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SOLUTION OF PARTIALLY SINGULARLY PERTURBED SYSTEM OF INITIAL AND BOUNDARY VALUE PROBLEMS USING NON-UNIFORM HAAR WAVELET

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Abstract. An efficient non-uniform Haar wavelet method is proposed for the numerical solution of system of first order linear partially singularly perturbed initial value problem on piecewise uniform Shishkin mesh and ρ-mesh. Further, we apply same technique for solving system of second order linear partially singularly perturbed boundary value problems on piecewise uniform Shishkin mesh and q-mesh. Our method produces better results in comparison to uniform Haar wavelet, classical finite difference operator method and parameter uniform methods. We demonstrated two test problems to support the theory, accuracy and efficiency of the non-uniform Haar Wavelet method.

Keywords: System of Differential Equations; Non-Uniform Haar Wavelet; Shishkin Mesh; Singular Perturbation; Initial and Boundary Value Problems.

2000 Mathematics Subject Classification: 65M99; 65N35; 65N55; 65L10.

1. Introduction

In this paper, we took a particular case of linear system of first and second order singularly perturbed initial and boundary value problems which contains perturbation parameter ϵ in both equations. Consider the linear system of first order singularly perturbed initial value problem as follows:

$$\epsilon y_1'(t) + a(t)y_1(t) + b(t)y_2(t) = f_1(t), \tag{1}$$

$$\epsilon y_2'(t) + c(t)y_1(t) + d(t)y_2(t) = f_2(t), \quad \forall \quad t \in (0, 1] \tag{2}$$

with initial conditions

$$y_1(0) = \alpha_1, \quad y_2(0) = \alpha_2 \tag{3}$$

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where $a(t)$, $b(t)$, $c(t)$, $d(t)$, $f_1(t)$ and $f_2(t)$ are continuous functions.
Further, we consider linear system of second order singularly perturbed boundary value problem as follows:

$$-\epsilon y_1''(t) + a(t)y_1(t) + b(t)y_2(t) = f_1(t), \tag{4}$$

$$\epsilon y_2''(t) + c(t)y_1(t) + d(t)y_2(t) = f_2(t), \quad \forall \quad t \in (0,1] \tag{5}$$

with boundary conditions

$$y_1(0) = \alpha_1, \quad y_2(0) = \alpha_2, \quad y_1(1) = \beta_1 \quad and \quad y_2(1) = \beta_2. \tag{6}$$

Now we consider the following linear system of partially singularly perturbed initial value problem

$$\epsilon y_1'(t) + a(t)y_1(t) + b(t)y_2(t) = f_1(t), \tag{7}$$

$$y_2'(t) + c(t)y_1(t) + d(t)y_2(t) = f_2(t), \quad \forall \quad t \in (0,1] \tag{8}$$

with initial conditions

$$y_1(0) = \alpha_1, \qquad y_2(0) = \alpha_2. \tag{9}$$

The matrix form of the system (7)-(8) can be written as

$$\begin{pmatrix} -\epsilon\frac{d}{dt} & 0 \\ 0 & \frac{d}{dt} \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} + \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}, \quad \forall \quad t \in (0,1] \tag{10}$$

with initial conditions

$$y(0) = (y_1(0), y_2(0))^T = (\alpha_1, \alpha_2)^T. \tag{11}$$

Here, we took one equation with perturbation parameter ϵ and the other equation which does not contain perturbation parameter ϵ or we can say that $\epsilon = 1$ in equations (2) and (5). Also, the system of partially singularly perturbed second order boundary value problem is written as follows:

$$-\epsilon y_1''(t) + a(t)y_1(t) + b(t)y_2(t) = f_1(t), \tag{12}$$

$$y_2''(t) + c(t)y_1(t) + d(t)y_2(t) = f_2(t), \quad \forall \quad t \in (0,1] \tag{13}$$

with boundary conditions

$$y_1(0) = \alpha_1, \quad y_2(0) = \alpha_2, \quad y_1(1) = \beta_1 \quad and \quad y_2(1) = \beta_2. \tag{14}$$

The matrix form of the system of (12)-(13) can be written as

$$\begin{pmatrix} -\epsilon\frac{d^2}{dt^2} & 0 \\ 0 & \frac{d^2}{dt^2} \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} + \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} \tag{15}$$

with the boundary conditions

$$\begin{aligned} y(0) &= (y_1(0), y_2(0))^T = (\alpha_1, \alpha_2)^T \\ and \quad y(1) &= (y_1(1), y_2(1))^T = (\beta_1, \beta_2)^T. \end{aligned} \tag{16}$$

Such systems occurs in various fields especially in electro-analytical chemistry, chemical reactions, semiconductor devices, circuit theory, diffusion process complicated by linear diffusion chemical reactions, linear state regulator problem, viscous fluid flow, mathematical miniature of turbulence in water wave when they interact with electricity and predator prey population dynamics.

A study of solution of linear system of first and second order partially singularly perturbed initial and boundary value problems can be seen in [7]. Standard finite difference method is not sufficient to give the solution of such problems due to presence of ϵ multiplied by highest order derivatives. Further, various numerical approximations were given to deal with these kind of problems, such as classical finite difference operator method, fitted operator method, parameter robust numerical method, fitted numerical method, parameter uniform finite difference method, uniformly accurate finite element method [5, 6, 11, 12, 15]. In the present paper, we have solved system of first order linear partially singularly perturbed initial value problem on piecewise uniform Shishkin mesh and ρ-mesh and system of second order linear partially singularly perturbed boundary value problem on piecewise uniform Shishkin mesh and q-mesh. For the numerical solution of these problems we have proposed an efficient non-uniform Haar wavelet method. In fact, our method produces better results in comparison to uniform Haar wavelet, classical finite difference operator method and parameter uniform methods. Moreover, we have demonstrated two problems to show the better efficiency of the non-uniform Haar wavelet method.

