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EXTENDING THE APPLICABILITY OF A FOURTH-ORDER METHOD UNDER LIPSCHITZ CONTINUOUS DERIVATIVE IN BANACH SPACES

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Abstract. We extend the applicability of a fourth-order convergent nonlinear system solver by providing its local convergence analysis under Lipschitz continuous Fréchet derivative in Banach spaces. Our analysis only uses the first-order Fréchet derivative to ensure the convergence and provides the uniqueness of the solution, the radius of convergence ball and the computable error bounds. This study is applicable in solving such problems for which earlier studies are not effective. Furthermore, the convergence region for the scheme to approximate the zeros of various polynomials is studied using basins of attraction tool. Various computational tests are conducted to validate that our analysis is beneficial when prior studies fail to solve problems.

Keywords: Local convergence, Iterative methods, Banach space, Lipschitz continuity condition, Basin of attraction

AMS Subject Classification: 47H99, 49M15, 65D99, 65G99, 65J15

1. Introduction

The main objective of the analysis discussed in this article is to extend the applicability of a fourth-order iterative scheme for obtaining a locally unique solution u^* of

$$H(u) = 0, \tag{1}$$

where $H : \Omega \subseteq X \to Y$ is a Fréchet differentiable operator with values in the Banach space Y and $\Omega(\neq \emptyset)$ is an open and convex subset of the Banach space X. Many problems can be solved in the domain of engineering and applied sciences, by reducing the form to nonlinear equations (1). Taking the reality into consideration that numerous problems in applied sciences and engineering such as the integral equations occur in radiative transfer theory, problems in optimization, the boundary value problems related to Kinetic theory of gases and many others can be solved by obtaining the solutions of nonlinear equations in the form (1), a lot of successful algorithms has been constructed. "Generally, the solutions for these nonlinear equations are not obtained in closed form" [1]. Iterative algorithms

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are therefore often used to avoid these issues. A simple and widely accepted iterative algorithm for solving (1) is Newton's scheme, which is expressed as:

$$u_{n+1} = u_n - [H'(u_n)]^{-1}H(u_n),\ n \geq 0. \tag{2}$$

Also, the cubically convergent Halley's, Chebyshev's and Super-Halley's schemes are generated by choosing ($\gamma = \frac{1}{2}$), ($\gamma = 0$) and ($\gamma = 1$) respectively in

$$u_{n+1} = u_n - \left(1 + \frac{1}{2}(1 - \gamma F_H(u_n))^{-1}F_H(u_n)\right)[H'(u_n)]^{-1}H(u_n), \tag{3}$$

where $F_H(u_n) = H'(u_n)^{-1}H''(u_n)H'(u_n)^{-1}H(u_n)$.

Several researchers have designed newtons-like methods [2, 3, 4, 5, 6, 7, 9, 10, 12, 11, 13, 8, 14, 15, 16] such as harmonic mean Newton's method, midpoint Newton's method and other variants of Newton's method to handle the computation of higher-order derivatives found in conventional third-order schemes. "The local convergence analysis of iterative schemes is based on the information around a solution and provides the radii of convergence balls"[1]. Numerous researchers [1, 17, 18, 19] discussed the local convergence study of several modifications of the schemes (3) including deformed Halley, modified Halley-like and improved Chebyshev-Halley type methods. In addition to that, the local convergence study for Newton-type, Jarratt-type, Weerakoon-type, etc. is studied in Banach spaces in [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35]. In this paper, our primary focus is to enhance the applicability of a fourth-order scheme using Lipschitz continuity condition only on H' in Banach spaces.

In [36], the authors derived a fourth-order convergent nonlinear systems solver, which is given as:

$$\begin{aligned} s_n &= u_n - H'(u_n)^{-1}H(u_n) \\ t_n &= u_n - H'(u_n)^{-1}(H(u_n) + H(s_n)) \\ u_{n+1} &= s_n - H'(t_n)^{-1}H(s_n) \end{aligned} \tag{4}$$

The implementation of this method needs the computation of only H'. However, the analysis of convergence is provided under the Taylor series approach using higher-order (up to fourth-order) derivatives in [36]. These techniques, which require the higher-order derivatives, restrict the algorithm applicability for the problems where the derivatives of higher order are undefined or unobtainable. Consider, for example, the function H defined on $\Omega = [-\frac{1}{2}, \frac{5}{2}]$ by

$$H(u) = \begin{cases} u^3 \log(u^2) + u^5 - u^4, & \text{if } u \neq 0 \\ 0, & \text{if } u = 0 \end{cases}.$$

It is important to note that H''' is not bounded on Ω. So, the earlier procedure [36], which needs at least fourth-order derivative, fails to show the convergence of the method (4) for above problem. In addition, one can get zero knowledge regarding the radii of convergence balls in [36]. In this study, we analyze the local convergence for the scheme (4) assuming the theory based on H' to remove the calculation of derivatives of higher order. In particular, we assume that the first-order Fréchet derivative belongs to the Lipschitz class. This analysis boosts the applicability of the scheme (4) to address such problems for which earlier analysis can not be used.

