-1}) - \tau \leq \cdots \leq F(b_0) - n\tau, \quad \text{for all } n \in \mathbb{N} \tag{8}$$

Letting $n \to \infty$ in (3.8), we get $\lim_{n\to\infty} F\left(s^n b_n\right) = -\infty$. Then, by property (F2), we have

$$\lim_{n\to\infty} s^n b_n = 0. \tag{9}$$

From $(F3)$, there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} (s^n b_n)^k F(s^n b_n) = 0. \tag{10}$$

By (8), for all $n \in \mathbb{N}$, we infer that

$$(s^n b_n)^k F(s^n b_n) - (s^n b_n)^k F(b_0) \leq -(s^n b_n)^k n\tau \leq 0. \tag{11}$$

Letting $n \to \infty$ in (11) and using (10), we get

$$\lim_{n\to\infty} n(s^n b_n)^k = 0.$$

By the definition of limit, there exists $n_1 \in \mathbb{N}$ such that $n(s^n b_n)^k \leq 1$, for all $n \geq n_1$. Thus, we have

$$s^n b_n \leq \frac{1}{n^{1/k}}, \qquad \text{for all } n \geq n_1. \tag{12}$$

To prove that $\{x_n\}$ is a Cauchy sequence, let $m > n \geq n_1$. Then, using the triangular inequality and (12), we have

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{j=n}^{m-1} s^j d(x_j, x_{j+1}) \\ &= \sum_{j=n}^{m-1} s^j b_j \leq \sum_{j=n}^{m-1} \frac{1}{j^{1/k}} \\ &\leq \sum_{j=n}^{\infty} \frac{1}{j^{1/k}} < \infty. \end{aligned}$$

Since it is a partial sum of a convergent series. For $n, m \to \infty$ we get $d(x_n, x_m) \to 0$. Hence $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete metric space, so $\{x_n\}$ is convergent to some $z \in X$.

Now we claim $z \in Tz$, we have $\alpha(x_n, z) \geq s^2$. If there exists $p \in \mathbb{N}$ such $d(x_{p+1}, Tz) = 0$, then from the uniqueness of limit, $d(z, Tz) = 0$ and so $z \in Tz$. Otherwise there exists $n_2 \in \mathbb{N}$ such that $d(x_{n+1}, Tz) > 0$ which gives $H(Tx_n, Tz) > 0$ for all $n > n_2$. Thus, we have

$$\begin{aligned} F(d(x_{n+1}, Tz)) &\leq F(H(Tx_n, Tz)) \\ &\leq F(s^3 H(Tx_n, Tz)) \\ &\leq F(\psi(M_s(x_n, z)) - \tau \\ &\leq F(M_s(x_n, z)) - \tau \end{aligned}$$

Since F is strictly increasing, we get

$$d(x_{n+1}, Tz) < M_s(x_n, z),$$

where

$$\begin{aligned} M_s(x_n, z) &= \max\left\{d(x_n, z), D(x_n, Tx_n), D(z, Tz), \frac{D(x_n, Tz) + D(z, Tx_n)}{2s}\right\} \\ &\leq \max\left\{d(x_n, z), d(x_n, x_{n+1}), D(z, Tz), \frac{D(x_n, Tz) + d(z, x_{n+1})}{2s}\right\}. \end{aligned}$$

for all $n > n_2$. Letting $n \to \infty$ in the previous inequality, we obtain

$$d(z, Tz) \leq d(z, Tz),$$

which gives that $d(z, Tz) = 0$. This completes the proof. □

Since each α_s^*-admissible mapping is also α_s-admissible, we obtain following result.

Corollary 3.1. *Let (X, d, s) be a complete b-metric space, $\alpha\colon X \times X \to [0, +\infty)$ be a function and $T\colon X \to CB(X)$ be a multivalued mapping. Assume that the following conditions hold:*

(*i*) *T is an α_s^*-admissible.*

(*ii*) *There exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq s^2$.*

(iii) *For every sequence* $\{x_n\} \subset X$ *converges to some* x *in* X *and* $\alpha^*(x_n, x_{n+1}) \geq s^2$, *for all* $n \in N$. *Then* $\alpha^*(x_n, x) \geq s^2$, *for all* $n \in \mathbb{N}$.

