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AN EXTENSION THEOREM ON DEGREE OF APPROXIMATION OF FOURIER SERIES BY $(E, q)B-$MEAN

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Abstract. Now-a-days, approximation of functions have great importance in the field of science and engineering because of its wider applicability. It is observed that the determination of trigonometric approximation of functions in various function spaces using summability techniques of Fourier series and conjugate Fourier series received a growing interest among the researchers and scientists. In the present article, we have established a new result on the degree of approximation of a Fourier series of weighted Lipchitz class $W(L^P, \xi(u))$ by using the product mean $(E, q)B$.

Keywords: Degree of Approximation, $W(L^P, \xi(u))$ class function, $(E, q)-$ mean, $B-$mean, $(E, q)B-$mean, Fourier series and Lebesgue integral.

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1. Introduction

The concept of approximating a function is due to the great mathematician Weierstrass. To minimize the error in the degree of approximation, different summation methods of Fourier series were introduced. Looking at its wider applicability in the field of science and engineering, various researchers have investigated on the degree of approximations for periodic functions belonging to different spaces like: Lipschitz, Hölder, Zygmund and Besov. The degree of approximation of functions belonging to different class of functions have been studied by various investigators like Nigam [6], Padhy et al. [7], Parida et al.[8], Das et al.([1],[2]), Pradhan et al.[9], Jena et al.[4] etc. Working in the direction to get a better approximation, we have established a new result on the degree of approximation of a Fourier series of weighted Lipchitz class $W(L^P, \xi(u))$ by using $(E, q)B$ mean.

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2. Definitions and Notations

Let $\sum b_n$ be the given series and $\{s_n\}$ be its partial sums. If $B = (b_{mn})$ be an infinite matrix, then the transformation

$$\tau_m = \sum_{n=0}^{m} b_{mn} s_n \ , m = 1, 2, ... \tag{1}$$

denotes the $B-$ transform of the sequence $\{s_n\}$.
If

$$\tau_n \to s, \text{ as } n \to \infty,$$

then $\sum b_n$ is $B-$summable to s.
The conditions for the regularity of $B-$summability are:

(i) $\sup_m \sum_{n=0}^{\infty} |b_{mn}| < L$, where L is an absolute constant,
(ii) $\lim_{n\to\infty} b_{m,n} = 0$ for every $m = 1, 2, 3, ...$, and
(iii) $\lim_{m\to\infty} \sum_{n=0}^{\infty} b_{m,n} = 1$.

Further, the transformation [3]

$$t_n = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k \tag{2}$$

represents the $(E, q)-$transform of the sequence $\{s_n\}$.
If $t_n \to s$, as $n \to \infty$, the series $\sum b_n$ is summable by $(E, q)-$ method.
It is known that (E, q) is regular [11].
Furthermore, the transformation

$$w_n = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \tau_k \tag{3}$$

$$= \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \{\sum_{\nu=0}^{k} b_{k\nu} s_\nu\} \tag{4}$$

defines the $(E, q)-$transform of the $B-$transform of $\{s_n\}$.
If $w_n \to s$, as $n \to \infty$, then the series $\sum b_n$ is summable $(E, q)B$ to s.
Let $g(t)$ be a 2π periodic function, which is integrable over $(-\pi, \pi)$ in Lebesgue's sense.
Let

$$g(x) \equiv \frac{c_0}{2} + \sum_{n=1}^{\infty} (c_n cos\ nx + d_n sin\ nx) \equiv \sum_{n=0}^{\infty} G_n(x) \tag{5}$$

be the Fourier series at any point $'x'$,where c_0, c_n and d_n are the Euler Fourier constants.
Let $s_n(g; x)$ be the nth partial sum of the Fourier series (5).
For a function $g : R \to R$, the $L_\infty-$norm of is defined by

$$\|g\|_\infty = \sup\{|g(x)| : x \in R\} \tag{6}$$

and the $L_\nu-$ norm is defined by

$$\|g\|_\nu = \Big\{ \int_0^{2\pi} |g(x)|^\nu \Big\}^{\frac{1}{\nu}}, \nu \geq 1. \tag{7}$$

The degree of approximation of the function g by a nth degree polynomial $Q_n(x)$ under the norm $\|.\|_\infty$ is given by [10]

$$\|Q_n - g\|_\infty = \sup\Big\{|Q_n(x) - g(x)| : x \in R\Big\} \tag{8}$$

and the trigonometric Fourier approximation under the norm L_ν is

$$E_n(g) = \min_{Q_n} \|Q_n - g(x)\|_\nu \tag{9}$$

For any real number α, $0 < \alpha \leq 1$, a function g is said to satisfy Lipschitz condition [5] i.e. $Lip\alpha$, if

$$|g(x+u) - g(x)| = O\Big\{|u|^\alpha\Big\},\ u > 0 \tag{10}$$

and for any real number $r \geq 1$, $0 \leq x \leq 2\pi$, $g(x) \in Lip(\alpha, r)$ if

$$\Big\{\int_0^{2\pi} |g(x+u) - g(x)|^r dx\Big\}^{\frac{1}{r}} = O\Big(|u|^\alpha\Big),\ u > 0. \tag{11}$$