Also, the region of convergence for an iterative scheme to obtain the zeros of complex polynomials can be explored using basins of attraction. In [37, 38], the authors presented the convergence region of different efficient schemes using attraction basins tool. We also provide convergence region of the algorithm (4) when applied to various complex polynomials.

The outline this manuscript is as follows: Sect. 2 deals with the local convergence study of the scheme (4). Basins of attraction related to the scheme (4) is studied in Sect. 3. Section 4 discusses the applicability of our analytical results on standard numerical tests. Conclusion is provided in the final section.

2. Local convergence analysis

In this section, the local convergence analysis of the fourth-order convergent method (4) is discussed. We use the notations $\bar{B}(c,\rho)$ and $B(c,\rho)$ for the closed and open balls in X with center c and radius $\rho > 0$ respectively. Also, the set of bounded linear operators from Y to X is denoted as $BL(Y,X)$. Considering the parameters $k_0 > 0$ and $k_1 > 0$ with $k_0 \leq k_1$, we define the function J_1 on the interval $[0, \frac{1}{k_0})$ by

$$J_1(w) = \frac{k_1 w}{2(1-k_0 w)} \tag{5}$$

and the parameter

$$\theta_1 = \frac{2}{2k_0 + k_1} < \frac{1}{k_0}.$$

Note that $J_1(\theta_1) = 1$. Again, we consider the functions J_2 and K_2 on $[0, \frac{1}{k_0})$ defined by

$$J_2(w) = \left[\frac{k_1(2 + J_1(w))w}{2(1-k_0 w)}\right] J_1(w) \tag{6}$$

and

$$K_2(w) = J_2(w) - 1.$$

Now, $K_2(0) = -1 < 0$ and $\lim\limits_{w \to (\frac{1}{k_0})^-} K_2(w) = +\infty$. The zeros of the function $K_2(w)$ lies in $(0, \frac{1}{k_0})$ due to the intermediate value theorem. The notation of the smallest zero of $K_2(w)$ in $(0, \frac{1}{k_0})$ is θ_2. Also, we consider functions J_3 and K_3 on $[0, \frac{1}{k_0})$ by

$$J_3(w) = k_0 J_2(w) w \tag{7}$$

and

$$K_3(w) = J_3(w) - 1.$$

Now, $K_3(0) = -1 < 0$ and $\lim\limits_{w \to (\frac{1}{k_0})^-} K_3(w) = +\infty$. The intermediate value theorem ensures the existence of the smallest zero θ_3 of $K_3(w)$ in $(0, \frac{1}{k_0})$. Again, we define J_4 and K_4 on $[0, \theta_3)$ by

$$J_4(w) = \left[1 + \frac{(1 + k_0 J_1(w) w)}{1 - J_3(w)}\right] J_1(w) \tag{8}$$

and

$$K_4(w) = J_4(w) - 1.$$

Now, $K_4(0) = -1 < 0$ and $\lim\limits_{w \to \theta_3^-} K_4(w) = +\infty$. So, the interval $(0, \theta_3)$ holds the smallest zero θ_4 of the function $K_4(w)$. Choosing

$$R = min\{\theta_1, \theta_2, \theta_4\}, \tag{9}$$

we get

$$0 \leq J_1(w) < 1, \tag{10}$$

$$0 \leq J_2(w) < 1, \tag{11}$$

$$0 \leq J_3(w) < 1 \tag{12}$$

and

$$0 \leq J_4(w) < 1, \tag{13}$$

$\forall w \in [0, R)$. Also, we assume the following conditions for the Fréchet differentiable operator $H : \Omega \subseteq X \to Y$.

$$H(u^*) = 0,\ H'(u^*)^{-1} \in BL(Y, X), \tag{14}$$

$$||H'(u^*)^{-1}(H'(u) - H'(u^*))|| \leq k_0||u - u^*||,\ \forall u \in \Omega \tag{15}$$

and

$$||H'(u^*)^{-1}(H'(u) - H'(s))|| \leq k_1||u - s||,\ \forall u,\ s \in \Omega. \tag{16}$$

Many authors [27, 26, 1, 17, 23] use an extra assumption

$$||H'(u^*)^{-1}H'(u)|| \leq M,\ \forall u \in B\left(u^*, \frac{1}{k_0}\right). \tag{17}$$

We eliminate this additional condition by using the following results.