(iv) *There exist* $F \in \mathcal{F}$, $\psi \in \Phi$ *and* $\tau > 0$ *such that*

$$\tau + F(s^3 H(Tx, Ty)) \leq F(\psi(M_s(x, y))),$$

where

$$M_s(x, y) = \max\left\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(y, Tx)}{2s}\right\}.$$

Then T *has a fixed point.*

Example 3.1. *Let* $X = \{1, 2, 4\}$ *and* $d(x, y) = |x - y|^2$. *Define* $T\colon X \to CB(X)$ *and* $\alpha : X \times X \to [0, \infty)$ *by*

$$Tx = \begin{cases} \{2\}, & x \in \{1, 2\} \\ \{1\}, & x = 4 \end{cases}$$

and

$$\alpha(x, y) = \begin{cases} 4, & (x, y) \in \{(1, 2), (1, 4)\} \\ 0, & \text{otherwise.} \end{cases}$$

Taking $F(x) = \ln x + x$, $\psi(t) = t$, $\tau = \frac{1}{5}$, *we need to show that*

$$8H(Tx, Ty)e^{8H(Tx,Ty)} \leq \psi(M_2(x, y))e^{\psi(M_2(x,y))}e^{-\frac{1}{5}} \quad ,$$

for all $x, y \in X$ *with* $H(Tx, Ty) > 0$ *and* $\alpha(x, y) \geq 4$.

(1) *For* $x = 1$ *and* $y = 2$, *we have*

$$H(T1, T2) = 0, \quad d(1, 2) = 1, \quad \psi(d(1, 2)) = d(1, 2) = 1,$$

then

$$\begin{aligned} 8H(T1, T2)e^{8H(T1,T2)} &\leq \psi(d(1, 2))e^{\psi(d(1,2))}e^{-\frac{1}{5}} \\ &\leq \psi(M(1, 2))e^{\psi(d(1,2))}e^{-\frac{1}{5}}. \end{aligned}$$

(2) *For* $x = 1$ *and* $y = 4$, *we have*

$$H(T1, T4) = 1, \quad d(1, 4) = 9, \quad \text{and } \psi(d(1, 4)) = d(1, 4) = 9,$$

then

$$\begin{aligned} 8H(T1, T4)e^{8H(T1,T4)} &\leq \psi(d(1, 4))e^{\psi((d(1,4))}e^{-\frac{1}{5}} \\ &\leq \psi(M_2(1, 4))e^{\psi((d(1,4))}e^{-\frac{1}{5}}. \end{aligned}$$

It is easy to see that T *is an* α_s*-admissible and there exist* $x_0 = 4$ *and* $x_1 = 1 \in Tx_0$ *such that* $\alpha(x_0, x_1) \geq 4$. *Also, it is obvious that* T *is* α*-lower semi-continuous. Consequently, all conditions of Theorem 3.1 are satisfied. Then* T *has a fixed point which is* 2.

Now, we give new fixed point results on a b-metric space endowed with a partial ordering/graph, by using the results provided in previous section. Define

$$\alpha\colon X \times X \to [0, +\infty), \quad \alpha(x, y) = \begin{cases} s^2, & \text{if } x \preceq y, \\ 0, & \text{otherwise.} \end{cases}$$

Then the following result is a direct consequence of our results.

Theorem 3.2. *Let* $(X, \preceq, d)$ *be a complete ordered b-metric space and* $T\colon X \to CB(X)$ *be a multivalued mapping. Assume that the following assertions hold.*

(1) *For each* $x \in X$ *and* $y \in Tx$ *with* $x \preceq y$, *we have* $y \preceq z$ *for all* $z \in Ty$;

(2) *There exist* $x_0 \in X$ *and* $x_1 \in Tx_0$ *such that* $x_0 \preceq x_1$;

(3) *For $x \in X$ and a sequence $\{x_n\}$ in X with* $\lim_{n\to\infty} d(x_n, x) = 0$ *and $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$, implies*

$$\liminf_{n\to\infty} d(x_n, Tx_n) \geq d(x, Tx)$$

or, for every sequence $\{x_n\}$ in X such that $x_n \to x \in X$ and $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$, we have $x_n \preceq x$ for all $n \in \mathbb{N}$;

(4) *There exist $F \in \mathcal{F}_s$, $\psi \in \Phi$ and $\tau > 0$ such that*

$$\tau + F(s^3 H(Tx, Ty)) \leq F(\psi(M_s(x, y))),$$

where

$$M_s(x, y) = \max\left\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(y, Tx)}{2s}\right\}.$$

Then T has a fixed point.