Let $\xi(u)$ be a positive increasing function, then for the real number $r \geq 1$, $g(x)$ is said to belong $Lip\Big(\xi(u), r\Big)$, if

$$\Big\{\int_0^{2\pi} |g(x+u) - g(x)|^r dx\Big\}^{\frac{1}{r}} = O\Big(\xi(u)\Big),\ r \geq 1,\ u > 0. \tag{12}$$

and for any integer $p > 1$, the function $g(x) \in W\Big(L^p, \xi(u)\Big)$ if

$$\Big(\int_0^{2\pi} |g(x+u) - g(x)|^p (sin^\beta x)^p dx\Big)^{\frac{1}{p}} = O\Big(\xi(u)\Big),\ \beta \geq 0. \tag{13}$$

From (10),(11),(12) and (13), it is clear that

$$Lip\alpha \subseteq Lip(\alpha, r) \subseteq Lip\Big(\xi(u), r\Big) \subseteq W\Big(L^p, \xi(u)\Big)$$

We use the following notations throughout the chapter:

$$\phi(u) = g(x+u) + g(x-u) - 2g(x) \tag{14}$$

and

$$K_n(u) = \frac{1}{2\pi(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \Big\{\sum_{\lambda=0}^{k} b_{k\lambda} \frac{sin(\lambda + \frac{1}{2})u}{sin(\frac{u}{2})}\Big\} \tag{15}$$

3. Known Theorems

Nigam [6] has proved the following theorem on degree of approximation by the product $(E, q)(C, 1)-$ mean of the Fourier series.

Theorem 3.1. *If g is a $2\pi-$ periodic function of class $Lip\alpha$, then degree of approximation by the product$(E, q)(C, 1)$ summability means on its Fourier series (5) is given by $\|E_n^q c_n^1 - g\|_\infty = O\Big(\frac{1}{(n+1)^\alpha}\Big)$, $0 < \alpha < 1$, where $E_n^q c_n^1$ represents the (E, q) transform of $(C, 1)$ transform of $s_n(g; x)$.*

Padhy et al. [7] proved the following theorem using $(E, q)B-$ mean of the Fourier series.

Theorem 3.2. *Let $B = (b_{mn})_{\infty\times\infty}$ be a regular matrix. If g is a $2\pi-$ periodic function of class $Lip\alpha$, then degree of approximation by the product $(E, q)B$ summability means on its Fourier series (5) is given by$\|w_n - g\|_\infty = O\Big(\frac{1}{(n+1)^\alpha}\Big)$, $0 < \alpha < 1$,where w_n is as defined in (4).*

In this paper, generalizing the result of Padhy et al., by taking the function belonging to weighted Lipschitz class, we establish the following result.

4. Main Theorem

Theorem 4.1. *The degree of trigonometric approximation of the Fourier series (5) of a* $2\pi-$*periodic function of class* $W\Big(L^p, \xi(u)\Big),\ p>1,\ u>0$ *by* $(E,q)B$ *summability is*

$$\|w_\nu - g\|_r = O\Big((\nu+1)^{\beta+\frac{1}{r}}\xi\Big(\frac{1}{\nu+1}\Big)\Big),\ r\geq 1, \tag{16}$$

provided

$$\Big(\int_0^{\frac{1}{\nu+1}}\Big(\frac{u|\phi(u)|sin^\beta u}{\xi(u)}\Big)^r du\Big)^{\frac{1}{r}} = O\Big(\frac{1}{\nu+1}\Big) \tag{17}$$

and

$$\Big(\int_{\frac{1}{\nu+1}}^{\pi}\Big(\frac{u^{-\delta}|\phi(u)|}{\xi(u)}\Big)^r du\Big)^{\frac{1}{r}} = O\Big((\nu+1)^\delta\Big) \tag{18}$$

hold uniformly with $\frac{1}{r}+\frac{1}{s}=1$ *and for an arbitrary* δ , $s(1-\delta)-1>0$ *and* w_n *is as defined in (4).*

5. Required Lemmas

Lemma 5.1. *[6]* $|K_n(t)| = O(n)$, *for* $0\leq t\leq \frac{1}{n+1}$.

Lemma 5.2. *[6]* $|K_n(t)| = O(\frac{1}{t})$, *for* $\frac{1}{n+1} < t \leq \pi$.

