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NONSTANDARD FINITE DIFFERENCE SCHEMES WITH APPLICATION TO BIOLOGICAL MODELS

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Abstract. This paper deals with the construction of nonstandard finite difference methods for solving a specific Rosenzweig-MacArthur predator-prey model. The reorganization of the denominator of the discrete derivatives and nonlocal approximations of nonlinear terms are used in the design of new schemes. We establish that the proposed nonstandard finite difference methods are elementary stable and satisfy the positivity requirement. We provide some numerical comparisons to illustrate our results.

Keywords: Predator–prey, Positivity, Elementary stability, Nonstandard finite difference.

AMS Subject Classification: 37M05, 39A11, 65L12, 65L20.

1. Introduction

Ordinary differential equations (ODEs) are used extensively in the modeling of many biological and physical applications. They constitute a central component in applied mathematics. Their numerical simulations are fundamental importance in gaining the correct qualitative and quantitative information on the systems. Numerical methods based on the finite difference approximations, Taylor series expansion, and interpolation, such as Euler, Runge-Kutta and Adams methods are widely used (see, for example [14]). Traditionally, important requirements in this context are, the investigation of the consistency of the discrete schemes with the original differential equation and linear stability analysis for problems with smooth solutions. These requirements are important, because they guarantee convergence of the discrete solution to the exact one, but the essential qualitative properties of the solution are not transferred to the numerical solution. On the other hand, several biological and physical problems involve the presence of variables that satisfy positivity constraints. For instance, when the variables are the population density in mathematical biology, it is a natural demand that the resulting numerical approximations should be non-negative. Furthermore, a negative value may cause undershoots near a steep gradient. Therefore, we need to analysis numerical methods from the point of view of positivity (preservation of nonnegativity)[13, 15, 16, 17, 18, 19, 20, 21, 25, 26]. One of the

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system of ODEs that satisfies the condition of positivity of its solutions is the well-known predator-prey model from mathematical biology. This system of differential equations was introduced by Rosenzweig-MacArthur in order to understanding the population dynamics of analyzed biological systems. There are many different kind of predator-prey models in the mathematical biology literature including continuous and discrete models. Several works have been devoted to investigate these models regarding periodicity, global stability boundedness and others features [7, 8, 11, 23]. The aim of this manuscript is to introduce two finite-difference schemes to approximate consistent solutions of the predator-prey models, which is an equation for which the existence of the non-negative solution is a well-known fact [6, 7, 8, 9]. Proposed methods are elementary stable and satisfy the positivity requirement. The remainder of this paper is organized as follows:
In Section 2, we give some preliminaries and definitions including nonstandard finite difference methods for ODEs, elementary stability and Rosenzweig-MacArthur predator-prey model. In Section 3, we propose new NSFD schemes and then we present the elementary stability conditions and positivity requirements. Furthermore, to illustrate the advantages of the new schemes we compare them with the results obtained from the second order Rung-Kutta (RK2) method, elementary stable nonstandard (ESN) Euler and explicit Euler methods. Finally, we end the paper with some conclusions in Section 4.

2. Preliminaries and definitions

We now give a brief summery of the nonstandard finite difference (NSFD) methods for the numerical solution of the following initial value problem

$$\frac{d}{dt}U(t) = F(U(t)), \ \ (t \geq 0), \qquad U(0) = U_0, \tag{1}$$

where $U(t)$ mapping $[t_0, T) \to \mathcal{C}^k$ and the corresponding F mapping $([t_0, T), \mathcal{C}^k) \to ([t_0, T), \mathcal{C}^k)$. Discretization of the differential equation with $t_n = t_0 + n\Delta t$, where Δt is a positive step size. So, we get

$$U_n \approx U(t_n), \tag{2}$$

and for the Eq. (1) we obtain

$$\mathcal{D}_{\Delta t} U_n = \mathcal{F}_n(F, U_n), \tag{3}$$

where $\mathcal{D}_{\Delta t} U_n$ discretization of $\frac{d}{dt}U(t)$ and $\mathcal{F}_n(F, U_n)$ approximation of $F(U(t))$ in t_n. We define the nonstandard one-step finite-difference method based on a definition given by Anguelov and Lubuma [1, 2, 3, 4].