Lemma 2.1. *If H obeys (15) and $\bar{B}(u^*, R) \subseteq \Omega$, then $\forall u \in B(u^*, R)$, we get*

$$||H'(u^*)^{-1}H'(u)|| \leq 1 + k_0||u - u^*|| \tag{18}$$

and

$$||H'(u^*)^{-1}H(u)|| \leq (1 + k_0||u - u^*||)||u - u^*|| \tag{19}$$

Proof. Applying (15), we get

$$||H'(u^*)^{-1}H'(u)|| \leq 1 + ||H'(u^*)^{-1}(H'(u) - H'(u^*))|| \leq 1 + k_0||u - u^*||.$$

For $\beta \in [0, 1]$,

$$||H'(u^*)^{-1}H'(u^* + \beta(u - u^*))|| \leq 1 + k_0\beta||u - u^*|| \leq 1 + k_0||u - u^*||$$

and the mean value theorem helps in obtaining

$$\begin{aligned} ||H'(u^*)^{-1}H(u)|| &= ||H'(u^*)^{-1}(H(u) - H(u^*))|| \\ &\leq ||H'(u^*)^{-1}H'(u^* + \beta(u - u^*))(u - u^*)|| \\ &\leq (1 + k_0||u - u^*||)||u - u^*||. \end{aligned}$$

□

Now, we move forward to describe the local convergence analysis of the algorithm (4) in the next Theorem.

Theorem 2.1. *Let $H : \Omega \subseteq X \to Y$ be a Fréchet differentiable operator. Assume that $u^* \in \Omega$, H satisfies (14)-(16) and*

$$\bar{B}(u^*, R) \subseteq \Omega, \tag{20}$$

where R is mentioned in (9). For $u_0 \in B(u^*, R)$ the scheme (4) provides the well defined sequence $\{u_n\}_{n\geq 0}$ such that $\{u_n\}_{n\geq 0} \in B(u^*, R)$ and converges to u^*. In addition, the followings are true $\forall n \geq 0$.

$$||s_n - u^*|| \leq J_1(||u_n - u^*||)||u_n - u^*|| < ||u_n - u^*|| < R, \tag{21}$$

$$||t_n - u^*|| \leq J_2(||u_n - u^*||)||u_n - u^*|| < ||u_n - u^*|| < R \tag{22}$$

and

$$||u_{n+1} - u^*|| \leq J_4(||u_n - u^*||)||u_n - u^*|| < ||u_n - u^*|| < R, \tag{23}$$

where J_1, J_2 and J_4 are provided in (5), (6) and (8) respectively. Also, u^* is the unique solution of $H(u) = 0$ in $\bar{B}(u^*, \Delta) \cap \Omega$ for $\Delta \in [R, \frac{2}{k_0})$.

Proof. From (9), (15) and the assumption $u_0 \in B(u^*, R)$, we obtain

$$||H'(u^*)^{-1}(H'(u_0) - H'(u^*))|| \leq k_0||u_0 - u^*|| < k_0 R < 1. \tag{24}$$

Now, Banach Lemma on invertible operators [2, 4, 5, 6, 7] ensures that $H'(u_0)^{-1} \in BL(Y, X)$ with

$$||H'(u_0)^{-1}H'(u^*)|| \leq \frac{1}{1 - k_0||u_0 - u^*||} < \frac{1}{1 - k_0 R} \tag{25}$$

and this confirms the existence of s_0. Again,

$$\begin{aligned} s_0 - u^* &= u_0 - u^* - H'(u_0)^{-1}H(u_0) \\ &= -\left[H'(u_0)^{-1}H'(u^*)\right]\left[\int_0^1 H'(u^*)^{-1}(H'(u^* + \beta(u_0 - u^*)) - H'(u_0))(u_0 - u^*)\, d\beta\right]. \end{aligned} \tag{26}$$

With the help of (5), (9), (10), (16), (25) and (26), we get

$$\begin{aligned} ||s_0 - u^*|| &\leq \left[||H'(u_0)^{-1}H'(u^*)||\right]\left[\left|\left|\int_0^1 H'(u^*)^{-1}(H'(u^* + \beta(u_0 - u^*)) - H'(u_0))(u_0 - u^*)\, d\beta\right|\right|\right] \\ &\leq \frac{k_1||u_0 - u^*||}{2(1 - k_0||u_0 - u^*||)}||u_0 - u^*|| \\ &= J_1(||u_0 - u^*||)||u_0 - u^*|| < ||u_0 - u^*|| < R. \end{aligned} \tag{27}$$