2. Non-Uniform Haar Wavelet

Basic non-uniform Haar wavelet and non-uniform multi-resolution analysis was introduced by F. Dubeau et al. [4] in 2004. Solution of integral and differential equations using non-uniform Haar wavelet was given by U.Lepik ([9], [10]), solution of two point boundary value problem using Haar wavelet was given by Siraj-ul-Islam et al. [8]. For further details on wavelets we refer to [1]-[3]. The non-uniform Haar Wavelet family for $t \in [0, 1]$ is defined as follows:

$$\mathcal{H}_i(t) = \begin{cases} 1, & \xi_1(i) \le t < \xi_2(i), \\ -n_i, & \xi_2(i) \le t < \xi_3(i), \\ 0, & \text{otherwise}, \end{cases} \tag{17}$$

where i indicates the wavelet number and

$$\xi_1(i) = x(2k\mu) \quad, \xi_2(i) = x((2k+1)\mu), \quad \xi_3(i) = x((2k+2)\mu),\ \mu = \frac{M}{m}$$

$m = 2^j, j = 0, 1, 2..., J,\ M = 2^J$ and integer $k = 0, 1..., m-1$.

Here J indicates the level of resolution and k represents the translations parameter. Index i is calculated as $i = m + k + 1$ which is true for $i \ge 2$. We have given the graph of non-uniform Haar wavelet for $i = 1, 2...8$. in figure 1.

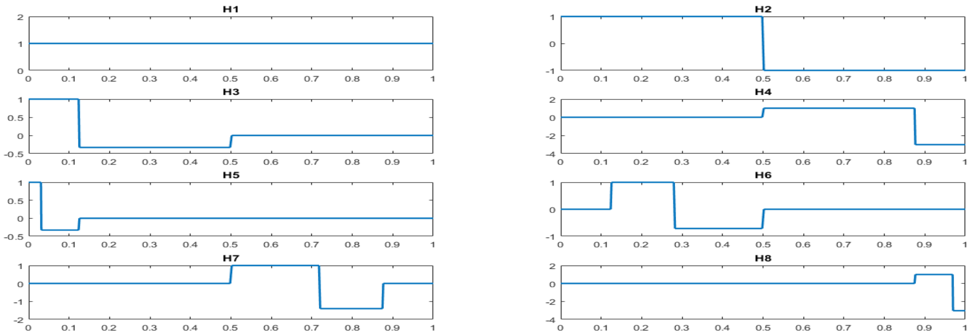

Figure 1. Graph of non-uniform Haar wavelet for $i = 1, 2...8$.

The integration of non-uniform Haar wavelet can be given as follows:

$$\mathcal{P}_i(t) = \begin{cases} t - \xi_1(i), & \xi_1(i) \le t < \xi_2(i), \\ (\xi_3(i) - t)n_i, & \xi_2(i) \le t < \xi_3(i), \\ 0, & \text{otherwise.} \end{cases} \quad (18)$$

The graph of integration of non-uniform Haar wavelet is given in figure 2.

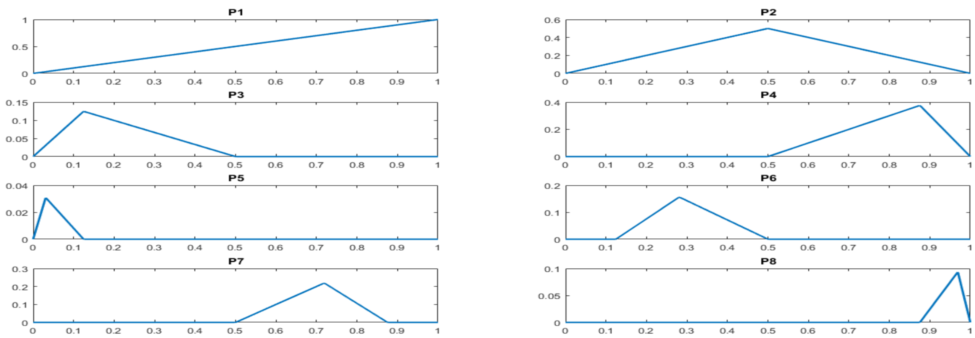

Figure 2. Graph of integration of non-uniform Haar wavelet for $i = 1, 2...8$.

The double integration of non-uniform Haar wavelet can be given as follows:

$$\mathcal{Q}_i(t) = \begin{cases} \frac{1}{2}(t - \xi_1(i))^2, & \xi_1(i) \le t < \xi_2(i), \\ \mathcal{K} - \frac{1}{2}(\xi_3(i) - t)^2 n_i, & \xi_2(i) \le t < \xi_3(i), \\ \mathcal{K}, & \xi_3(i) \le t < 1 \\ 0, & \text{otherwise.} \end{cases} \quad (19)$$

The graph of double integration of non-uniform Haar wavelet is given in figure 3.

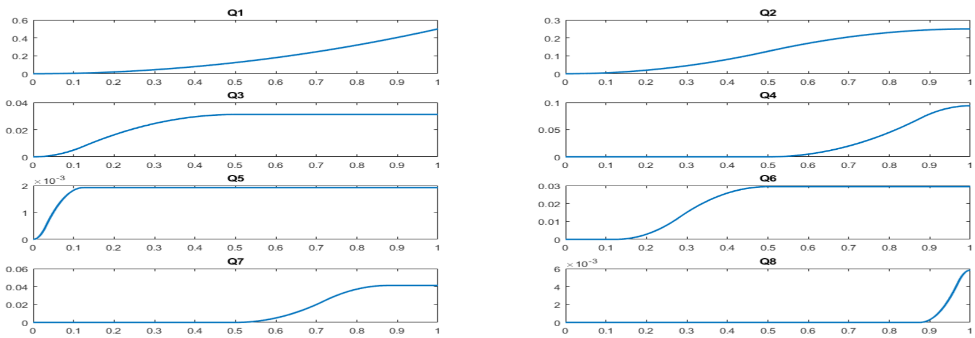

Figure 3. Graph of double integration of non-uniform Haar wavelet for $i = 1, 2...8$.

