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GENERALIZATION OF RANDIĆ ENERGY AND SUM-CONNECTIVITY ENERGY

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ABSTRACT. In this paper we define the generalization of Randić energy and sum connectivity energy of a graph. Then we obtain upper and lower bounds for $E(A_{rs})$, generalization of Randić energy and sum connectivity energy of a graph. Further we compute the generalization of Randić energy and sum connectivity energies of complete graph, star graph, complete bipartite graph, $(S_m \wedge P_2)$ graph and crown graph.

Keywords: Generalization of Randić energy and sum-connectivity energy.

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1. Introduction

In 2010, Bo Zhou and Nenad Trinajstic [3] introduced the sum-connectivity energy of a graph as follows. Let G be a simple graph and let $v_1, v_2, \ldots, v_n$ be its vertices. For $i = 1, 2, \ldots, n$, let d_i denote the degree of the vertex v_i. Then the sum-connectivity matrix of G is defined as $S = (S_{ij})$, where

$$S_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \frac{1}{\sqrt{d_i + d_j}}, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{if the vertices } v_i \text{ and } v_j \text{ are not adjacent.} \end{cases}$$

The sum-connectivity energy of G is defined as the sum of absolute values of the eigenvalues of the sum-connectivity matrix of G arranged in a non-increasing order.

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In the same year, Burcu Bozkurt, Dilek Güngör, Gutman and Sinan Çevik [2], have defined the Randić energy of a graph G as the sum of the absolute values of the eigenvalues of the Randić matrix (R_{ij}) where

$$R_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \frac{1}{\sqrt{d_i d_j}}, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{if the vertices } v_i \text{ and } v_j \text{ are not adjacent.} \end{cases}$$

Motivated by these works, we introduce the generalization of Randić energy and sum-connectivity energy of a simple graph G as follows. Let a and b be two nonnegative real numbers with $a + b \neq 0$. The generalization of Randić and sum-connectivity matrix of G is the $n \times n$ matrix $A_{rs} = (a_{ij})$ where

$$a_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \frac{1}{\sqrt{a(d_i+d_j)+b(d_i d_j)}}, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{if the vertices } v_i \text{ and } v_j \text{ are not adjacent.} \end{cases}$$

The eigenvalues of the graph G are the eigenvalues of A_{rs}. Since A_{rs} is real and symmetric, its eigenvalues are real numbers which are denoted by $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$, where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq ... \geq \lambda_n$. Then the generalization of Randić energy and sum-connectivity energy of G is defined as

$$E_{rs}(G) = \sum_{i=1}^{n} \mid \lambda_i \mid .$$

Since A_{rs} is a real symmetric matrix, we have

$$\sum_{i=1}^{n} \lambda_i = tr(A_{rs}) = 0 \tag{1}$$

and

$$\sum_{i=1}^{n} \lambda_i^2 = tr(A_{rs}^2) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2 = 2\sum_{i\sim j} \frac{1}{a(d_i + d_j) + b(d_i d_j)}. \tag{2}$$

Remark: 1. If $a = 1$ and $b = 0$, then $E_{rs}(G)$ is the sum-connectivity energy.
2. If $a = 0$ and $b = 1$, then $E_{rs}(G)$ is the Randić energy.

In this paper we compute $E_{rs}(G)$, the generalization of Randić energy and sum-connectivity energy of complete graph, star graph, complete bipartite graph, $(S_m \wedge P_2)$ graph and crown graph. Also we obtain the upper and lower bounds for $E_{rs}(G)$.

2. Upper and lower bounds for $E_{rs}(G)$

In this section we obtain Upper and lower bounds for $E_{rs}(G)$.