Definition 2.1. *Method* (2) *is called a nonstandard finite-difference method if at least one of the following conditions is met:*

- *In the discrete derivatives $\mathcal{D}_{\Delta t} U_n$ the traditional denominator Δt is replaced by a nonnegative function $\varphi(\Delta t)$ such that*

$$\varphi(\Delta t) = \Delta t + O(\Delta t^2) \ \ as \ \ 0 < \Delta t \to 0, \tag{4}$$

- *Nonlinear terms in $F(U(t))$ are approximated in a nonlocal way, i.e. by a suitable function of several points of the mesh. For instance, the non-linear terms U^2 and U^3 can be modelled as follows as in Anguelov and Lubuma* [2]*:*

$$\begin{aligned} U &\approx aU_k + (1-a)U_{k+1}, && a \in \mathbb{R} \\ U^2 &\approx aU_k^2 + bU_kU_{k+1}, && a, b \in \mathbb{R} \ and \ a + b = 1, \\ U^3 &\approx aU_k^3 + (1-a)U_k^2U_{k+1}, && a \in \mathbb{R}. \end{aligned}$$

Definition 2.2. *Any constant-vector $\tilde{U}$ satisfying*

$$F(\tilde{U}) = 0,$$

is called equilibrium point (fixed-point or critical point) of the differential equation in (1).

Definition 2.3. *The finite difference method*

$$\mathcal{D}_{\Delta t} U_n = \mathcal{F}_n(F, U_n),$$

is called elementary stable, if for any value of the step size Δt, its only equilibrium point $\tilde{U}$ are those of the differential system (1). *The linear stability properties of each $\tilde{U}$ being the same for both the differential system and the discrete method.*

Lemma 2.1. *For the quadratic equation $\lambda^2 + \alpha\lambda + \beta = 0$ by using the well-known Jury condition* [10, 24] *both roots satisfy $|\lambda_i| < 1, i = 1, 2$ iff the following conditions are satisfied:*

- $1 - \alpha + \beta > 0$,
- $1 + \alpha + \beta > 0$, *and*
- $\beta < 1$.

Predator-prey systems are among the most discussed and analyzed topics in mathematical biology. Their relatively simple form as a system of two differential equations allows for detailed understanding of their underlying behavior, even though explicit solutions are not available in a closed form. The general Rosenzweig–MacArthur predator-prey model [5, 9, 27] with a logistic intrinsic growth of the prey population has the following form:

$$\begin{aligned} \frac{dx}{dt} &= bx(1-x) - ag(x)xy, \qquad && x(t_0) = x_0 \geq 0, \\ \frac{dy}{dt} &= g(x)xy - dy, && y(t_0) = y_0 \geq 0, \end{aligned} \tag{5}$$

where x and y represent the prey and predator population sizes, respectively, $b > 0$ represents the intrinsic growth rate of the prey, $a > 0$ stands for the capturing rate and $d > 0$ is the predator death rate. In (5) it is reasonable to assume

$$g(x) \geq 0, \qquad g'(x) \leq 0, \qquad [xg(x)]' \leq 0, \tag{6}$$

and that functional response $xg(x)$ is bounded as $x \to \infty$. From [2, 9] system (1) have the following equilibria:

1: $E_0 = (0, 0)$,
2: $E_1 = (1, 0)$ and
3: $E^* = (x^*, y^*)$ where x^* is the solution of $x^*g(x^*) = d$ and $y^* = \frac{bx^*(1-x^*)}{ad}$. The equilibrium E^* exists if and only if $g(1) > d$.

The equilibrium $(0, 0)$ is always linearly unstable. The equilibrium $(1, 0)$ is linearly stable if $g(1) < d$ and linearly unstable if $g(1) > d$. Finally, the equilibrium (x^*, y^*) is linearly stable if $b + ay^*g'(x^*) > 0$ and linearly unstable if $b + ay^*g'(x^*) < 0$.

3. Construction of new schemes

In this section, our main aim is apply Mickens rules [22] to construct two positive and elementary stable nonstandard (PESN) schemes for solving the system (1).