Proceeding in similar manner the n^{th} integration of non-uniform Haar wavelet see ([8], [9],[10],[13] and [14]) can be obtained as follows :

$$I_n\mathcal{H}_i(t) = \begin{cases} 0, & t < \xi_1(i), \\ \frac{1}{n!}[t - \xi_1(i)]^n, & \xi_1(i) \le t < \xi_2(i), \\ \frac{1}{n!}[(t - \xi_1(i))^n - (1 + n_i)(t - \xi_2(i))^n], & \xi_2(i) \le t < \xi_3(i), \\ \frac{1}{n!}[(t - \xi_1(i))^n - (1 + n_i)(t - \xi_2(i))^n + n_i(t - \xi_3(i))^n], & \xi_3(i) \le t, \end{cases} \quad (20)$$

where, $\mathcal{K} = \frac{(\xi_2-\xi_1)(\xi_3-\xi_1)}{2}$ and $n_i = \frac{(\xi_2-\xi_1)}{(\xi_3-\xi_1)}$.

Grid Formation: Here we discuss the non-uniform and piecewise uniform fitted mesh/grids: The non-uniform grid, i.e., q-mesh is defined as

$$\hat{t}(j) = \frac{1-q^j}{1-q^N}, \quad j = 0,1,2,...,N \tag{21}$$

$$t(j) = \frac{\hat{t}(j-1)+\hat{t}(j)}{2}, \quad j = 1,2,...,N. \tag{22}$$

The ρ-mesh is given as follows

$$M = \frac{N}{2}, \quad \rho = \frac{5-\epsilon}{5M}, \quad \delta = \frac{N\rho - 1}{M(N-1)}$$

$$t_0 = 0, \quad t_j = 1-(N-j)\rho + \delta(N-j+1)\frac{N-j}{2}, \quad j = 1,2,...N$$

$$Tc_j = \frac{t_{j+1}+t_j}{2}, \quad j = 1,2,...N \tag{23}$$

and the Shishkin mesh is given as follows

$$\sigma = min(\frac{1}{\alpha}, \epsilon log(N))$$

$$t_j = \frac{2\sigma(j-\frac{1}{2})}{N}, \quad j = 1,2,...\frac{N}{2}-1 \tag{24}$$

$$t_j = \sigma + \frac{2(1-\sigma)(j-M-\frac{1}{2})}{N}, \quad j = \frac{N}{2},...N. \tag{25}$$

3. Solution of Partially Singularly Perturbed System of Initial and Boundary Value Problems

3.1. Method for Solving System of First Order Linear Partially Singularly Perturbed Initial Value Problem. We consider the system as follows

$$\epsilon y_1'(t) + a(t)y_1(t) + b(t)y_2(t) = f_1(t), \tag{26}$$

$$y_2'(t) + c(t)y_1(t) + d(t)y_2(t) = f_2(t), \quad \forall \quad t \in (0,1] \tag{27}$$

with initial conditions

$$y_1(0) = \alpha_1, \quad y_2(0) = \alpha_2. \tag{28}$$

To solve this system, we assume that

$$y_1'(t) = \sum_{i=1}^{N} a_i \mathcal{H}_i(t), \tag{29}$$

and

$$y_2'(t) = \sum_{i=1}^{N} b_i \mathcal{H}_i(t). \tag{30}$$

Now, integrating (29) from 0 to t, we get

$$y_1(t) = \sum_{i=1}^{N} a_i \mathcal{P}_i(t) + y_1(0). \tag{31}$$

Further, integrating (30) from 0 to t, we get

$$y_2(t) = \sum_{i=1}^{N} b_i \mathcal{P}_i(t) + y_2(0). \tag{32}$$

Now, putting the values of $y_1'(t)$, $y_1(t)$, $y_2'(t)$ and $y_2(t)$ from equations (29)-(32), in equations (26) and (27), respectively, we get

$$\epsilon \sum_{i=1}^{N} a_i \mathcal{H}_i(t) + a(t)(\sum_{i=1}^{N} a_i \mathcal{P}_i(t) + y_1(0)) + b(t)(\sum_{i=1}^{N} b_i \mathcal{P}_i(t) + y_2(0)) = f_1(t), \tag{33}$$

and

$$\sum_{i=1}^{N} b_i \mathcal{H}_i(t) + c(t)(\sum_{i=1}^{N} a_i \mathcal{P}_i(t) + y_1(0)) + d(t)(\sum_{i=1}^{N} b_i \mathcal{P}_i(t) + y_2(0)) = f_2(t), \quad \forall \quad t \in (0,1] \tag{34}$$

On simplifying (33) and (34), we get

$$\sum_{i=1}^{N} a_i(\epsilon \mathcal{H}_i(t) + a(t)\mathcal{P}_i(t)) + b(t) \sum_{i=1}^{N} b_i \mathcal{P}_i(t) = f_1(t) - a(t)y_1(0) - b(t)y_2(0), \tag{35}$$

and

$$\sum_{i=1}^{N} b_i(\mathcal{H}_i(t) + d(t)\mathcal{P}_i(t)) + c(t) \sum_{i=1}^{N} a_i \mathcal{P}_i(t) = f_2(t) - c(t)y_1(0) - d(t)y_2(0), \quad \forall \quad t \in (0,1] \tag{36}$$

The equations (35) and (36) are system of linear equations with unknown non-uniform Haar wavelet coefficients $a_i's$ and $b_i's$, which can be solved using any method present in literature, such as Gauss elimination method and then we put the values of non-uniform Haar wavelet coefficients $a_i's$ in equation (31) and $b_i's$ in equation (32), which is the non-uniform Haar wavelet approximate solution of the system of first order linear partially singularly perturbed initial value problem.