Now, we present the existence of fixed point for multivalued mappings from a b-metric space X, endowed with a graph, into the space of nonempty closed and bounded subsets of the metric space. Consider a graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops; that is, $E(G) \supseteq \Delta$, where $\Delta = \{(x, x), x \in X\}$. We assume G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$.

We define the function

$$\alpha\colon X \times X \to [0, +\infty), \quad \alpha(x, y) = \begin{cases} s^2, & \text{if } (x, y) \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 3.3. *Let (X, d, s) be a complete b-metric space endowed with a graph G and $T\colon X \to CB(X)$ be a multivalued mapping. Assume that the following conditions hold:*

(1) *For each $x \in X$ and $y \in Tx$ with $(x, y) \in E(G)$, we have $(y, z) \in E(G)$ for all $z \in Ty$;*

(2) *There exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E(G)$;*

(3) *For every sequence $\{x_n\}$ in X such that $x_n \to x \in X$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, we have $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$;*

(4) *There exist $F \in \mathcal{F}_s$, $\psi \in \Phi$ and $\tau > 0$ such that*

$$\tau + F(s^3 H(Tx, Ty)) \leq F(\psi(M_s(x, y))), \tag{13}$$

where

$$M_s(x, y) = \max\left\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(y, Tx)}{2s}\right\}.$$

Then T has a fixed point.

4. **Application**

In this section, we apply our obtained results to prove existence theorem of solution for an integral inclusion of Fredholm-type . For this purpose, let $X := C([a, b], \mathbb{R})$ be the space of all continuous real valued functions on $[a, b]$. Note that X is complete b-metric space by considering $d(x, y) = \sup_{t\in[a,b]} |x(t) - y(t)|^2$ with s = 2.

Consider now the following problem

$$x(t) \in h(t) + \int_a^b P(t, s, x(s))ds, \qquad t \in J = [a, b]. \tag{14}$$

where $h \in X$ and $P : J \times J \times \mathbb{R} \to CB(\mathbb{R})$.

Our hypotheses are on the following data:

(A) : For each $x \in X$, the multivalued operator $P_x(t,s) := P(t,s,x(s))$, $(t,s) \in J \times J$ is lower semi-continuous;

(B) : There exists a continuous function $\eta : J \times J \to [0,+\infty)$ such that

$$|q_{x_1}(t,s) - q_{x_2}(t,s)|^2 \leq \eta(t,s)|x_1(s) - x_2(s)|^2.$$

For all $x_1, x_2 \in X$ with $(x_1,x_2) \in E(G)$ and $x_1 \neq x_2$, all $q_{x_1} \in P_{x_1}, q_{x_2} \in P_{x_2}$ and for each $(t,s) \in J \times J$;

(C) : there exists $\tau > 0$ such that

$$\sup_{t\in J} \int_a^b |\eta(t,s)| ds \leq \frac{e^{-\tau}}{8};$$

(D) : There exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0,x_1) \in E(G)$.

(E) : For each $x \in X$ and $y \in Tx$ with $(x,y) \in E(G)$, we have $(y,z) \in E(G)$ for all $z \in Ty$;

(F) : For every sequence $\{x_n\}$ in X such that $x_n \to x \in X$ and $(x_n,x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, we have $(x_n,x) \in E(G)$ for all $n \in \mathbb{N}$.

Theorem 4.1. *Under assumptions* $(A)-(F)$ *the integral inclusion* (14) *has a solution in* X.

Proof. We have to show that the operator T satisfies all conditions of Theorem 3.3.
Consider the set-valued operator $T : X \to CB(X)$ as follows

$$Tx(t) = \left\{ y \in X : y \in h(t) + \int_a^b P(t,s,x(s))ds, \quad t \in J \right\}.$$

