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# THE NON-HOMOGENEOUS OF SEMI-LINEAR PARABOLIC EQUATION WITH INTEGRAL CONDITIONS

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Abstract. In this paper we introduce another method to establish finite time blow-up. This method, introduce by Levine–Payne in the papers [3], [4] and is due to Levine (1973), uses the concavity of an auxiliary function $I(t)$.

Keywords: Integral Condition, blow-up, Semi-linear parabolic equations, concavity method.

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## 1. Introduction

Livene, H. A., (1973) produced a clever argument (concavity method) to show the phenomenon of blow-up in a finite time to establish finite time blow-up. This argument has been investigated by Hu. B., (2011) who treated the same phenomenon and other very rapid instabilities occur in situations in mechanics and other areas of applied mathematics. However, the work of Hu. B., (2011) shows how this method could be applied on semi linear parabolic equation, dependent of the initial boundary condition and the concavity of an auxiliary function. This concavity technique is actually powerful enough to be applied to many other types of second parabolic equations as well as other types of evolution equations and may be applied to establish nonexistence for a solution to a many equation. Which is useful in calculating the blow-up of a solution in finite time.

Another illustration of the method of concavity is given in the book by Straughan. B., (1998) which employed the geometrical interpretation of the concavity method. There is a vast literature devoted to study to the blow up effect in solution of equations. An review of the method and known resultas can found in the papers and books [4], [5], [6] and [7].

Here, we are confined the case of semi-linear equation, in this case it is possible to derive sufficient conditions for the finite time blow up.

The basic idea is to construct a positive definite functional $I(t)$ and then show $I^{-\alpha}(t)$ is a concave function of $t$, for some number $\alpha > 0$.
The function $I(t)$ is thus bounded below by a function which blows up in finite time provided $I'(0) > 0$.

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This argument by itself does not establish that $I(t)$ actually blows up. How ever, it certainly shows the solution cannot exist for all time in a classical sense.
We illustrate this method by application to the semilinear parabolic problems of the form

$$\frac{\partial u}{\partial t} - a(t)\frac{\partial^2 u}{\partial x^2} - \mid u \mid^2 u = f(x,t) \tag{1}$$

with the initial condition

$$u(x,0) = \varphi(x) \tag{2}$$

The solution u of (1) is subject to null boundary condition

$$u(0,t) = u(1,t) = 0 \tag{3}$$

where $\varphi(x)$ is a nonincreasing, and $f$ is a $C^2$ function satisfying

- $\int_\Omega f\frac{\partial \overline{u}}{\partial t}(x,t)dx < 0$
- $\int_\Omega \overline{u}\frac{\partial f}{\partial t}(x,t)dx < 0$

It makes no use of maximum principles and the following theorem is only a special case discussed in [1], [2] and [3].Then we can prove the following:

**Theorem 1.1.** *Let $\Omega$ be a smooth domain. Assume that for some positive constants $a_1$,$a_2$ the initial datum $\varphi$ satisfies the conditions*

$$a_2 \int_\Omega \mid \frac{\partial \varphi}{\partial x} \mid^2 dx > \int_\Omega \mid \varphi \mid^4 dx > 2a_1 \int_\Omega \mid \frac{\partial \varphi}{\partial x} \mid^2 dx \tag{4}$$

*then the solution of (1) – (3) must blow-up in finite time.*

*Proof.* To study the possibility of the explosion (blow-up) we apply the concavity method of Levine [3]. Multiplying the equation by $\overline{u}$ and $\overline{\frac{\partial u}{\partial t}}$ respectively, and then integrating over $\Omega$, we obtain

$$\frac{\partial u}{\partial t}\overline{u} - a(t)\frac{\partial^2 u}{\partial x^2}\overline{u} - \mid u \mid^4 = f(x,t)\overline{u}$$

we shall have

$$\int_\Omega \frac{\partial u}{\partial t}\overline{u}dx = \frac{1}{2}\frac{d}{dt}\int_\Omega \mid u \mid^2 dx$$

this becomes,

$$-a(t)\int_\Omega \frac{\partial^2 u}{\partial x^2}\overline{u}dx = a(t)\int_\Omega \mid \frac{\partial u}{\partial x} \mid^2 dx$$

this may be written

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega \mid u \mid^2 dx\right) + a(t)\int_\Omega \mid \frac{\partial u}{\partial x} \mid^2 dx = \int_\Omega \mid u \mid^4 dx + \int_\Omega f\overline{u}dx \tag{5}$$