3.2. Method for Solving System of Second Order Linear Partially Singularly Perturbed Boundary Value Problems. We consider the system as follows

$$-\epsilon y_1''(t) + a(t)y_1(t) + b(t)y_2(t) = f_1(t), \tag{37}$$

$$y_2''(t) + c(t)y_1(t) + d(t)y_2(t) = f_2(t), \quad \forall \quad t \in (0,1] \tag{38}$$

with boundary conditions

$$y_1(0) = \alpha_1, \quad y_2(0) = \alpha_2, \quad y_1(1) = \beta_1 \quad and \quad y_2(1) = \beta_2. \tag{39}$$

To solve this system, we assume that

$$y_1''(t) = \sum_{i=1}^{N} a_i \mathcal{H}_i(t), \tag{40}$$

and

$$y_2''(t) = \sum_{i=1}^{N} b_i \mathcal{H}_i(t). \tag{41}$$

Now, integrating (40) from 0 to t, we get

$$y_1'(t) = \sum_{i=1}^{N} a_i \mathcal{P}_i(t) + y_1'(0). \tag{42}$$

Also, integrating (41) from 0 to t we get,

$$y_2'(t) = \sum_{i=1}^{N} b_i \mathcal{P}_i(t) + y_2'(0). \tag{43}$$

Again integrating (42) and (43) from 0 to t, respectively, we get

$$y_1(t) = \sum_{i=1}^{N} a_i \mathcal{Q}_i(t) + t y_1'(0) + y_1(0), \tag{44}$$

$$y_2(t) = \sum_{i=1}^{N} b_i \mathcal{Q}_i(t) + t y_2'(0) + y_2(0). \tag{45}$$

Further, to find $y_1'(0)$ and $y_2'(0)$ we integrate the equations (40) and (41) from 0 to 1, respectively, and then we get

$$y_1'(0) = y_1(1) - y_1(0) - \sum_{i=1}^{N} a_i \mathcal{C}_i(t) \tag{46}$$

and

$$y_2'(0) = y_2(1) - y_2(0) - \sum_{i=1}^{N} b_i \mathcal{C}_i(t) \tag{47}$$

where $\mathcal{C}(t) = \int_0^1 \mathcal{P}(t)dt$.
Now, putting the values of $y_1'(0)$ and $y_2'(0)$ from equations (46) and (47) in equations (44) and (45), respectively, we get

$$y_1(t) = \sum_{i=1}^{N} a_i \mathcal{Q}_i(t) + t(y_1(1) - y_1(0) - \sum_{i=1}^{N} a_i \mathcal{C}_i(t)) + y_1(0), \tag{48}$$

$$y_2(t) = \sum_{i=1}^{N} b_i \mathcal{Q}_i(t) + t(y_2(1) - y_2(0) - \sum_{i=1}^{N} b_i \mathcal{C}_i(t)) + y_2(0). \tag{49}$$

Substituting the values of $y_1''(t)$, $y_1(t)$, $y_2''(t)$ and $y_2(t)$ from equations (40), (48), (41)and (49) in equations (37) and (38), respectively, we get the following system of linear equations

$$\begin{aligned}-\epsilon(\sum_{i=1}^{N} a_i \mathcal{H}_i(t)) + a(t)(\sum_{i=1}^{N} a_i \mathcal{Q}_i(t) + t(y_1(1) - y_1(0) - \sum_{i=1}^{N} a_i \mathcal{C}_i(t)) + y_1(0)) +\\ b(t)(\sum_{i=1}^{N} b_i \mathcal{Q}_i(t) + t(y_2(1) - y_2(0) - \sum_{i=1}^{N} b_i \mathcal{C}_i(t)) + y_2(0)) = f_1(t),\end{aligned} \tag{50}$$

and

$$\begin{aligned}\sum_{i=1}^{N} b_i \mathcal{H}_i(t) + c(t)(\sum_{i=1}^{N} a_i \mathcal{Q}_i(t) + t(y_1(1) - y_1(0) - \sum_{i=1}^{N} a_i \mathcal{C}_i(t)) + y_1(0)) +\\ d(t)(\sum_{i=1}^{N} b_i \mathcal{Q}_i(t) + t(y_2(1) - y_2(0) - \sum_{i=1}^{N} b_i \mathcal{C}_i(t)) + y_2(0)) = f_2(t).\end{aligned} \tag{51}$$

On simplifying the system (50) and (51), we get

$$\sum_{i=1}^{N} a_i(-\epsilon\mathcal{H}_i(t)+a(t)(\mathcal{Q}_i(t)-t\mathcal{C}_i(t)))+\sum_{i=1}^{N} b_i(b(t)(\mathcal{Q}_i(t)-t\mathcal{C}_i(t)))$$
$$=f_1(t)-a(t)(t(y_1(1)-y_1(0))+y_1(0))-b(t)(t(y_2(1)-y_2(0))-y_2(0)), \quad (52)$$

and

$$\sum_{i=1}^{N} b_i(\mathcal{H}_i(t)+d(t)(\mathcal{Q}_i(t))-t\mathcal{C}_i(t))+\sum_{i=1}^{N} a_i c(t)(\mathcal{Q}_i(t)-t\mathcal{C}_i(t))$$
$$=f_2(t)-c(t)(t(y_1(1)-y_1(0))+y_1(0))-d(t)(t(y_2(1)-y_2(0))+y_2(0)). \quad (53)$$

The equations (52) and (53) are system of linear equations with unknown non-uniform Haar wavelet coefficients $a_i's$ and $b_i's$, which can be solved using any method present in literature, such as Gauss elimination method and then we put the values of non-uniform Haar wavelet coefficients $a_i's$ in equation (48) and $b_i's$ in equation (49), which is the non-uniform Haar wavelet approximate solution of the system of second order linear partially singularly perturbed boundary value problems.

4. Error Analysis

Lemma. Let $y(x)$ be a square integrable function with bounded first derivative on $(0,1)$ and $y(x_j)$, be Haar wavelet approximation of $y(x)$, then the error norm at J^{th} level satisfies the inequality

$$\|E\|=|y(t)-y(t_j)|\leq 2D\sqrt{K}\Big(\frac{2^{-2(J+1)}}{3}\Big)^2 \quad (54)$$

Proof. For the proof see [8, 13].

In case, if we do not have the exact solution of the system of first order linear partially singularly perturbed initial value problem, then the maximum absolute residual error is calculated by the following formula

$$E=\max|\epsilon y_1'(t_j)+a(t_j)y_1(t_j)+b(t_j)y_2(t_j)-f_1(t_j)|, \quad (55)$$

where $y_1'(t_j)$, $y_1(t_j)$ and $y_2(t_j)$ are given as in equations (29), (31) and (32), respectively, $j=1,2,...N.$, $N=2^{J+1}$.