Theorem 2.1. *Let G be a simple graph of order n with no isolated vertices and a, b be as defined above. Then*

$$E_{rs}(G) \leq \sqrt{2n \sum_{i\sim j} \frac{1}{a(d_i + d_j) + b(d_i d_j)}}. \tag{3}$$

Proof. Let $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$, be the eigenvalues of A_{rs}. Then using (2) and the Cauchy-Schwartz inequality, we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right) \cdot \left(\sum_{i=1}^{n} b_i^2\right)$$

with $a_i = 1$, $b_i = \mid \lambda_i \mid$, we obtain

$$E_{rs}(G) = \sum_{i=1}^{n} |\lambda_i| = \sqrt{(\sum_{i=1}^{n} |\lambda_i|)^2} \leq \sqrt{n \sum_{i=1}^{n} \lambda_i^2} = \sqrt{2n \sum_{i \sim j} \frac{1}{a(d_i + d_j) + b(d_i d_j)}}.$$

□

Theorem 2.2. *Let G be a simple graph of order n with no isolated vertices and a, b be as defined above. Then*

$$E_{rs}(G) \geq 2\sqrt{\sum_{i \sim j} \frac{1}{a(d_i + d_j) + b(d_i d_j)}}. \tag{4}$$

Proof. From (1), we have

$$\sum_{i=1}^{n} \lambda_i^2 + 2 \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j = 0$$

and therefore

$$-\sum_{i=1}^{n} \lambda_i^2 = 2 \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j. \tag{5}$$

Thus

$$(E_{rs}(G))^2 = \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 = \sum_{i=1}^{n} \lambda_i^2 + 2 \sum_{1 \leq i < j \leq n} |\lambda_i \lambda_j|$$

$$\geq \sum_{i=1}^{n} \lambda_i^2 + 2 \left| \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j \right|$$

$$= 2 \sum_{i=1}^{n} \lambda_i^2, \quad \textit{on using } (5).$$

This together with (2) implies that

$$(E_{rs}(G))^2 \geq 4 \sum_{i \sim j} \frac{1}{a(d_i + d_j) + b(d_i d_j)},$$

which gives (4). □

3. Generalization of Randić energy and sum-connectivity energies of some families of graphs

We recall that the complete graph is one in which every pair of its distinct vertices are adjacent. A complete graph on n vertices is denoted by K_n. A bigraph or bipartite graph G is a graph whose vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every line of G joins V_1 with V_2. (V_1, V_2) is a bipartition of G. If G contains every line joining V_1 and V_2, then G is a complete bigraph. If V_1 and V_2 have m and n points, we write $G = K_{m,n}$. A star is a complete bigraph $K_{1,n}$[4].

The following definitions and notations, will be used in the remainder of this paper.

Definition 3.1. *[1] The Crown graph S_n^0 for an integer $n \geq 3$ is the graph with vertex set $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and edge set $\{u_i v_j; 1 \leq i, j \leq n, i \neq j\}$. S_n^0 is the complete bipartite graph $K_{n,n}$ with the horizontal edges removed.*

Definition 3.2. *[5] The conjunction $(S_m \wedge P_2)$ of $S_m = \overline{K}_m + K_1$ and P_2 is the graph having the vertex set $V(S_m) \times V(P_2)$ and edge set $\{(v_i, v_j)(v_k, v_l) | v_i v_k \in E(S_m)$ and $v_j v_l \in E(P_2)$ and $1 \leq i, k \leq m+1, 1 \leq j, l \leq 2\}$. In fact S_m is the star $K_{1,m}$ and P_2 is K_2.*

Now we compute generalization of Randić energy and sum-connectivity energies of complete graph, star graph, complete bipartite graph, $(S_m \wedge P_2)$ graph and crown graph.