Hence, (21) holds for $n = 0$. Also,

$$\begin{aligned} t_0 - u^* &= u_0 - u^* - H'(u_0)^{-1}(H(u_0) + H(s_0)) \\ &= u_0 - u^* - H'(u_0)^{-1}H(u_0) - H'(u_0)^{-1}H(s_0) \\ &= s_0 - u^* - H'(u_0)^{-1}H(s_0) \\ &= -\left[H'(u_0)^{-1}H'(u^*)\right]\left[\int_0^1 H'(u^*)^{-1}(H'(u^* + \beta(s_0 - u^*)) - H'(u_0))(s_0 - u^*)\, d\beta\right]. \end{aligned} \tag{28}$$

We use (6), (9), (11), (19), (25), (27) and (28) and deduce that

$$\begin{aligned} ||t_0 - u^*|| &\leq \left[||H'(u_0)^{-1}H'(u^*)||\right]\left[\left|\left|\int_0^1 H'(u^*)^{-1}(H'(u^* + \beta(s_0 - u^*)) - H'(u_0))(s_0 - u^*)\, d\beta\right|\right|\right] \\ &\leq \frac{k_1 \int_0^1 (||u^* + \beta(s_0 - u^*) - u_0||)\, d\beta}{(1 - k_0||u_0 - u^*||)}||s_0 - u^*|| \\ &\leq \left[\frac{k_1\left(||u_0 - u^*|| + \frac{||s_0 - u^*||}{2}\right)}{(1 - k_0||u_0 - u^*||)}\right]||s_0 - u^*|| \\ &\leq \left[\frac{k_1\left(||u_0 - u^*|| + \frac{J_1(||u_0 - u^*||)||u_0 - u^*||}{2}\right)}{(1 - k_0||u_0 - u^*||)}\right]J_1(||u_0 - u^*||)||u_0 - u^*|| \\ &= \left[\frac{k_1(2||u_0 - u^*|| + J_1(||u_0 - u^*||)||u_0 - u^*||)}{2(1 - k_0||u_0 - u^*||)}\right]J_1(||u_0 - u^*||)||u_0 - u^*|| \\ &= \left[\frac{k_1(2 + J_1(||u_0 - u^*||))||u_0 - u^*||}{2(1 - k_0||u_0 - u^*||)}\right]J_1(||u_0 - u^*||)||u_0 - u^*|| \\ &= J_2(||u_0 - u^*||)||u_0 - u^*|| < ||u_0 - u^*|| < R. \end{aligned} \tag{29}$$

Again,

$$||H'(u^*)^{-1}(H'(t_0)-H'(u^*))|| \leq k_0||t_0-u^*|| < k_0 J_2(||u_0-u^*||)||u_0-u^*|| = J_3(||u_0-u^*||) < 1. \tag{30}$$

So, $H'(t_0)^{-1} \in BL(Y, X)$ with

$$||H'(t_0)^{-1}H'(u^*)|| \leq \frac{1}{1 - J_3(||u_0 - u^*||)}. \tag{31}$$

This ensures the existence of u_1. At last, we use the definition of R, (8), (13), (19), (29) and (31) to yield

$$\begin{aligned}
||u_1 - u^*|| &\leq ||s_0 - u^*|| + ||H'(t_0)^{-1}H(s_0)|| \\
&= ||s_0 - u^*|| + ||H'(t_0)^{-1}H'(u^*)|| \; ||H'(u^*)^{-1}H(s_0)|| \\
&\leq J_1(||u_0 - u^*||)||u_0 - u^*|| + \frac{(1 + k_0||s_0 - u^*||)||s_0 - u^*||}{1 - J_3(||u_0 - u^*||)} \\
&\leq J_1(||u_0 - u^*||)||u_0 - u^*|| + \frac{(1 + k_0 J_1(||u_0 - u^*||)||u_0 - u^*||)J_1(||u_0 - u^*||)||u_0 - u^*||}{1 - J_3(||u_0 - u^*||)} \\
&= \left[1 + \frac{(1 + k_0 J_1(||u_0 - u^*||)||u_0 - u^*||)}{1 - J_3(||u_0 - u^*||)}\right] J_1(||u_0 - u^*||)||u_0 - u^*|| \\
&= J_4(||u_0 - u^*||)||u_0 - u^*|| < ||u_0 - u^*|| < R.
\end{aligned} \tag{32}$$

Hence, we get (23) for $n = 0$. The estimates (21)-(23) emerge from the substitution of u_n, s_n, t_n and u_{n+1} instead of u_0, s_0, t_0 and u_1 respectively in the prior estimates. The inequality $||u_{n+1} - u^*|| \leq J_4(R)||u_n - u^*|| < R$ ensures that $u_{n+1} \in B(u^*, R)$ and $\lim_{n\to\infty} u_n = u^*$. Now, we are left with the uniqueness part. If another solution $s^*(\neq u^*)$ of $H(u) = 0$ exists in $B(u^*, \Delta)$, then using $A = \int_0^1 H'(s^* + \beta(u^* - s^*))\, d\beta$ and (15), we arrive at

$$\begin{aligned}
||H'(u^*)^{-1}(A - H'(u^*))|| &\leq \int_0^1 k_0||s^* + \beta(u^* - s^*) - u^*||\, d\beta \\
&\leq \frac{k_0}{2}||u^* - s^*|| \\
&\leq \frac{k_0\Delta}{2} < 1.
\end{aligned}$$