In case, if we do not have the exact solution of the system of second order linear partially singularly perturbed boundary value problem, then the maximum absolute residual error is calculated by the following formula

$$E=\max|-\epsilon y_1''(t_j)+a(t_j)y_1(t_j)+b(t_j)y_2(t_j)-f_1(t_j)|, \quad (56)$$

where $y_1''(t_j)$, $y_1(t_j)$ and $y_2(t_j)$ are given as in equations (40), (48) and (49), respectively, $j=1,2,...N.$, $N=2^{J+1}$.

5. Numerical Examples

To illustrate the non-uniform Haar wavelet method on different meshes, we demonstrate one numerical example of the system of first order linear partially singularly perturbed initial value problem and one numerical example of the system of second order linear partially singularly perturbed boundary value problem. The maximum absolute residual errors are tabulated and also compared with the classical finite difference operator method and parameter uniform methods presented in [11] and [12], respectively.

Problem 1. Let us consider the following partially singularly perturbed initial value problem

$$\epsilon y_1'(t) + (2+t)y_1(t) - (1+\frac{t}{2})y_2(t) = 5t + \frac{1}{2}, \tag{57}$$

$$y_2'(t) - (1+t)y_1(t) + (2+t)y_2(t) = te^t, \quad \forall \quad t \in (0,1) \tag{58}$$

with initial conditions

$$y_1(0) = 2, \quad y_2(0) = 2. \tag{59}$$

Comparison of maximum absolute residual errors obtained by non-uniform Haar wavelet method with different resolutions level and classical finite difference operator method have been given in the Tables 1.1, 1.2, 1.3, 1.4 and 1.5 for various values of perturbation parameter. Also, graph of non-uniform Haar wavelet solution is given in Figures 4, 5, 6 and 7.

Table 1.1 Maximum absolute residual errors obtained by non-uniform Haar wavelet on ρ-mesh for various values of ϵ

ϵ \N	64	128	256	512	1024	2048
2^{-2}	4.1078e-15	2.8866e-14	9.4147e-14	5.9952e-15	2.2204e-14	3.4195e-14
2^{-4}	1.0436e-14	7.9492e-14	7.7716e-14	3.9746e-14	1.0703e-13	1.0836e-13
2^{-6}	7.3275e-15	1.1502e-13	7.0832e-14	7.4163e-14	1.2657e-13	2.6290e-13
2^{-8}	9.9920e-15	1.2101e-14	3.2419e-14	1.1902e-13	7.5495e-14	2.7267e-13
2^{-10}	1.3212e-14	4.4076e-14	6.0618e-14	6.7946e-14	3.5616e-13	1.4999e-12
2^{-12}	1.7875e-14	1.1480e-13	3.3423e-13	3.3984e-13	3.2663e-13	1.0367e-12
2^{-14}	2.4203e-14	1.5599e-13	7.7754e-13	2.3839e-12	9.5279e-13	1.5477e-12
2^{-16}	1.6209e-14	1.8452e-13	9.9498e-13	8.4892e-12	4.9925e-12	3.6597e-12
2^{-18}	2.7311e-14	1.6853e-13	1.0961e-12	1.1143e-11	1.9294e-11	7.5797e-12
2^{-20}	1.7319e-14	1.5854e-13	1.1033e-12	1.2064e-11	2.5990e-11	8.2536e-11
2^{-22}	1.7764e-14	2.1072e-13	1.0854e-12	1.1987e-11	2.7449e-11	4.0708e-11
2^{-24}	1.7764e-14	1.7253e-13	1.1520e-12	1.2177e-11	2.8213e-11	4.3110e-11
2^{-26}	2.3093e-14	2.0806e-13	1.1336e-12	1.2306e-11	2.8385e-11	4.4727e-11
2^{-28}	1.5765e-14	2.1139e-13	1.1326e-12	1.2246e-11	2.8827e-11	4.5758e-11
2^{-30}	4.1744e-14	1.9384e-13	1.2054e-12	1.2563e-11	2.8260e-11	4.4787e-11
2^{-32}	2.0317e-14	1.9207e-13	1.0894e-12	1.2280e-11	2.8895e-11	8.1765e-11
2^{-34}	1.7542e-14	1.8163e-13	1.0513e-12	1.2288e-11	2.8594e-11	4.5211e-11
2^{-36}	1.9096e-14	1.8829e-13	1.0154e-12	1.2091e-11	2.8591e-11	4.4827e-11
2^{-38}	1.7764e-14	1.8097e-13	1.2020e-12	1.2412e-11	2.8871e-11	4.5883e-11
2^{-40}	3.8636e-14	1.4766e-13	1.1727e-12	1.2459e-11	2.8853e-11	4.4534e-11

Table 1.2 Maximum absolute residual errors obtained by non-uniform Haar wavelet on ρ-mesh for various values of ϵ

ϵ \N	32	64	128	256	512	1024
10^{-3}	1.3767e-14	4.6407e-14	8.6597e-14	9.3259e-14	5.0093e-13	1.3487e-12
10^{-5}	2.8089e-14	1.7120e-13	9.7722e-13	9.4564e-12	9.7020e-12	6.0334e-12
10^{-7}	2.8200e-14	1.6720e-13	1.0703e-12	1.2225e-11	2.8184e-11	4.3185e-11

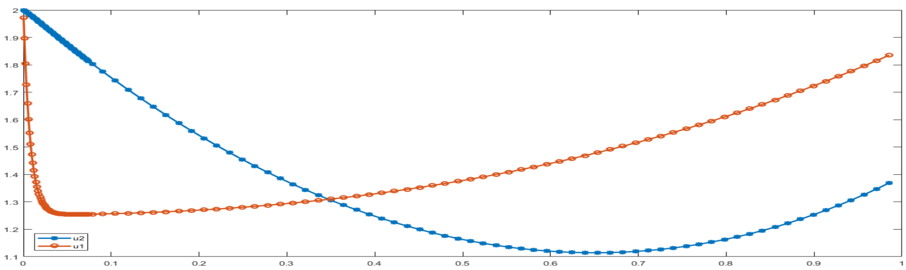

Figure 4. Non-uniform Haar wavelet solution on shishkin mesh of problem 1 for $\epsilon = 2^{-6}$ with $J = 6$.