Theorem 3.1. *Let a and b be as defined above. Then the generalization of Randić energy and sum-connectivity energy of complete bipartite graph $K_{m,n}$ is $2\sqrt{\frac{mn}{a(m+n)+b(mn)}}$.*

Proof. Let the vertex set of the complete bipartite graph be $V(K_{m,n}) = \{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\}$. Then the generalization of Randić and sum-connectivity matrix of complete bipartite graph is given by

$$A_{rs} = \begin{pmatrix} 0 & \cdots & 0 & \frac{1}{\sqrt{a(m+n)+b(mn)}} & \cdots & \frac{1}{\sqrt{a(m+n)+b(mn)}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{1}{\sqrt{a(m+n)+b(mn)}} & \cdots & \frac{1}{\sqrt{a(m+n)+b(mn)}} \\ \frac{1}{\sqrt{a(m+n)+b(mn)}} & \cdots & -\frac{1}{\sqrt{a(m+n)+b(mn)}} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{a(m+n)+b(mn)}} & \cdots & \frac{1}{\sqrt{a(m+n)+b(mn)}} & 0 & \cdots & 0 \end{pmatrix}.$$

Its characteristic polynomial is

$$|\lambda I - A_{rs}| = \begin{vmatrix} \lambda I_m & -\frac{1}{\sqrt{a(m+n)+b(mn)}} J^T \\ -\frac{1}{\sqrt{a(m+n)+b(mn)}} J & \lambda I_n \end{vmatrix},$$

where J is an $n \times m$ matrix with all the entries are equal to 1. Hence the characteristic equation is given by

$$\begin{vmatrix} \lambda I_m & -\frac{1}{\sqrt{a(m+n)+b(mn)}} J^T \\ -\frac{1}{\sqrt{a(m+n)+b(mn)}} J & \lambda I_n \end{vmatrix} = 0,$$

which can be written as

$$|\lambda I_m| \left| \lambda I_n - \left(-\frac{1}{\sqrt{a(m+n)+b(mn)}} J \right) \frac{I_m}{\lambda} \left(-\frac{1}{\sqrt{a(m+n)+b(mn)}} J^T \right) \right| = 0.$$

On simplification, we obtain

$$\frac{\lambda^{m-n}}{(a(m+n)+b(mn))^n}\left|(a(m+n)+b(mn))\lambda^2 I_n - JJ^T\right| \quad = \quad 0,$$

which can be written as

$$\frac{\lambda^{m-n}}{(a(m+n)+b(mn))^n} P_{JJ^T}((a(m+n)+b(mn))\lambda^2) \quad = \quad 0,$$

where $P_{JJ^T}(\lambda)$ is the characteristic polynomial of the matrix JJ^T. Thus, we have

$$\frac{\lambda^{m-n}}{(a(m+n)+b(mn))^n}((a(m+n)+b(mn))\lambda^2 - mn)((a(m+n)+b(mn))\lambda^2)^{n-1} = 0,$$

which is same as

$$\lambda^{m+n-2}(\lambda^2 - \frac{mn}{a(m+n)+b(mn)}) = 0.$$

Therefore, the spectrum of $K_{m,n}$ is given by

$$Spec\,(K_{m,n}) = \begin{pmatrix} 0 & \sqrt{\frac{mn}{a(m+n)+b(mn))}} & -\sqrt{\frac{mn}{a(m+n)+b(mn)}} \\ m+n-2 & 1 & 1 \end{pmatrix}.$$

Hence the generalization of Randić energy and sum-connectivity energy of complete bipartite graph is

$$E_{rs}(K_{m,n}) = 2\sqrt{\frac{mn}{a(m+n)+b(mn)}},$$

as desired. ☐

Theorem 3.2. *Let a and b be as defined above. Then the generalization of Randić energy and sum-connectivity energy of S_n is $2\sqrt{\frac{n-1}{an+(n-1)b}}$.*