Thus, $A^{-1} \in BL(Y, X)$ for $\Delta < \frac{2}{k_0}$. Now, the identity $0 = H(u^*) - H(s^*) = A(u^* - s^*)$ guarantees that $u^* = s^*$. This finishes the proof. □

3. Basins of attraction

Suppose $P : \mathbb{C} \to \mathbb{C}$ be a complex polynomial with degree greater than or equal to two. Choosing $z_0 \in \mathbb{C}$ as an initial guess, let $\{z_n\}_{n=0}^{\infty}$ be the sequence of successive iterates produced by an iterative algorithm. We say the point z_0 is attracted to z^* if the sequence $\{z_n\}_{n=0}^{\infty}$ converges to the root z^*. The set of all starting points z_0 which are attracted to a zero z^* of the polynomial $P(z)$ is the basin of attraction corresponding to z^*. We consider a region $S \subset \mathbb{C}$ and a grid of 400×400 points covering $S = [-10, 10] \times [-10, 10]$ to produce the basins of attraction of the scheme (4) corresponding to the zeros of a polynomial. In this numerical experiment, is considered and we apply the algorithm (4) treating every $z_0 \in S$ as an initial guess. If the corresponding sequence $\{z_n\}_{n=0}^{\infty}$ converges to a zero of the test function, then we conclude that z_0 is in the attraction basin of that zero and this initial point is painted with a color related to the zero. The point $z_0 \in S$ is painted in black color if the method (4) starting from z_0 does not converge to any zero of the test polynomial. We set $e = 10^{-6}$ as the tolerance error and 1000 as the maximum number of iterations. The test polynomials are chosen from [37, 38]. The region S holds all zeros of considered test functions. We used MATLAB 2019a to produce the figures.

Experiment 1: At first, we show the convergence region for the scheme (4) to obtain the zeros of the polynomial $P_1(z) = z^2 - 1$. In Fig. 1, convergence to the zeros -1 and 1 of $P_1(z)$ is painted in red and green color respectively.

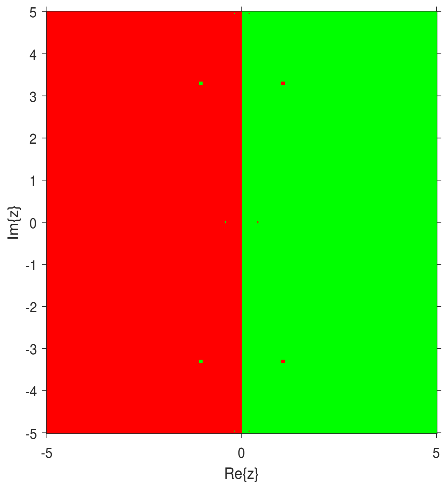

FIGURE 1. Attraction basins related to the zeros of $P_1(z)$

Experiment 2: In Fig. 2, the region of convergence for the scheme (4) to approximate the zeros of the polynomial $P_2(z) = z^3 - 1$ is presented. Convergence to zeros $-0.500000 - 0.866025i$, $-0.500000 + 0.866025i$ and 1 of $P_2(z)$ is painted in green, red and yellow color respectively.

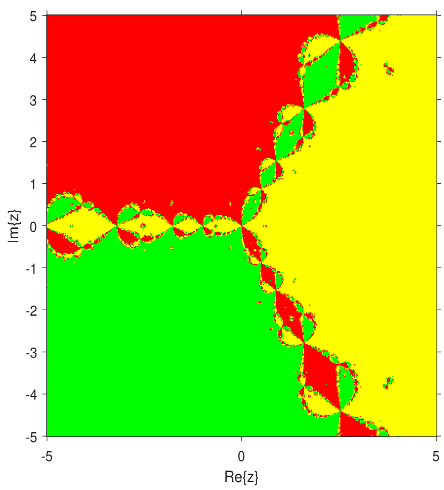

FIGURE 2. Attraction basins related to the zeros of $P_2(z)$

Experiment 3: In Fig. 3, we provide the convergence region for the scheme (4) to obtain the zeros of the polynomial $P_3(z) = z^3 + (0.275 + 1.65i - 1)z + 0.275 + 1.65i$. Convergence to the zeros $0.401440 - 0.915201i$, $-1.401440 + 0.915201i$ and 1 of $P_3(z)$ is painted in green, magenta and yellow color respectively.