Table 1.3 Maximum absolute residual errors obtained by non-uniform Haar wavelet on Shishkin mesh for various values of ϵ

$\epsilon \setminus N$	64	128	256	512	1024	2048	4096
2^{-2}	2.1538e-14	6.6613e-15	1.5099e-14	1.6209e-14	1.5099e-14	3.6859e-14	7.0610e-14
2^{-4}	2.0872e-14	4.4409e-15	1.9096e-14	1.5543e-14	5.0404e-14	7.5939e-14	1.3323e-13
2^{-6}	2.1982e-14	3.5749e-14	6.6391e-14	2.5313e-14	9.1260e-14	5.8908e-13	3.0465e-13
2^{-8}	9.5923e-14	2.3492e-13	1.4300e-13	2.5047e-13	2.8022e-13	1.0683e-12	2.4984e-12
2^{-10}	4.4476e-13	5.8753e-13	3.2374e-13	8.4777e-13	1.0953e-12	5.8855e-12	6.7606e-12
2^{-12}	6.7479e-13	1.8021e-12	3.3475e-12	4.2180e-12	5.8935e-12	1.1380e-11	3.5781e-11
2^{-14}	5.4818e-12	8.0473e-12	7.9565e-12	2.0488e-11	2.1784e-11	3.6403e-11	8.1301e-11
2^{-16}	5.3906e-11	1.5910e-11	1.5997e-11	8.0267e-11	3.3118e-11	7.3043e-11	2.2942e-10
2^{-18}	1.1533e-10	6.9644e-11	3.2157e-10	4.8666e-10	2.4769e-10	1.0448e-09	4.9136e-10
2^{-20}	6.1724e-10	7.8969e-10	7.7818e-10	2.8672e-09	1.1941e-09	2.2140e-09	2.9340e-09
2^{-22}	1.2860e-09	2.6925e-09	4.1590e-09	2.4290e-09	5.7606e-09	1.0989e-08	1.4569e-08
2^{-24}	5.7088e-09	1.3774e-08	8.8248e-09	1.0128e-08	2.9396e-08	3.2977e-08	4.6273e-08
2^{-26}	4.6880e-08	1.9145e-08	6.1711e-08	9.7442e-08	9.8635e-08	1.0465e-07	1.7249e-07
2^{-28}	9.8958e-08	1.0455e-07	1.3310e-07	1.5702e-07	3.4662e-07	1.0747e-06	6.4958e-07
2^{-30}	1.8406e-07	4.8247e-07	3.5228e-07	1.6018e-06	1.9999e-06	3.4653e-06	2.3661e-06
2^{-32}	2.2135e-06	9.5038e-07	1.1344e-06	5.2311e-06	6.8724e-06	8.9694e-06	1.0042e-05
2^{-34}	3.7412e-06	3.6463e-06	1.6677e-05	1.7547e-05	3.7073e-05	4.9961e-05	3.4575e-05
2^{-36}	2.5019e-05	3.9447e-05	5.5062e-05	9.4696e-05	1.7000e-04	2.7315e-04	1.4333e-04

Table 1.4 Maximum absolute residual errors obtained by non-uniform Haar wavelet on Shishkin mesh for various values of ϵ

$\epsilon \setminus N$	32	64	128	256	512	1024	2048
10^{-3}	2.2404e-13	4.5097e-13	7.4585e-13	1.1644e-12	3.2627e-12	3.9213e-13	9.1205e-12
10^{-5}	2.7947e-11	3.0884e-11	2.5485e-11	1.0856e-10	3.7653e-11	5.2737e-11	6.3557e-10
10^{-7}	1.0022e-08	3.2361e-09	4.5378e-09	5.1181e-09	1.4684e-08	5.0842e-08	2.6979e-08

Table 1.5 Maximum absolute residual errors obtained by classical finite difference operator method [12] for various values of ϵ.

$\epsilon \setminus N$	64	128	256	512	1024	2048
10^{-3}	3.4000e - 002	1.8100e - 002	1.11700e - 002	7.1800e - 02	4.2700e - 03	1.3700e - 03
10^{-5}	3.4000e - 002	1.8100e - 002	1.11700e - 002	7.1800e - 02	4.2700e - 03	1.3700e - 03
10^{-7}	3.4000e - 002	1.8100e - 002	1.11700e - 002	7.1800e - 02	4.2700e - 03	1.3700e - 03

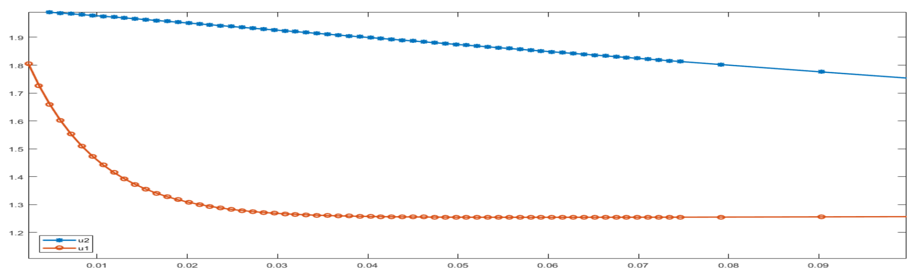

Figure 5. Blowup of the initial layer in the sub-domain for $\epsilon = 2^{-6}$ with $J = 6$ in the interval [0, 0.1].

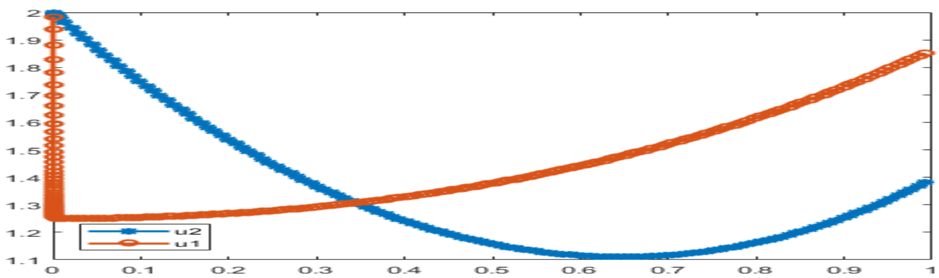

Figure 6. Non-uniform Haar wavelet solution of problem 1 for $\epsilon = 2^{-10}$ with $J = 7$.