Note that the integral inclusion (14) has a solution if and only if T has a fixed point in X. For the set-valued operator $P_x(t,s) : J \times J \to CB(\mathbb{R})$, it follows from Michaels selection theorem for $x \in X$ there exists a continuous operator $q_x : J \times J \to \mathbb{R}$ such that $q_x(t,s) \in P_x(t,s)$ for all $t,s \in J \times J$. It follows that $h(t) + \int_a^b q_x(t,s)ds \in Tx$, so Tx is non-empty for all $x \in X$. Since h and q_x are continuous on J, resp. J^2, their ranges are bounded and closed and hence Tx is bounded, i.e., $T : X \to CB(X)$.
Let $x_1, x_2 \in X$ with $(x_1,x_2) \in E(G)$ and $x_1 \neq x_2$, and let $v_1 \in Tx_1$. Then

$$v_1(t) \in h(t) + \int_a^b P(t,s,x_1(s))ds, \quad t \in J.$$

It follows that

$$v_1(t) = h(t) + \int_a^b q_{x_1}(t,s)ds, \quad (t,s) \in J \times J,$$

where $q_{x_1}(t,s) \in P_{x_1}(t,s)$.
From (B), there exists $w(t,s) \in P_{x_2}(t,s)$ such that

$$d(q_{x_1}(t,s) - w(t,s)) \leq \eta(t,s) \cdot |x_1(s) - x_2(s)|^2,$$

for all $(t,s) \in J \times J$. Consider the multivalued operator L defined by

$$L(t,s) = P_{x_2}(t,s) \cap \{z \in \mathbb{R} : |q_{x_1}(t,s) - z| \leq \eta(t,s) \cdot |x_1(s) - x_2(s)|^2\},$$

for all $(t,s) \in J \times J$. Since, by (A), L is lower semi-continuous, there exists a continuous function $q_{x_2}(t,s) \in L(t,s)$ for $t,s \in J$. Thus, we have

$$v_2(t) = h(t) + \int_a^b q_{x_2}(t,s)ds \in h(t) + \int_a^b P(t,s,x_2(s))ds, \quad t \in J$$

and

$$
\begin{aligned}
|v_1(t,s) - v_2(t,s)|^2 &\leq \int_a^b |q_{x_1}(t,s) - q_{x_2}(t,s)|^2 ds \\
&\leq \int_a^b \eta(t,s)|x_1(s) - x_2(s)|^2 ds \\
&\leq \sup_{s\in[a,b]} |x_1(s) - x_2(s)|^2 \int_a^b \eta(t,s)ds \\
&= d(x_1,x_2) \int_a^b \eta(t,s)ds \\
&\leq \frac{e^{-\tau}}{8} d(x_1,x_2).
\end{aligned}
$$

Consequently, we have

$$8d(v_1,v_2) \leq e^{-\tau} d(x_1,x_2),$$

which implies that

$$8H(Tx_1,Tx_2) \leq e^{-\tau} d(x_1,x_2).$$

Taking logarithm of two sides in above inequality we get

$$
\begin{aligned}
\tau + \ln(8H(Tx_1,Tx_2)) &\leq \ln(d(x_1,x_2)) \\
&\leq \ln(M_2(x_1,x_2)),
\end{aligned}
$$

for all $x_1, x_2 \in X$ with $(x_1,x_2) \in E(G)$ and $x_1 \neq x_2$, Thus, we observe that the operator T satisfies condition (13) with $F(t) = \ln t$ and $\psi(t) = t$. All other conditions of Theorem 3.3 immediately follows by the hypothesis. Therefore, T has a fixed point, that is, the Fredholm-type integral inclusion (14) has a solution in X. □

5. Conclusions

In this study, we have established some fixed point results for a set valued contraction of Wordowski type combined with α-admissibility property and Geraghty contractive condition in the setting of b-metric spaces. An example has been given to illustrate the usability of our results. We have also gave some consequences on b-metric spaces endowed with partial ordering, graph. We have also furnished an application of the existence of solutions for fredholm-type integral inclusions results.

References

[1] Ali, M. U., Kamran, T., (2016), Multivalued F-Contractions and Related Fixed Point Theorems with an Application, Filomat 30:14, pp. 3779-3793.

[2] Ali, M. U., Kamran, T., Postolache, M., (2017), Solution of Volterra integral inclusion in b-metric spaces via new fixed point theorem, Nonlinear Analysis, Modelling and Control,22, No. 1, pp. 17-30.

[3] Amer, E., Arshad, M., Shatanawi, W., (2017), Common fixed point results for generalized -y-contraction multivalued mappings in b-metric spaces, J. Fixed Point Theory Appl. 19 (4), pp. 3069-3086.