Similarly we find

$$\frac{\partial u}{\partial t}\overline{\frac{\partial u}{\partial t}} - a(t)\frac{\partial^2 u}{\partial x^2}\overline{\frac{\partial u}{\partial t}} - \mid u \mid^2 u\overline{\frac{\partial u}{\partial t}} = f(x,t)\overline{\frac{\partial u}{\partial t}}$$

we are able to obtain

$$\int_\Omega \mid \frac{\partial u}{\partial t} \mid^2 dx - a(t)\int_\Omega \frac{\partial^2 u}{\partial x^2}\overline{\frac{\partial u}{\partial t}}dx - \int_\Omega \mid u \mid^2 u\overline{\frac{\partial u}{\partial t}}dx = \int_\Omega f(x,t)\overline{\frac{\partial u}{\partial t}}dx$$

Note that by integrating by parts we can show that

$$\int_\Omega \mid \frac{\partial u}{\partial t} \mid^2 dx + \frac{a(t)}{2}\frac{d}{dt}\left(\int_\Omega \mid \frac{\partial u}{\partial x} \mid^2 dx\right) = \frac{1}{4}\frac{d}{dt}\left(\int_\Omega \mid u \mid^4 dx\right) \tag{6}$$

$$+\int_\Omega f(x,t)\overline{\frac{\partial u}{\partial t}}dx$$

We then pick

$$2a_1 < 2a(t) < a_2, \quad -c_1 < \tfrac{\partial a(t)}{\partial t} < -c_2 0 < a_1, 0 < a_2, 0 < c_1, 0 < c_1.$$

Let us introduce the functions

$$J(t) = -\frac{1}{2}a(t)\int_\Omega \mid \frac{\partial u}{\partial x} \mid^2 dx + \frac{1}{4}\int_\Omega \mid u \mid^4 dx + \int_0^t \int_\Omega f(x,t)\overline{\frac{\partial u}{\partial t}}dxdt \tag{7}$$

Thus, we differentiate (7) we arrive at the relation

$$J'(t) = -\frac{1}{2}a(t)\frac{d}{dt}\left(\int_\Omega \mid \frac{\partial u}{\partial x} \mid^2 dx\right) + \frac{1}{4}\frac{d}{dt}\left(\int_\Omega \mid u \mid^4 dx\right) + \int_\Omega f(x,t)\overline{\frac{\partial u}{\partial t}}dx$$

$$-\frac{1}{2}\frac{da(t)}{dt}\left(\int_\Omega \mid \frac{\partial u}{\partial x} \mid^2 dx\right) \geq \int_\Omega \mid \frac{\partial u}{\partial t} \mid^2 dx + \frac{c_2}{2}\left(\int_\Omega \mid \frac{\partial u}{\partial x} \mid^2 dx\right).$$

which yield the inequality,

$$J'(t) \geq 0 \tag{8}$$

Integration of this inequality gives

$$J(t) \geq J(0) + \int_0^t \int_\Omega \mid \frac{\partial u}{\partial t} \mid^2 dxdt + \frac{c_2}{2}\int_0^t \int_\Omega \mid \frac{\partial u}{\partial x} \mid^2 dx \tag{9}$$

where

$$J(0) = -\frac{a_1}{2}\int_\Omega \mid \frac{\partial \varphi}{\partial x} \mid^2 dx + \frac{1}{4}\int_\Omega \mid \varphi \mid^4 dx > 0 \tag{10}$$

Therefore,

$$J(t) \geq 0,$$

Using the representation

$$I(t) = \int_0^t \int_\Omega \mid u \mid^2 dxdt + A \tag{11}$$

$$-2\int_0^t \int_0^\xi \int_0^\tau \int_\Omega \frac{\partial f}{\partial \tau}(u,\tau)\overline{u}dxd\tau d\xi dt - 2\int_0^t \int_0^\xi \int_\Omega f_0\overline{\varphi}dxd\xi dt$$

where $A > 0$ is to be determined.
And the differential equality for $I(t)$ takes on the form

$$I'(t) = \int_\Omega \mid u \mid^2 dx - 2\int_0^t \int_0^\xi \int_\Omega \frac{\partial f}{\partial \xi}(u,\xi)\overline{u}dxd\xi dt - 2\int_0^t \int_\Omega f_0\overline{\varphi}dxdt$$

It follows that

$$I''(t) = \frac{d}{dt}\left(\int_\Omega \mid u \mid^2 dx\right) - 2\int_0^t \int_\Omega \frac{\partial f}{\partial \xi}(x,\xi)\overline{u}dxd\xi - 2\int_\Omega f_0\overline{\varphi}dx$$

The term $\frac{d}{dt}\left(\int_\Omega \mid u \mid^2 dx\right)$ is substituted using (5) to then find

$$I''(t) = -2a(t)\int_\Omega \mid \frac{\partial u}{\partial x} \mid^2 dx + 2\int_\Omega \mid u \mid^4 dx + 2\int_\Omega f\overline{u}dx$$

$$-2\int_0^t \int_\Omega \frac{\partial f}{\partial \xi}(x,\xi)\overline{u}dxd\xi - 2\int_\Omega f_0\overline{\varphi}dx$$