Proof. Let the vertex set of star graph be given by $V(S_n) = \{v_1, v_2, \ldots, v_n\}$. Then the generalization of Randić and sum-connectivity matrix of star graph S_n is given by

$$A_{rs} = \begin{pmatrix} 0 & \frac{1}{\sqrt{an+(n-1)b}} & \frac{1}{\sqrt{an+(n-1)b}} & \cdots & \frac{1}{\sqrt{an+(n-1)b}} & \frac{1}{\sqrt{an+(n-1)b}} \\ \frac{1}{\sqrt{an+(n-1)b}} & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{\sqrt{an+(n-1)b}} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\sqrt{an+(n-1)b}} & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{\sqrt{an+(n-1)b}} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Hence the characteristic polynomial is given by

$$|\lambda I - A_{rs}| = \begin{vmatrix} \lambda & -\frac{1}{\sqrt{an+(n-1)b}} & -\frac{1}{\sqrt{an+(n-1)b}} & \cdots & -\frac{1}{\sqrt{an+(n-1)b}} \\ -\frac{1}{\sqrt{an+(n-1)b}} & \lambda & 0 & \cdots & 0 \\ -\frac{1}{\sqrt{an+(n-1)b}} & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{\sqrt{an+(n-1)b}} & 0 & 0 & \cdots & \lambda \end{vmatrix}$$

$$= \left(\frac{1}{\sqrt{an+(n-1)b}}\right)^n \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & 0 & \cdots & 0 & 0 \\ -1 & 0 & \mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & \mu & 0 \\ -1 & 0 & 0 & \cdots & 0 & \mu \end{vmatrix},$$

where $\mu = \lambda\sqrt{an+(n-1)b}$. Then $|\lambda I - A_{rs}| = \phi_n(\mu)\left(\frac{1}{\sqrt{an+(n-1)b}}\right)^n$,

where $\phi_n(\mu) = \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ 1 & \mu & 0 & \cdots & 0 & 0 \\ 1 & 0 & \mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & \mu & 0 \\ 1 & 0 & 0 & \cdots & 0 & \mu \end{vmatrix}$.

Using the properties of the determinants, we obtain after some simplifications

$$\phi_n(\mu) = (\mu\phi_{n-1}(\mu) - \mu^{n-2}).$$

Iterating this, we obtain

$$\phi_n(\mu) = \mu^{n-2}(\mu^2 - (n-1)).$$

Therefore

$$|\lambda I - A_{rs}| = \left(\frac{1}{\sqrt{an+(n-1)b}}\right)^n \left[\left((an+(n-1)b)\lambda^2 - (n-1)\right)(\lambda\sqrt{an+(n-1)b})^{n-2}\right].$$

Thus the characteristic equation is given by

$$\lambda^{n-2}\left(\lambda^2 - \frac{n-1}{an+(n-1)b}\right) = 0.$$

Hence

$$Spec\,(S_n) = \begin{pmatrix} 0 & \sqrt{\frac{n-1}{an+(n-1)b}} & -\sqrt{\frac{n-1}{an+(n-1)b}} \\ n-2 & 1 & 1 \end{pmatrix}.$$

Hence the generalization of Randić energy and sum-connectivity energy of S_n is

$$E_{rs}(S_n) = 2\sqrt{\frac{n-1}{an+(n-1)b}}.$$

□

Theorem 3.3. *Let a and b be as defined above. Then the generalization of Randić energy and sum-connectivity energy of K_n is* $2\sqrt{\frac{n-1}{2a+b(n-1)}}$.

Proof. Let the vertex set of Complete graph be given by $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Then the generalization of Randić and sum-connectivity matrix of complete graph K_n is given by

$$A_{rs} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2a(n-1)+(n-1)^2b}} & \cdots & \frac{1}{\sqrt{2a(n-1)+(n-1)^2b}} \\ \frac{1}{\sqrt{2a(n-1)+(n-1)^2b}} & 0 & \cdots & \frac{1}{\sqrt{2a(n-1)+(n-1)^2b}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{2a(n-1)+(n-1)^2b}} & \frac{1}{\sqrt{2a(n-1)+(n-1)^2b}} & \cdots & 0 \end{pmatrix}.$$