6. Proof of Main Theorem

Proof. By making use of Riemann-Lebesgue's theorem and following Titchmarsh[10], we have

$$s_n(g;x) - g(x) = \frac{1}{2\pi}\int_0^\pi \phi(u)\frac{sin\Big(n+\frac{1}{2}\Big)u}{sin\Big(\frac{u}{2}\Big)}du,$$

and the $B-$transform of $s_n(g;x)$ using (1) is given by

$$\tau_n - g(x) = \frac{1}{2\pi}\int_0^\pi \phi(u)\sum_{k=0}^{n} b_{nk}\frac{sin\Big(n+\frac{1}{2}\Big)u}{sin\Big(\frac{u}{2}\Big)}du,$$

Since, w_n is $(E,q)B-$ mean of the sequence $\{s_n(g;x)\}$, we have

$$\begin{aligned}\|w_n - g\| &= \frac{1}{2\pi}\int_0^\pi \phi(u)\sum_{k=0}^{n}\binom{n}{k}\frac{q^{n-k}}{(1+q)^n}\sum_{\lambda=0}^{k} b_{k\lambda}\frac{sin\Big(n+\frac{1}{2}\Big)u}{sin\Big(\frac{u}{2}\Big)}du\\ &= \int_0^\pi K_n(u)\ \phi(u)\ du = \Big(\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\Big)\phi(u)K_n(u)du\\ &= I_1 + I_2,\ \text{say}\end{aligned} \tag{19}$$

Now

$$|I_1| = \frac{1}{2\pi}\int_0^{\frac{1}{n+1}} \phi(u)\sum_{k=0}^{n}\binom{n}{k}\frac{q^{n-k}}{(1+q)^n}\sum_{\lambda=0}^{k} b_{k\lambda}\frac{sin\Big(n+\frac{1}{2}\Big)u}{sin\Big(\frac{u}{2}\Big)}du$$

$$= \Big|\int_0^{\frac{1}{n+1}} \phi(u)K_n(u)du\Big|$$

By using H$\ddot{o}$lder's inequality,

$$|I_1| \le \Big(\int_0^{\frac{1}{n+1}}\Big|\frac{u\phi(u)sin^{\beta}u}{\xi(u)}\Big|^r du\Big)^{\frac{1}{r}}\Big(\int_0^{\frac{1}{n+1}}\Big|\frac{\xi(u)K_n(u)}{usin^{\beta}u}\Big|^s du\Big)^{\frac{1}{s}}, \text{where} \frac{1}{r}+\frac{1}{s}=1$$

$$= O(1)\Big(\int_0^{\frac{1}{n+1}}\Big(\frac{\xi(u)}{u^{1+\beta}}\Big)^s du\Big)^{\frac{1}{s}}, \text{using lemma 4.1 and (17)}$$

$$= O\Big(\xi\Big(\frac{1}{n+1}\Big)\Big(\int_{\epsilon}^{\frac{1}{n+1}}\frac{du}{u^{(\beta+1)s}}\Big)^{\frac{1}{s}}, 0\le\epsilon\le\frac{1}{1+n}.$$

$$= O\Big(\xi\Big(\frac{1}{1+n}\Big)\Big)O\Big((1+n)^{\beta+\frac{-1}{s}+1}\Big)$$

$$= O\Big(\xi\Big(\frac{1}{1+n}\Big)(1+n)^{\beta+\frac{1}{r}}\Big) \tag{20}$$

Similarly, by using H$\ddot{o}$lder's inequality,

$$|I_2| \le \Big(\int_{\frac{1}{n+1}}^{\pi}\Big|\frac{u^{-\delta}|\phi(u)|sin^{\beta}u}{\xi(u)}\Big|^r du\Big)^{\frac{1}{r}}\times\Big(\int_{\frac{1}{n+1}}^{\pi}\Big|\frac{\xi(u)K_n(u)}{u^{-\delta}sin^{\beta}u}\Big|^s du\Big)^{\frac{1}{s}}, \text{where}\frac{1}{r}+\frac{1}{s}=1,$$

$$= O\Big((n+1)^{\delta}\Big)\Big(\int_{\frac{1}{n+1}}^{\pi}\Big(\frac{\xi(u)}{u^{\beta+1-\delta}}\Big)^s du\Big)^{\frac{1}{s}}, \text{using lemma 4.2 and (18).}$$

$$= O\Big((1+n)^{\delta}\Big)\Big(\int_{\frac{1}{1+n}}^{\pi}\Big(\frac{\xi(\frac{1}{y})}{y^{\delta-\beta-1}}\Big)^s\frac{dy}{y^2}\Big)^{\frac{1}{s}},$$

$$= O\Big((1+n)^{1+\delta}\xi\Big(\frac{1}{1+n}\Big)\Big)\Big(\int_{\epsilon}^{1+n}\frac{dy}{y^{s(\delta-\beta-1)+2}}\Big)^{\frac{1}{s}},\ \frac{1}{\pi}\le\epsilon\le 1+n,$$