Using this, we may write the following chain of relations:

$$I''(t) = -2a(t)\int_\Omega \mid \frac{\partial u}{\partial x} \mid^2 dx + 2\int_\Omega \mid u \mid^4 dx + 2\int_\Omega \int_0^t \frac{\partial (f\overline{u})}{\partial \xi} dx d\xi \tag{12}$$

$$-2\int_0^t \int_\Omega \frac{\partial f}{\partial \xi}(x,\xi)\overline{u} dx d\xi + 2\int_\Omega f_0 \overline{\varphi} dx - 2\int_\Omega f_0 \overline{\varphi} dx$$

$$= -2a(t)\int_\Omega \mid \frac{\partial u}{\partial x} \mid^2 dx + 2\int_\Omega \mid u \mid^4 dx$$

$$+2\int_0^t \int_\Omega f \frac{\partial \overline{u}}{\partial \xi}(x,\xi) dx d\xi$$

$$I''(t) = -2a(t)\int_\Omega \mid \frac{\partial u}{\partial x} \mid^2 dx + 2\int_\Omega \mid u \mid^4 dx + 2\int_0^t \int_\Omega f \frac{\partial \overline{u}}{\partial \xi}(x,\xi) dx d\xi \tag{13}$$

Using (6), we know

$$J(t) = -\frac{1}{2}a(t)\int_\Omega \mid \frac{\partial u}{\partial x} \mid^2 dx + \frac{1}{4}\int_\Omega \mid u \mid^4 dx + \int_0^t \int_\Omega f \frac{\partial \overline{u}}{\partial \xi}(x,\xi) dx d\xi \tag{14}$$

By comparing the terms in (13) and (14). That is to say,

$$I''(t) - 6J(t) = \int_\Omega \mid u \mid^4 dx - 2\int_0^t \int_\Omega f \frac{\partial \overline{u}}{\partial \xi}(x,\xi) dx d\xi \tag{15}$$

$$+a(t)\int_\Omega \mid \frac{\partial u}{\partial x} \mid^2 dx + \frac{1}{2}\int_\Omega \mid u \mid^4 dx \geq 0$$

Since

$\int_0^t \int_\Omega f \frac{\partial \overline{u}}{\partial \xi}(x,\xi) dx d\xi < 0$

happens that the two functions $I''(t)$ and $J(t)$ are so related that

$$I''(t) \geq 6J(t)$$

so that. The assumptions (15) and (9) above imply that

$$I''(t) \geq 6\left( J(0) + \int_0^t \int_\Omega \mid \frac{\partial u}{\partial t} \mid^2 dx dt + \frac{c_2}{2}\int_0^t \int_\Omega \mid \frac{\partial u}{\partial x} \mid^2 dx \right) \tag{16}$$

Moreover since $I'(t)$ is related to $u$ by the equation

$$I'(t) = \int_\Omega \mid u \mid^2 dx - B - C$$

We define $B$ and $C$ by

$B = 2\int_0^t \int_0^\xi \int_\Omega \frac{\partial f}{\partial \xi}(x,\xi)\overline{u} dx d\xi dt, \quad C = 2\int_0^t \int_\Omega f_0 \overline{\varphi} dx dt.$

Therefore,

$$I'(t) = 2Re\left( \int_0^t \int_\Omega u \frac{\partial \overline{u}}{\partial t} dx dt \right) + \int_\Omega \mid \varphi \mid^2 dx - B - C$$

Note that

$$\left[I'(t)\right]^2 = \left[ 2Re\left( \int_0^t \int_\Omega u \frac{\partial \overline{u}}{\partial t} dx dt \right) + \int_\Omega \mid \varphi \mid^2 dx - B - C \right]^2$$

$$= \left[ 2Re\left( \int_0^t \int_\Omega u \frac{\partial \overline{u}}{\partial t} dx dt \right) + \int_\Omega \mid \varphi \mid^2 dx \right]^2$$

$$-4(B+C)Re\left(\int_0^t\int_\Omega u\frac{\partial\overline{u}}{\partial t}dxdt\right)-2(B+C)\int_\Omega\mid\varphi\mid^2 dx+(B+C)^2.$$