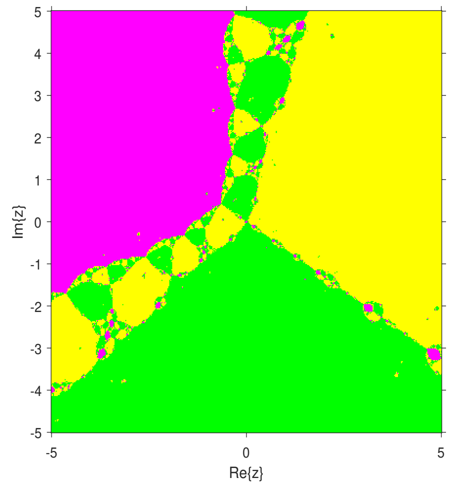

FIGURE 3. Attraction basins related to the zeros of $P_3(z)$

Experiment 4: In Fig. 4, the region of convergence for the scheme (4) to approximate the zeros of the polynomial $P_4(z) = z^3 - 4z^2 + 10$ is shown. Convergence to the zeros $-2.682615 - 0.358259i$, $-2.682615 + 0.358259i$ and 1.365230 of $P_4(z)$ is painted in green, magenta and yellow color respectively.

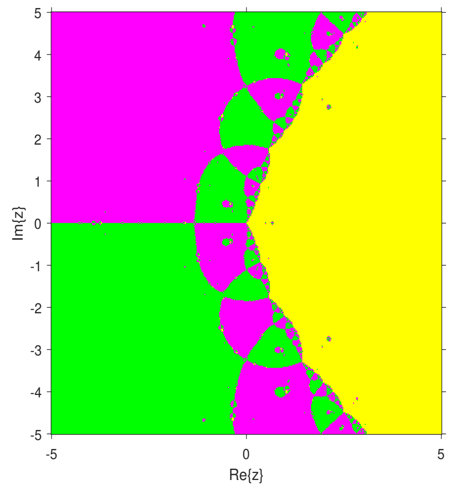

FIGURE 4. Attraction basins related to the zeros of $P_4(z)$

Experiment 5: In Fig. 5, we present the convergence region for the scheme (4) to obtain the zeros of the polynomial $P_5(z) = z^4 - 1$. Convergence to the zeros $-i$, -1, i and 1 of $P_5(z)$ is painted in green, magenta, yellow and red color respectively. Fig. 6 provides a zoomed view of Fig. 5.