Scheme 1. Our first proposed scheme is

$$
\begin{aligned}
\frac{x_{k+1}-x_k}{\varphi(h)} &= 2bx_k - (b + bx_k + ag(x_k)y_k)x_{k+1},\\
\frac{y_{k+1}-y_k}{\varphi(h)} &= (d + g(x_k)x_k)y_k - 2dy_{k+1},
\end{aligned}
\tag{7}
$$

where

$$
\varphi(h) = \frac{\phi(hq)}{q} < \frac{1}{q}, \qquad 0 < \phi(h) < 1.
$$

The explicit form of the scheme (7) can be written as follow:

$$
\begin{aligned}
x_{k+1} &= \frac{(1+2b\varphi(h))x_k}{1 + b\varphi(h)(1+x_k) + a\varphi(h)g(x_k)y_k},\\
y_{k+1} &= \frac{(1+\varphi(h)(d + g(x_k)x_k))y_k}{1+2d\varphi(h)}.
\end{aligned}
\tag{8}
$$

Theorem 3.1. *The sufficient condition for the scheme* (7) *to be PESN is*

$$
\varphi(h) < \frac{b + ayg'(x^*)}{bg(x^*)(1-3x^*) - hbx^*(1-x^*)g'(x^*)}.
$$

Proof. Since the constants a, b, d and the function g are positive then the system (8) is unconditionally positive and its equilibrium points are exactly the same equilibria E_0, E_1 and E^* of system (5). The Jacobian matrix of the scheme (8) has the following form:

$J(\bar{x}, \bar{y}) =$

$$
\begin{pmatrix}
\dfrac{(1+2b\varphi(h))\,(1 + b\varphi(h) + a\varphi(h)\bar{y}(g(\bar{x}) - g'(\bar{x})\bar{x}))}{(1 + b\varphi(h)(1+\bar{x}) + \varphi(h)a\bar{y}g(\bar{x}))^2} & -\dfrac{(1+2b\varphi(h))a\varphi(h)\bar{x}g(\bar{x})}{(1 + b\varphi(h)(1+\bar{x}) + \varphi(h)a\bar{y}g(\bar{x}))^2} \\[3ex]
\dfrac{\varphi(h)\bar{y}(g(\bar{x}) + \bar{x}g'(\bar{x}))}{1+2d\varphi(h)} & \dfrac{1+\varphi(h)(d + \bar{x}g(\bar{x}))}{1+2d\varphi(h)}
\end{pmatrix}.
$$

Substituting E_0 in J we derive

$$
J(0,0) = \begin{pmatrix}
\dfrac{1+2b\varphi(h)}{1+b\varphi(h)} & 0 \\[2ex]
0 & \dfrac{1+d\varphi(h)}{1+2d\varphi(h)}
\end{pmatrix},
$$

from which

$$
\lambda_1 = \frac{1+2b\varphi(h)}{1+b\varphi(h)} \quad \text{and} \quad \lambda_2 = \frac{1+d\varphi(h)}{1+2d\varphi(h)},
$$

since $|\lambda_1| > 1$ therefore, the equilibrium point E_0 is unstable. Now, by substituting E_1 in the Jacobian we have

$$
J(1,0) = \begin{pmatrix}
\dfrac{1+b\varphi(h)}{1+2b\varphi(h)} & -\dfrac{\varphi(h)ag(1)}{1+2b\varphi(h)} \\[3ex]
0 & \dfrac{1+\varphi(h)(d+g(1))}{1+2d\varphi(h)}
\end{pmatrix},
$$

the eigenvalues of $J(1,0)$ are

$$
\lambda_1 = \frac{1+b\varphi(h)}{1+2b\varphi(h)} \quad \text{and} \quad \lambda_2 = \frac{1+\varphi(h)(d+g(1))}{1+2d\varphi(h)}.
$$

If the equilibrium E_1 is stable then $g(1) < d$ and therefore $|\lambda_1| < 1$ and $|\lambda_2| < 1$. Thus E_1 is a stable fixed point of scheme (8).
For the equilibrium $E^* = (x^*, y^*)$, the Jacobian matrix is as follow:

$$J(x^*, y^*) = \begin{pmatrix} 1 - \dfrac{x^*\varphi(h)\left(g(x^*) + ay^*g'(x^*)\right)}{1+2b\varphi(h)} & \dfrac{-ad\varphi(h)}{1+2b\varphi(h)} \\ \\ \dfrac{\varphi(h)y^*\left(g(x^*) + x^*g'(x^*)\right)}{1+2d\varphi(h)} & 1 \end{pmatrix},$$

the eigenvalues of $J(x^*, y^*)$ are roots of the quadratic equation $\lambda^2 - \alpha\lambda + \beta = 0$ where

$$\alpha = C + 1, \quad \beta = C + AB,$$

with

$$A = \frac{ad\varphi(h)}{1+2b\varphi(h)}, \quad B = \frac{\varphi(h)y^*\left(g(x^*) + x^*g'(x^*)\right)}{1+2d\varphi(h)},$$
$$C = \frac{1 + b\varphi(h)(2 - x^*) - a\varphi(h)x^*y^*g'(x^*)}{1+2b\varphi(h)}.$$