Hence the characteristic polynomial is given by

$$|\lambda I - A_{rs}| = \begin{vmatrix} \lambda & -\frac{1}{\sqrt{2a(n-1)+(n-1)^2b}} & \cdots & -\frac{1}{\sqrt{2a(n-1)+(n-1)^2b}} \\ -\frac{1}{\sqrt{2a(n-1)+(n-1)^2b}} & \lambda & \cdots & -\frac{1}{\sqrt{2a(n-1)+(n-1)^2b}} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{\sqrt{2a(n-1)+(n-1)^2b}} & -\frac{1}{\sqrt{2a(n-1)+(n-1)^2b}} & \cdots & \lambda \end{vmatrix}$$

$$= \left(\frac{1}{\sqrt{2a(n-)+(n-1)^2b}}\right)^n \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ -1 & -1 & -1 & \cdots & -1 & \mu \end{vmatrix},$$

where $\mu = \lambda\sqrt{2a(n-1)+(n-1)^2b}$. Then $|\lambda I - A_{rs}| = \phi_n(\mu)\left(\frac{1}{\sqrt{2a(n-1)+(n-1)^2b}}\right)^n$,

$$\text{where } \phi_n(\mu) = \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ -1 & -1 & -1 & \cdots & -1 & \mu \end{vmatrix}$$

$$= \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ 0 & 0 & 0 & \cdots & -1-\mu & \mu+1 \end{vmatrix}$$

$$= (\mu+1)\begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ -1 & -1 & -1 & \cdots & -1 & -1 \end{vmatrix} + (\mu+1)\begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ -1 & -1 & -1 & \cdots & -1 & \mu \end{vmatrix}.$$

$$\phi_n(\mu) = -(\mu+1)^{n-1} + (\mu+1)\left[(\mu+1)^{n-2}(\mu-(n-2))\right]$$

$$= -(\mu+1)^{n-1} + (\mu+1)^{n-1}(\mu-(n-2)).$$

Iterating this, we obtain

$$\phi_n(\mu) = \mu^{n-2}(\mu^2-(n-1)),$$

thus the characteristic equation is given by

$$\left(\frac{1}{\sqrt{2a(n-1)+(n-1)^2b}}\right)^n (\mu+1)^{n-1}(\mu-(n-1)) = 0.$$

Hence

$$Spec\,(K_n) = \begin{pmatrix} \frac{-1}{\sqrt{2a(n-1)+(n-1)^2b}} & \frac{n-1}{\sqrt{2a(n-1)+(n-1)^2b}} \\ n-1 & 1 \end{pmatrix}.$$

Hence the generalization of Randić energy and sum-connectivity energy of K_n is

$$E_{rs}(K_n) = 2\sqrt{\frac{n-1}{2a+b(n-1)}}.$$

□

Theorem 3.4. *Let a and b be as defined above. Then the generalization of Randić energy and sum-connectivity energy of* $(S_m \wedge P_2)$ *is* $4\sqrt{\frac{n-1}{an+(n-1)b}}$.

Proof. Let the vertex set of $(S_m \wedge P_2)$ graph be given by $V(S_m \wedge P_2) = \{v_1, v_2, \ldots, v_{2m+2}\}$. Then the generalization of Randić and sum-connectivity matrix of $(S_m \wedge P_2)$ graph is given by

$$A_{rs} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & \frac{1}{\sqrt{na+(n-1)b}} \\ 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{na+(n-1)b}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{na+(n-1)b}} & \cdots & 0 \\ 0 & \frac{1}{\sqrt{na+(n-1)b}} & \cdots & \frac{1}{\sqrt{na+(n-1)b}} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{na+(n-1)b}} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{na+(n-1)b}} & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}_{2n\times 2n},$$

where $m+1=n$. Its characteristic polynomial is given by

$$|\lambda I - A_{rs}| = \begin{vmatrix} \lambda & \cdots & 0 & 0 & -\frac{1}{\sqrt{na+(n-1)b}} & -\frac{1}{\sqrt{n}} \\ 0 & \cdots & 0 & -\frac{1}{\sqrt{na+(n-1)b}} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda & -\frac{1}{\sqrt{na+(n-1)b}} & \cdots & 0 \\ 0 & \cdots & -\frac{1}{\sqrt{na+(n-1)b}} & \lambda & \cdots & 0 \\ -\frac{1}{\sqrt{na+(n-1)b}} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{\sqrt{na+(n-1)b}} & \cdots & 0 & 0 & \cdots & \lambda \end{vmatrix}_{2n\times 2n}.$$