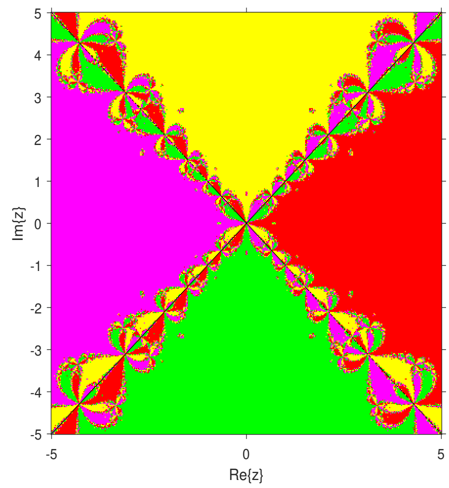

FIGURE 5. Attraction basins related to the zeros of P_5

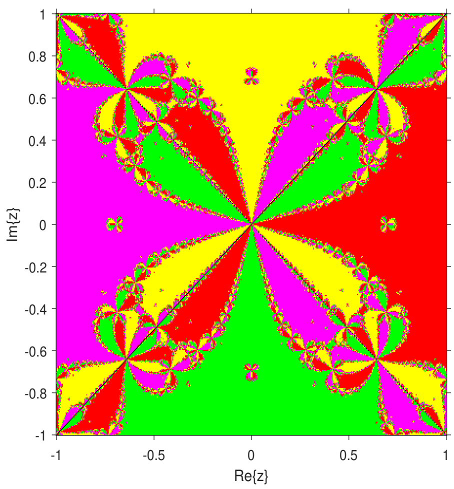

FIGURE 6. Zoomed view of Fig. 5

Experiment 6: In Fig. 7, the region of convergence for the scheme (4) to approximate the zeros of the polynomial $P_6(z) = z^4 - 10z^2 + 9$ is given. Convergence to the zeros -3, -1, 1 and 3 of $P_6(z)$ is painted in green, magenta, yellow and red color respectively.
Experiment 7: In Fig. 8, we show the convergence region for the scheme (4) to obtain the zeros of the polynomial $P_7(z) = z^5 - 1$. Convergence to the zeros $-0.809016 - 0.587785i$, $-0.809016+0.587785i$, $0.309016-0.951056i$, $0.309016+0.951056i$ and 1 of $P_7(z)$ is painted in green, red, yellow, magenta and cyan color respectively. Fig. 9 shows a zoomed view of Fig. 8.
Experiment 8: In Fig. 10, the region of convergence for the scheme (4) to obtain the zeros of the polynomial $P_8(z) = z^5 + 5z^3 - 4z$ is provided. Convergence to the zeros -2, -1, 0 , 1 and 2 of $P_8(z)$ is painted in green, magenta, yellow, blue and red color respectively.
Experiment 9: At last, we present the convergence region for the scheme (4) to obtain the zeros of the polynomial $P_9(z) = z^6 - 0.5z^5 + \frac{11}{4}(1+i)z^4 - \frac{1}{4}(19+3i)z^3 + \frac{1}{4}(11+i)z^2 -$

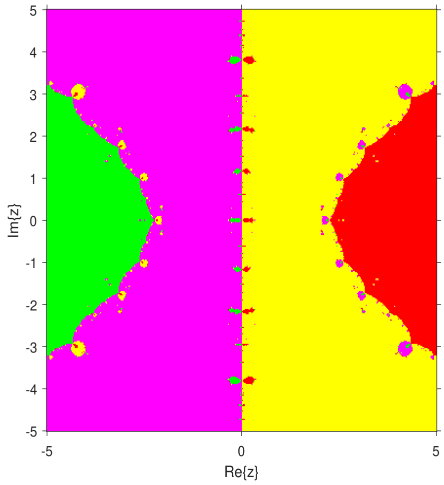

FIGURE 7. Attraction basins related to the zeros of $P_6(z)$

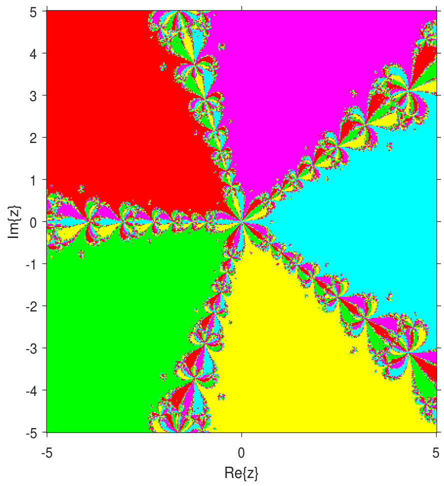

FIGURE 8. Attraction basins related to the zeros of $P_7(z)$

$\frac{1}{4}(19+3i)z+\frac{3}{2}-3i$. In Fig. 11, convergence to the zeros $-\frac{3}{2}i$, $-1+2i$, $-\frac{1}{2}-\frac{i}{2}$, i, 1 and $1-i$ of $P_9(z)$ is painted in green, magenta, yellow, blue, red and orange color respectively. Fig. 12 presents a zoomed view of Fig. 11.

4. Numerical Examples

In this section, we compute the radii of convergence balls of the method (4) for standard numerical problems. Also, we compare the convergence radii with that of the fourth and

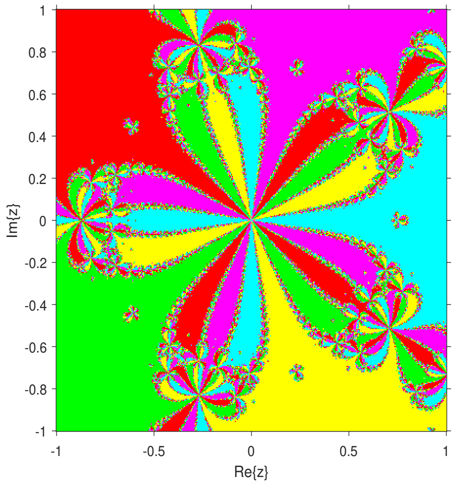

FIGURE 9. Zoomed view of Fig. 7

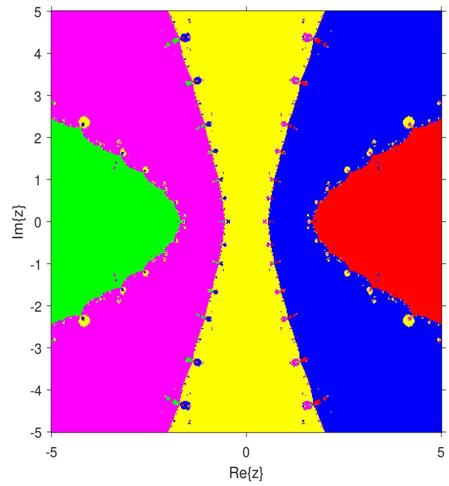

FIGURE 10. Attraction basins related to the zeros of $P_8(z)$

fifth order convergent family of methods,

$$\begin{aligned} s_n &= u_n - aH'(u_n)^{-1}H(u_n),\ a \in (-\infty, \infty) \setminus \{0\} \\ t_n &= s_n - H'(u_n)^{-1}H(s_n) \\ u_{n+1} &= t_n - \left(\frac{1}{a}H'(s_n)^{-1} + \left(1 - \frac{1}{a}\right)H'(u_n)^{-1}\right)H(t_n), \end{aligned} \tag{33}$$

obtained by the technique of Singh et al. discussed in [20]. We obtain larger domain of convergence for the method (4) using our technique in all cases.