To estimate $[I'(t)]^2$ from below we use the inverse Hölder inequality, and estimate as follows:

$$\left[I'(t)\right]^2\leq 4(1+\epsilon)\left(\int_0^t\int_\Omega\mid u\mid^2 dxdt\right)\left(\int_0^t\int_\Omega\mid\frac{\partial u}{\partial t}\mid^2 dxdt\right)$$
$$+(1+\frac{1}{\epsilon})\left(\int_\Omega\mid\varphi\mid^2 dx\right)^2+(B+C)^2$$
$$+4(-(B+C))Re\left(\int_0^t\int_\Omega u\frac{\partial\overline{u}}{\partial t}dxdt\right)+2(-(B+C))\int_\Omega\mid\varphi\mid^2 dx,$$

we obtain the needed estimate on $[I'(t)]^2$:

$$\left[I'(t)\right]^2\leq 4(1+\epsilon)\left(\int_0^t\int_\Omega\mid u\mid^2 dxdt\right)\left(\int_0^t\int_\Omega\mid\frac{\partial u}{\partial t}\mid^2 dxdt\right)$$
$$+(B+C)^2+(1+\frac{1}{\epsilon})\left(\int_\Omega\mid\varphi\mid^2 dx\right)^2$$
$$+4\left[\frac{1}{4\epsilon}(B+C)^2+\epsilon\left(\int_0^t\int_\Omega\mid u\mid^2 dxdt\right)\left(\int_0^t\int_\Omega\mid\frac{\partial u}{\partial t}\mid^2 dxdt\right)\right]$$
$$+2\left[\frac{\epsilon}{2}(B+C)^2+\frac{1}{2\epsilon}\left(\int_0^t\int_\Omega\mid\varphi\mid^2 dxdt\right)^2\right].$$

Writing this inequality in the equivalent form,

$$\left[I'(t)\right]^2\leq 4(1+2\epsilon)\left(\int_0^t\int_\Omega\mid u\mid^2 dxdt\right)\left(\int_0^t\int_\Omega\mid\frac{\partial u}{\partial t}\mid^2 dxdt\right) \tag{17}$$
$$+(1+\frac{2}{\epsilon})\left(\int_\Omega\mid\varphi\mid^2 dx\right)^2+(B+C)^2(1+\epsilon+\frac{1}{\epsilon})$$

On the basis of (16) and (17), we conclude that

$$I''(t)I(t)-(1+\alpha)\left[I'(t)\right]^2\geq 6\left(J(0)+\int_0^t\int_\Omega\mid\frac{\partial u}{\partial t}\mid^2 dxdt+\frac{c_2}{2}\int_0^t\int_\Omega\mid\frac{\partial u}{\partial x}\mid^2 dx\right)$$
$$\times\left(\int_0^t\int_\Omega\mid u\mid^2 dxdt+A-B-C\right)$$
$$-(1+\alpha)\left[4(1+2\epsilon)\left(\int_0^t\int_\Omega\mid u\mid^2 dxdt\right)\left(\int_0^t\int_\Omega\mid\frac{\partial u}{\partial t}\mid^2 dxdt\right)\right]$$
$$-(1+\alpha)\left[(1+\frac{2}{\epsilon})\left(\int_\Omega\mid\varphi\mid^2 dx\right)^2\right]$$
$$-(1+\alpha)(B+C)^2(1+\epsilon+\frac{1}{\epsilon})$$

and, consequently,

$$I''(t)I(t)-(1+\alpha)\left[I'(t)\right]^2\geq 6J(0)\int_0^t\int_\Omega\mid u\mid^2 dxdt$$
$$+(6-(1+\alpha)4(1+2\epsilon))\left(\int_0^t\int_\Omega\mid u\mid^2 dxdt\right)\left(\int_0^t\int_\Omega\mid\frac{\partial u}{\partial t}\mid^2 dxdt\right)$$

$$-(1+\alpha)(1+\frac{2}{\epsilon})\left(\int_{\Omega}\mid\varphi\mid^{2}dx\right)^{2}$$

$$+6\left(J(0)+\int_{0}^{t}\int_{\Omega}\mid\frac{\partial u}{\partial t}\mid^{2}dxdt\right)(A-(B+C))-(1+\alpha)(B+C)^{2}(1+\epsilon+\frac{1}{\epsilon})$$

We now select $\alpha$ to make the second term on the right disappear, i.e. we choose $\alpha$ and $\epsilon$ small enough such that

$$\frac{3}{2}\geq(1+\alpha)(1+2\epsilon)$$

By our assumption, $J(0)>0$, and $-B>0$. Thus we can choose $A$ to be large enough so that

$$I''(t)I(t)-(1+\alpha)\left[I'(t)\right]^{2}>0 \tag{18}$$

The inequality (18) implies that for $t>0$. It follows that

$$I(t)=\int_{0}^{t}\int_{\Omega}\mid u\mid^{2}dxdt+A-2\int_{0}^{t}\int_{0}^{\xi}\int_{0}^{\tau}\int_{\Omega}\frac{\partial f}{\partial\tau}(x,\tau)\overline{u}dxd\tau d\xi dt$$

$$-2\int_{0}^{t}\int_{0}^{\xi}\int_{\Omega}f(x,0)\overline{\varphi}dxd\xi dt$$

cannot remain finite for all t.This completes the proof of Theorem. □

However, many of the questions related to the detailed description of the solution profile the blow-up instant remain open.

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