Hence the characteristic equation is given by

$$\left(\frac{1}{\sqrt{na+(n-1)b}}\right)^{2n} \begin{vmatrix} \Lambda & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 \\ 0 & \Lambda & \cdots & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Lambda & -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{2n\times 2n} = 0,$$

$$where\ \Lambda = \sqrt{na+(n-1)b}\lambda.$$

Let

$$\phi_{2n}(\Lambda) = \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \Lambda & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \Lambda & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 & \Lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{2n\times 2n}.$$

$$= (-1)^{2n+2n}\Lambda \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \Lambda & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \Lambda & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 & \Lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{(2n-1)\times(2n-1)}$$

$$+(-1)^{2n+2} \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 & -1 \\ \Lambda & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & \Lambda & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & -1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda & 0 \end{vmatrix}_{(2n-1)\times(2n-1)}.$$

Let

$$\Psi_{2n-1}(\Lambda) = (-1)^{2n+2} \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 & -1 \\ \Lambda & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & \Lambda & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & -1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda & 0 \end{vmatrix}_{(2n-1)\times(2n-1)}$$

Using the properties of the determinants, we obtain, after some simplifications

$$\Psi_{2n-1}(\Lambda) = -\Lambda^{n-2}\Theta_n(\Lambda),$$

where $\Theta_n(\Lambda)=\begin{vmatrix} \Lambda & 0 & 0 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & -1 \\ 0 & 0 & \Lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \Lambda \end{vmatrix}_{n\times n}$.

Then

$$\phi_{2n}(\Lambda) = -\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda\phi_{2n-1}(\Lambda).$$

Now, proceeding as above, we obtain

$$\begin{aligned}\phi_{2n-1}(\Lambda) &= (-1)^{(2n-1)+2}\Psi_{2n-2}(\Lambda) + (-1)^{(2n-1)+(2n-1)}\Lambda\phi_{2n-2}(\Lambda) \\ &= -\Lambda^{n-3}\Theta_n(\Lambda) + \Lambda\phi_{2n-2}(\Lambda).\end{aligned}$$

Proceeding like this, we obtain at the $(n-1)^{th}$ step

$$\phi_{2n}(\Lambda) = -(n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{(n-1)}\xi_{n+1}(\Lambda),$$

where $\xi_{n+1}(\Lambda) = \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 0 \\ 0 & \Lambda & 0 & \cdots & -1 \\ 0 & 0 & \Lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1 & -1 & \cdots & \Lambda \end{vmatrix}_{(n+1)\times(n+1)}$.

$$\begin{aligned}\phi_{2n}(\Lambda) &= -(n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{n-1}\Lambda\Theta_n(\Lambda) \\ &= -(n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{n}\Theta_n(\Lambda) \\ &= (\Lambda^n - (n-1)\Lambda^{n-2})\Theta_n(\Lambda).\end{aligned}$$

Using the properties of the determinants, we obtain

$$\Theta_n(\Lambda) = \Lambda^n - (n-1)\Lambda^{n-2}.$$

Therefore

$$\phi_{2n}(\Lambda) = (\Lambda^n - (n-1)\Lambda^{n-2})^2.$$

Hence characteristic equation becomes

$$\left(\frac{1}{\sqrt{na+(n-1)b}}\right)^{2n}\phi_{2n}(\Lambda) = 0,$$

which is same as

$$\left(\frac{1}{\sqrt{na+(n-1)b}}\right)^{2n}(\Lambda^n - (n-1)\Lambda^{n-2})^2 = 0.$$