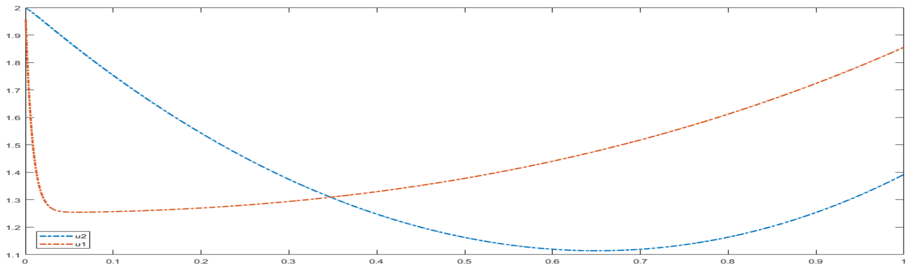

Figure 7. Non-uniform Haar wavelet solution on ρ-mesh of problem 1 for $\epsilon = 2^{-6}$ with $J = 9$.

Problem 2. Let us consider the following partially singularly perturbed boundary value problem

$$-\epsilon y_1''(t) + 2(1+t)^2 y_1(t) - (1+t^3) y_2(t) = 2e^t, \tag{60}$$

$$y_2''(t) - 2cos(\frac{\pi t}{4}) y_1(t) + 2.2e^{1-t} y_2(t) = 10t + 1, \quad \forall \quad t \in (0,1) \tag{61}$$

with boundary conditions

$$y_1(0) = y_2(0) = 0 \quad and \quad y_1(1) = y_2(1) = 0. \tag{62}$$

Computed maximum absolute residual errors obtained by non-uniform Haar wavelet method on different mesh with different resolutions level and parameter uniform method have been given in the Tables 2.1, 2.2, 2.3 and 2.4 for various values of perturbation parameter. Also, graph of non-uniform Haar wavelet solution is given in Figures 8, 9 and 10.

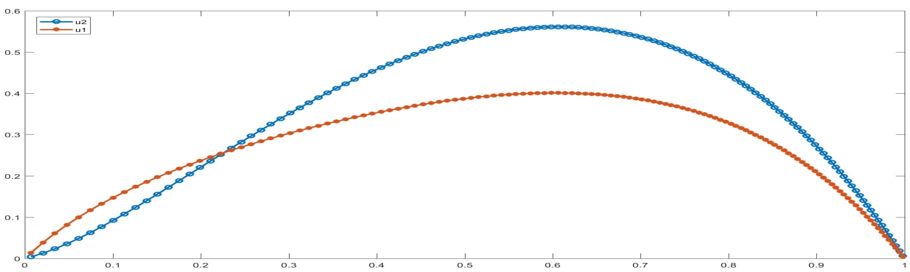

Figure 8. Non-uniform Haar wavelet solution on q-mesh of problem 2 for $\epsilon = 2^{-1}$ with $J = 6$.

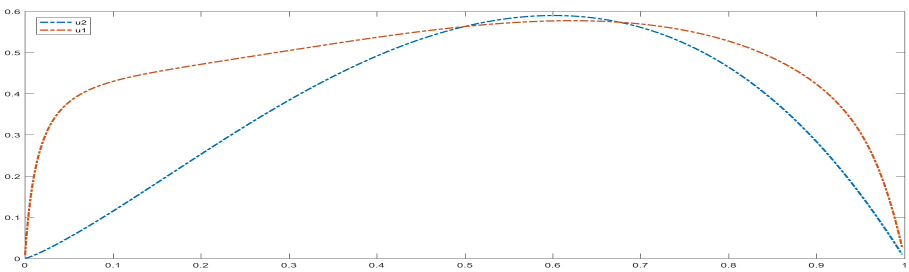

Figure 9. Non-uniform Haar wavelet solution on shishkin mesh of problem 2 for $\epsilon = 2^{-5}$ with $J = 8$.

Table 2.1 Maximum absolute residual errors obtained by non-uniform Haar wavelet on Shishkin mesh for various values of ϵ

$\epsilon \setminus N$	32	64	128	256	512	1024
2^{-2}	1.5543e-14	8.8818e-15	9.3259e-15	9.7700e-15	3.4639e-14	3.2863e-14
2^{-6}	1.5099e-14	1.0658e-14	1.1102e-14	9.7700e-15	3.7303e-14	3.4639e-14
2^{-10}	3.3751e-14	3.2863e-14	1.4655e-14	1.9096e-14	2.8422e-14	1.3323e-14
2^{-14}	1.7764e-14	1.4655e-14	2.7089e-14	1.4211e-14	2.4869e-14	1.4211e-14
2^{-18}	1.8652e-14	3.7748e-14	1.7764e-14	4.3077e-14	2.3981e-14	5.3291e-14
2^{-22}	2.5313e-14	1.2434e-14	2.9310e-14	4.1744e-14	3.3307e-14	3.9524e-14
2^{-26}	1.9096e-14	1.1990e-14	3.7303e-14	3.4195e-14	3.9524e-14	2.6201e-14
2^{-30}	1.9540e-14	1.1102e-14	4.0412e-14	1.3767e-14	2.8422e-14	4.8406e-14
2^{-34}	2.4869e-14	1.2434e-14	3.5971e-14	1.2434e-14	3.1974e-14	2.6201e-14
2^{-38}	2.0428e-14	1.1990e-14	3.8636e-14	1.2434e-14	3.1086e-14	2.4869e-14
2^{-42}	2.3093e-14	1.0658e-14	3.5971e-14	1.2434e-14	2.0428e-14	4.4853e-14
2^{-46}	2.1760e-14	1.1102e-14	3.3307e-14	1.2434e-14	2.9310e-14	2.3093e-14
2^{-50}	2.2204e-14	1.1990e-14	3.6859e-14	1.2434e-14	2.5313e-14	4.6185e-14

Table 2.2 Maximum absolute residual errors obtained by parameter uniform method [11] for various values of ϵ