[4] Asl, H., Rezapour, J., Shahzad, S., (2012), On fixed points of $\alpha - \psi$-contractive multifunctions, Fixed Point Theory Appl. 2012, Article ID 212.

[5] Aubin, J. P., Frankowska, H., (1990), Set-Valued Analysis, Birkhäuser, Boston, 1990.

[6] Aydi, H, Bota, M. F., Karapinar, E., Moradi, S., (2012), A common fixed point for weak-ϕ-contractions on b-metric spaces, Fixed Point Theory, 13(2), pp. 33-76.

[7] Bakhtin, I. A., (1989), The contraction mapping principle in almost metric spaces, Journal of Functional Analysis, vol. 30, pp. 26-37.

[8] Batra, R., Vashistha, S., (2014), Fixed points of an F-contraction on metric spaces with a graph, Int. J. Comput. Math., 91(12), pp. 2483-2490.

[9] Biles, D. C., Robinson, M. P., Spraker, J. S., (2005), Fixed point approaches to the solution of integral inclusions, Top. Methods. Non Linear. Anal, vol 25, pp. 297-311.

[10] Cosentino, M., Jleli, M., Samet, B., Vetro, C., (2015), Solvability of integrodifferential problem via fixed point theory in b-metric spaces, Fixed Point Theory Appl., 2015(70).

[11] Czerwik, S., (1993), Contraction mappings in b-metric spaces, Acta Math. Inform. Univ. Ostrav., 1, pp. 5-11.

[12] Czerwik, S., (1998), Nonlinear set-valued contraction mappings in b-metric spaces, Atti del Seminario Matematico e Fisico dell'Università di Modena, vol. 46, no. 2, pp. 263-276.

[13] Hussain, N., Salimi, P., (2014), Suzuki-Wardowski type fixed point theorems for α-GF-contractions, Taiwanese. J. Math., Vol. 18, No. 6, pp. 1879-1895.

[14] Iqbal, I., Hussain, N., (2016), Fixed point results for generalized multivalued nonlinear F-contractions, J. Nonlinear Sci. Appl. 9, pp. 5870-5893.

[15] Isik, H., Ionescu, C., (2018), New type of multivalued contractions with related results and applications, U.P.B. Sci. Bull., Series A, Vol. 80, Iss. 2, pp. 13-22.

[16] Jleli, M., Samet, B., Vetro, C., Vetro, F., (2015), Fixed Points for Multi-valued Mappings in b..Metric Spaces, Abstr. Appl. Anal. 2015.

[17] Kaddouri, H., Isik, H., Beloul, S., (2019), On new extensions of F-contraction with an application to integral inclusions, U.P.B. Sci. Bull., Series A, Vol.81(3), pp. 31-42.

[18] Klim, D., Wardowski, D., (2015), Fixed points of dynamic processes of set-valued F-contractions and application to functional equations, Fixed Point Theory Appl., 2015:22.

[19] Mohammadi, B., Rezapour, S., Shahzad, N., (2013), Some results on fixed points of $\alpha - \psi$ -Ćirić generalized multifunctions. Fixed Point Theory Appl., 2013, Art. No. 24.

[20] Nadler, S. B., (1969), Multi-valued contraction mappings, Pacific J. Math. 30 (1969), 475-488.

[21] Samet, B., Vetro, C., Vetro, P., (2012), Fixed point theorems for $\alpha - \psi$ -contractive type mappings, Nonlinear Analysis, vol. 75, no. 4, pp. 2154-2165.

[22] Samreen, M., Kamran, T., Shahzad, N., (2013), Some fixed point theorems in b-metric space endowed with graph, Abstr. Appl. Anal., 2013, ID 967132.

[23] Sgroi, M., Vetro, C., (2013), Multi-valued F-contractions and the solution of certain mappings and integral equations, Filomat, 27:7, pp. 1259-1268.

[24] Sintunavarat, W., (2016), Nonlinear integral equations with new admissibility types in b-metric spaces, J. Fixed Point Theory Appl., 18, pp. 397-416

[25] Wardowski, D., (2012), Fixed points of a new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl. 2012:94.

Heddi Kaddouri is a PhD student at El Oued University Algeria. She has comleted her master's on fundamental mathematics from University of El Oued. Her research area is in fixed point theory and applied mathematics.

Said Beloul for the photography and short autobiography, see TWMS J. Appl. Eng. Maths., V.6, N.1a, 2018.