Example 1 [20]: Let H is defined for $(u_1, u_2, u_3)^t$ on $\bar{B}(0,1)$ by

$$H(u) = (e^{u_1} - 1, \frac{e-1}{2}u_2^2 + u_2, u_3)^t$$

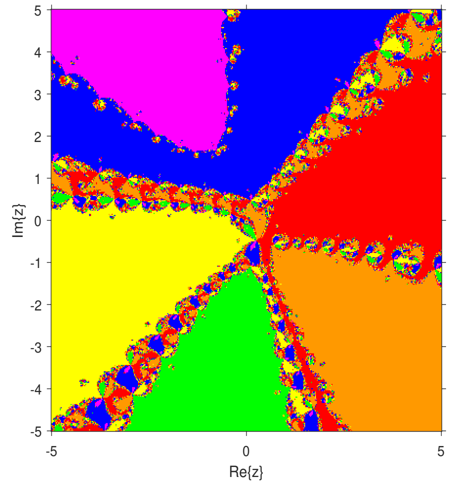

FIGURE 11. Attraction basins related to the roots of $P_9(z)$

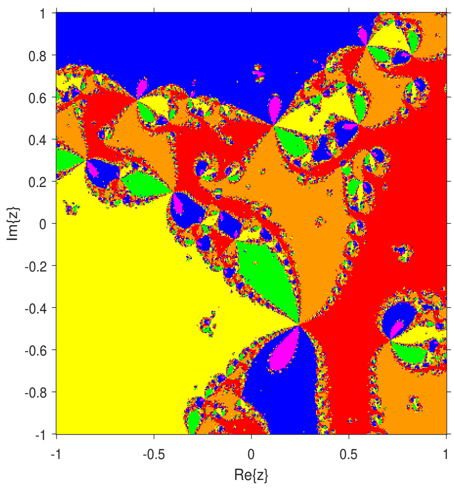

FIGURE 12. Zoomed view of Fig. 9

We have $u^* = (0,0,0)^t$. Also, we have $k_0 = e - 1$ and $k_1 = e$. The radius R is computed using "J" functions.

Table 1: Parameters for example 1

Method (4)				*Family of methods (33)* [20]
θ_1	θ_2	θ_4	R	ρ
0.324947	0.255270	0.203279	0.203279	0.118532

Example 2 [20]: Let us define H on $\Omega = [-1, 1]$ by

$$H(u) = \sin(u)$$

We have $u^* = 0$. Also, we have $k_0 = k_1 = 1$. R is calculated using "J" functions.

Table 2: Parameters for example 2

Method (4)				*Family of methods (33)* [20]
θ_1	θ_2	θ_4	R	ρ
0.666667	0.552789	0.449859	0.449859	0.272419

Example 3 [21]: Consider the nonlinear Hammerstein type integral equation given by

$$H(u)(x) = u(x) - 5\int_0^1 x\ y\ u(y)^3\ dy,$$

where $u(x) \in C[0,1]$. We have $u^* = 0$. Also, $k_0 = 7.5$ and $k_1 = 15$. "J" functions are used to find the radius R.

Table 3: Parameters for example 3

Method (4)				*Family of methods (33)* [20]
θ_1	θ_2	θ_4	R	ρ
0.066667	0.050929	0.040269	0.040269	0.022502

Example 4 [20]: We consider the motivational problem mentioned in the first section. We have $u^* = 1$. Also, $k_0 = k_1 = 96.6628$. The radius R is obtained from the "J" functions.

Table 4: Parameters for example 4

Method (4)				*Family of methods (33)* [20]
θ_1	θ_2	θ_4	R	ρ
0.006896	0.005719	0.004654	0.004654	0.002818

We ensure the convergence of the scheme (4) with radius $R = 0.004654$.

5. Conclusions

Local convergence analysis of the higher-order convergent scheme (4) is studied. We used the only assumption that the first-order Fréchet derivative is Lipschitz continuous to expand the application of the algorithm. The analysis presented in this study is applicable for solving such problems for which earlier analysis are not useful. Also, the region of convergence for the the method (4) to find the zeros of different polynomials is presented by means of basins of attraction. At last, various examples like a nonlinear system and the Hammerstein integral equation are solved to show the convergence of the algorithm.

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