$\epsilon \setminus N$	8	16	32	64	128	256	512
2^{0}	2.6110e-03	6.5500e-04	1.6440e-04	4.1100e-05	1.0270e-05	2.5670e-06	6.3980e-07
2^{-2}	2.5760e-03	6.4630e-04	1.6170e-04	4.0440e-05	1.0110e-05	2.5260e-06	6.2960e-07
2^{-4}	2.7920e-03	7.0150e-04	1.7660e-04	4.4170e-05	1.1040e-05	2.7580e-06	6.8760e-07
2^{-6}	2.9200e-03	7.2900e-04	1.8240e-04	4.5620e-05	1.1410e-05	2.8490e-06	7.1030e-07
2^{-8}	3.0350e-03	7.5260e-04	1.8830e-04	4.7070e-05	1.1770e-05	2.9390e-06	7.3270e-07
2^{-10}	3.1160e-03	7.8060e-04	1.9390e-04	4.8420e-05	1.2100e-05	3.0200e-06	7.5430e-07
2^{-12}	3.1520e-03	7.9670e-04	1.9810e-04	4.9350e-05	1.2300e-05	3.0550e-06	7.5960e-07
2^{-14}	3.1630e-03	8.0310e-04	2.0040e-04	4.9930e-05	1.2430e-05	3.1080e-06	7.7140e-07
2^{-16}	3.1670e-03	8.0520e-04	2.0120e-04	5.0210e-05	1.2520e-05	3.1130e-06	7.7260e-07
2^{-18}	3.1680e-03	8.0570e-04	2.0150e-04	5.0320e-05	1.2550e-05	3.1240e-06	7.7030e-07
2^{-20}	3.1680e-03	8.0590e-04	2.0160e-04	5.0370e-05	1.2570e-05	3.1330e-06	7.7100e-07
2^{-22}	3.1680e-03	8.0590e-04	2.0160e-04	5.0370e-05	1.2580e-05	3.1350e-06	7.7420e-07
2^{-24}	3.1680e-03	8.0600e-04	2.0160e-04	5.0380e-05	1.2580e-05	3.1360e-06	7.7460e-07
2^{-26}	3.1680e-03	8.0590e-04	2.0160e-04	5.0370e-05	1.2550e-05	3.0960e-06	7.6450e-07
2^{-28}	3.1680e-03	8.0600e-04	2.0160e-04	5.0380e-05	1.2560e-05	3.1120e-06	7.6440e-07

Table 2.3 Maximum absolute residual errors obtained by non-uniform Haar wavelet on q-mesh for various values of ϵ

$\epsilon \setminus N$	8	16	32	64	128	256	512
2^{-2}	5.3291e-15	3.9968e-15	7.1054e-15	6.2172e-15	8.4377e-15	3.3751e-14	2.4425e-14
2^{-6}	5.3291e-15	4.4409e-15	7.1054e-15	5.7732e-15	8.4377e-15	3.4639e-14	2.5757e-14
2^{-10}	5.3291e-15	3.9968e-15	6.6613e-15	5.7732e-15	8.8818e-15	3.5527e-14	2.5757e-14
2^{-14}	5.3291e-15	3.5527e-15	7.1054e-15	5.3291e-15	8.8818e-15	3.5083e-14	2.5757e-14
2^{-18}	5.3291e-15	4.4409e-15	7.1054e-15	5.7732e-15	8.8818e-15	3.5971e-14	2.5757e-14
2^{-22}	5.3291e-15	4.4409e-15	7.1054e-15	5.7732e-15	8.8818e-15	3.5083e-14	2.5757e-14
2^{-26}	5.3291e-15	3.5527e-15	6.2172e-15	5.7732e-15	8.8818e-15	3.6415e-14	2.5757e-14
2^{-30}	5.3291e-15	4.4409e-15	7.1054e-15	6.2172e-15	8.8818e-15	3.5971e-14	2.5757e-14
2^{-34}	5.3291e-15	3.9968e-15	6.6613e-15	5.7732e-15	8.8818e-15	3.5527e-14	2.5757e-14
2^{-38}	5.3291e-15	3.9968e-15	7.1054e-15	5.7732e-15	8.8818e-15	3.5527e-14	2.5757e-14
2^{-42}	5.3291e-15	3.9968e-15	6.6613e-15	5.7732e-15	8.8818e-15	3.5083e-14	2.5757e-14
2^{-46}	5.3291e-15	4.4409e-15	7.1054e-15	5.7732e-15	8.8818e-15	3.5083e-14	2.5757e-14
2^{-50}	4.8850e-15	3.9968e-15	6.6613e-15	5.7732e-15	8.8818e-15	3.5083e-14	2.5757e-14

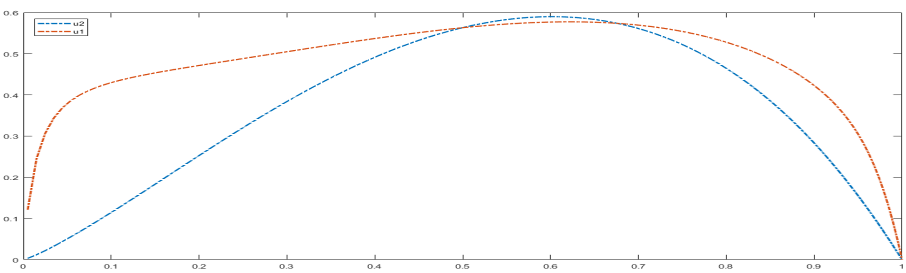

Figure 10. Non-uniform Haar wavelet solution on q-mesh of problem 2 for $\epsilon = 2^{-5}$ with $J = 8$.

Conclusion

In the present paper, we have solved the system of first and second order linear partially singularly perturbed initial and boundary value problems respectively using non-uniform Haar wavelet on different meshes such as Shishkin mesh, ρ-mesh and q-mesh. In fact, we obtained the approximate solution and computed maximum absolute residual errors, which are tabulated in the Tables $1.1 - 1.5$ and $2.1 - 2.3$. Also, we have compared our results with the existing methods given in [11, 12]. Further, the graphs of examples have been demonstrated in the Figures 1-7, which clearly indicate that non-uniform Haar wavelet produces better results in comparison of classical finite difference operator method and parameter uniform methods. The technique introduced here is easy to apply as well as yields more accurate results.

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