This reduces to

$$\lambda^{2n-4}((na+(n-1)b)\lambda^2 - (n-1))^2 = 0.$$

Therefore

$$Spec\,((S_m \wedge P_2)) = \begin{pmatrix} 0 & \sqrt{\frac{(n-1)}{na+(n-1)b}} & -\sqrt{\frac{(n-1)}{na+(n-1)b}} \\ 2n-4 & 2 & 2 \end{pmatrix}.$$

Hence the generalization of Randić energy and sum-connectivity energy of $(S_m \wedge P_2)$ graph is

$$E_{rs}((S_m \wedge P_2)) = 4\sqrt{\frac{n-1}{na+(n-1)b}}.$$

□

Theorem 3.5. *Let a and b be as defined as above. Then the generalization of Randić energy and sum-connectivity energy of crown graph is* $4\sqrt{\frac{(n-1)}{2a+b(n-1)}}$.

Proof. The vertex set of the crown graph be given by $V(S_n^0) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$. Then the generalization of Randić and sum-connectivity matrix of crown graph is given by

$$A_{rs} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & X & \cdots & X \\ 0 & 0 & \cdots & 0 & X & 0 & \cdots & X \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & X & X & \cdots & 0 \\ 0 & X & \cdots & X & 0 & 0 & \cdots & 0 \\ X & 0 & \cdots & X & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ X & X & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Where $X = \frac{1}{\sqrt{2a(n-1)+b(n-1)^2}}$. Its characteristic polynomial is

$$|\lambda I - A_{rs}| = \begin{vmatrix} \lambda I_n & -\frac{1}{\sqrt{2a(n-1)+b(n-1)^2}}K^T \\ -\frac{1}{\sqrt{2a(n-1)+b(n-1)^2}}K & \lambda I_n \end{vmatrix},$$

where K is an $n \times n$ matrix. Hence the characteristic equation is given by

$$\begin{vmatrix} \lambda I_n & -\frac{1}{\sqrt{2a(n-1)+b(n-1)^2}}K^T \\ -\frac{1}{\sqrt{2a(n-1)+b(n-1)^2}}K & \lambda I_n. \end{vmatrix} = 0.$$

This is same as

$$|\lambda I_n| \left| \lambda I_n - \left(-\frac{K}{\sqrt{2a(n-1)+b(n-1)^2}}\right) \frac{I_n}{\lambda} \left(-\frac{K^T}{\sqrt{2a(n-1)+b(n-1)^2}}\right) \right| = 0,$$

which can be written as

$$\frac{1}{(2a(n-1)+b(n-1)^2)^n} P_{KK^T}((2a(n-1)+b(n-1)^2)\lambda^2) = 0,$$

where $P_{KK^T}(\lambda)$ is the characteristic polynomial of the matrix KK^T. Thus we have

$$\frac{1}{(2a(n-1)+b(n-1)^2)^n}[2a(n-1)+b(n-1)^2\lambda^2-(n-1)^2][2a(n-1)+b(n-1)^2\lambda^2-1]^{n-1} = 0,$$

which is same as

$$\left(\lambda^2 - \frac{n-1}{2a+b(n-1)}\right)\left(\lambda^2 - \frac{1}{2a(n-1)+b(n-1)^2}\right)^{n-1} = 0.$$

Therefore

$$Spec\,(S_n^0) = \begin{pmatrix} \sqrt{\frac{(n-1)}{2a+b(n-1)}} & -\sqrt{\frac{(n-1)}{2a+b(n-1)}} & \frac{1}{\sqrt{2a(n-1)+b(n-1)^2}} & -\frac{1}{\sqrt{2a(n-1)+b(n-1)^2}} \\ 1 & 1 & n-1 & n-1 \end{pmatrix}.$$

Hence the generalization of Randić energy and sum-connectivity energy of crown graph is

$$E_{rs}(S_n^0) = 4\sqrt{\frac{(n-1)}{2a+b(n-1)}},$$

as desired. □

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