by second mean value theorem, (since, $\dfrac{\xi(\frac{1}{y})}{\frac{1}{y}}$ is positive and increasing)

$$= O\Big((1+n)^{\delta+1}\xi\Big(\frac{1}{1+n}\Big)\Big)O\Big((1+n)^{1+\beta-\delta-\frac{1}{s}}\Big)$$

$$= O\Big((1+n)^{\beta+\frac{1}{r}}\xi\Big(\frac{1}{1+n}\Big)\Big) \tag{21}$$

Then, by using (20) and (21), we get

$$|w_n - g(x)| = O\Big((1+n)^{\beta+\frac{1}{r}}\xi\Big(\frac{1}{1+n}\Big)\Big), \text{for}, r\ge 1,$$

$$\|w_n - g(x)\| = \Big(\int_0^{2\pi} O\Big((1+n)^{\beta+\frac{1}{r}}\xi\Big(\frac{1}{1+n}\Big)\Big)^r dx\Big)^{\frac{1}{r}}, r\ge 1$$

$$= O\Big((1+n)^{\beta+\frac{1}{r}}\xi\Big(\frac{1}{1+n}\Big)\Big)\Big(\int_0^{2\pi} dx\Big)^{\frac{1}{r}}$$

$$= O\Big((1+n)^{\beta+\frac{1}{r}}\xi\Big(\frac{1}{1+n}\Big)\Big)$$

This completes the proof of the theorem. □

7. Corollaries

Corollary 7.1. *The degree of approximation of a function g belonging to the class $Lip(\alpha, r), 0 < \alpha \leq 1, r \geq 1$ is given by*

$$\|w_n - g\|_\infty = O\Big((n+1)^{-\alpha+\frac{1}{r}}\Big)$$

.

Proof. The corollary follows by putting $\beta = 0$ and $\xi(u) = u^\alpha$ in the main theorem. □

Corollary 7.2. *The degree of approximation of a function g belonging to the class $Lip(\alpha), 0 < \alpha \leq 1,$ is given by*

$$\|w_n - g\|_\infty = O\Big((n+1)^{-\alpha}\Big)$$

Proof. The corollary follows when we take $r \to \infty$ in the corollary 6.1. □

8. Conclusion

Our result established here is more general than some earlier existing results. Also it generalizes the result of Padhy et.[7] al and Nigam [6].

References

[1] Das, A. A., Paikray, S. K., Pradhan, T. and Dutta, H., (2018), Statistical $(C,1)(E,\mu)$ summability and associated fuzzy approximation theorems with statistical fuzzy rates, Soft Comput., doi.org/10.1007/s00500-019-04591-2.

[2] Das, A. A., Paikray, S. K., Pradhan, T. and Dutta, H., (2019), Approximation of signals in the weighted Zygmund class via Euler-Huasdroff product summabilitymean of Fourier series, Journal of Indian Mathematical Society, 86, pp. 296-314.

[3] Hardy, G. H., (1970), Divergent Series, first edition, Oxford university press.

[4] Jena, B. B., Mishra, L. N., Paikray, S. K. and Misra, U. K., (2020), Approximation of signals by general matrix summability with effect of Gibbs phenomenon, Bol. Soc. Paran. Math., 38, pp. 141-158.

[5] McFadden, L., (1942), Absolute Norlund Summability, Duke Math. J., 9(1), pp. 168-207.

[6] Nigam, H. K., (2010), The degree of approximtion by product means, Ultra scientist, 22(3)M, pp. 889-894.

[7] Padhy, B. P., Mallik, B., Misra, U. K. and Misra, M., (2013), On degree of approximation of Fourier series by product means $(E,q)A$, International Journal of Mathematics and Computation, 19(2), pp. 34-41.

[8] Parida, P., Paikray, S. K., Dash, M. and Misra, U. K., (2019), Degree of approximation by product $(\overline{N}, p_n, q_n)(E,q)$ summability of Fourier series of a signal belonging to $Lip(\alpha, r)$-class, TWMS J. App. Eng. Math., 9, pp. 901-908.

[9] Pradhan, T., Paikray, S. K., Das, A. A. and Dutta, H., (2019), On approximation of signals in the generalized Zygmund class via $(E,1)(N,p_n)$ summability means of conjugate Fourier series, Proyecciones Journal of Mathematics, 38, pp. 1015-1033.

[10] Titchmarsh, E. C., (1939), The theory of functions, Oxford University press, pp. 402-403.

[11] Zygmund, A., (1959), Trigonometric series, Second edition, 1, Cambridge University press, Cambridge.

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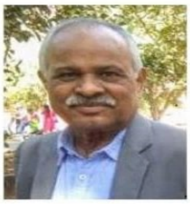

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