The equilibrium (x^*, y^*) is stable iff all conditions of Lemma 2.1 hold and is unstable if at least one of the conditions fails. The first condition of Lemma 2.1 is as follow:

$$1 - \alpha + \beta = AB.$$

Since

$$A = \frac{ad\varphi(h)}{1+2b\varphi(h)} > 0, \quad B = \frac{\varphi(h)y^*\left(g(x^*) + x^*g'(x^*)\right)}{1+2d\varphi(h)} > 0,$$

then $AB > 0$. The first condition of Lemma 2.1 is always true. In addition, since $0 < x^* < 1$ and $g'(x^*) < 0$, then $C > 0$ and we derive $1 + \alpha + \beta > 0$. Finally, the condition $\beta < 1$ is satisfied, provided that

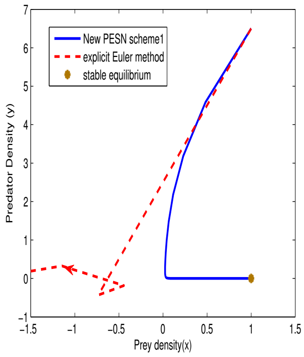
$$\varphi(h)(bg(x^*)(1 - 3x^*) - hx^*b(1 - x^*)g'(x^*)) < b + ayg'(x^*). \tag{9}$$

If $x^* \geq \frac{1}{3}$, there is nothing to prove and (9) is hold, then $\beta < 1$. Next, if $x^* < \frac{1}{3}$, a way to ensure (9) is to demand

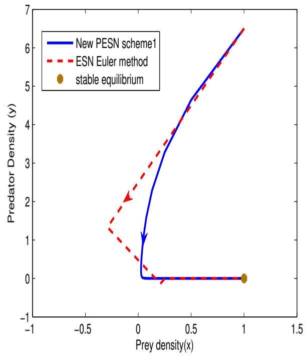
$$\varphi(h) < \frac{b + ayg'(x^*)}{bg(x^*)(1 - 3x^*) - hx^*b(1 - x^*)g'(x^*)}, \tag{10}$$

Hence, the equilibrium point E^* is stable. These results guarantee dynamical consistency between the system (5) and the numerical scheme (8) around all equilibria. Therefore, the new proposed scheme is elementary stable. □

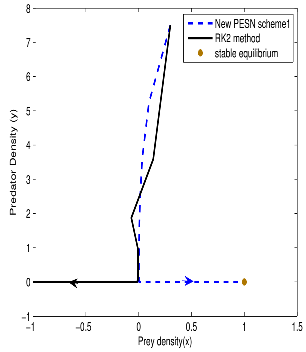
The scheme developed above was tested for system (5) with a Holling-type II predator functional response of the form $xg(x) = \dfrac{x}{c+x}$, corresponding parameters $a = 2$, $b = 1$, $c = 0.5$, $d = 6$ and $\varphi(h) = \frac{1-e^{-hq}}{q}$ [9, 12]. The graphical representations below show the positivity property of the new scheme while the approximations obtained by the explicit Euler method, RK2 method and ESN Euler method are not positive, see Figure 1((a),(b),(c)). Furthermore, the explicit Euler method and RK2 method diverge, Figure 1((a),(c)). Next, we examine system (5) in the case that the parameters are $a = 2$, $b = 1$, $c = 1$ and $d = 0.2$. It can be easily observed that new scheme consistent with the positive behavior of the system (5) and it is linearly stable and has only the same equilibrium point as the (5). Furthermore, unlike the RK2 method the new scheme is not sensitivity to changes in step size h as well as the initial values, see Figure 2.



(a) $h = 0.2$, $x_0 = 1$, $y_0 = 6.5$

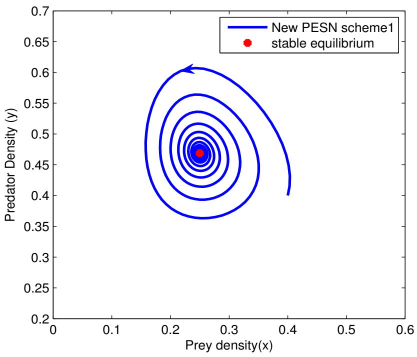


(b) $h = 0.2$, $x_0 = 1$, $y_0 = 26.5$

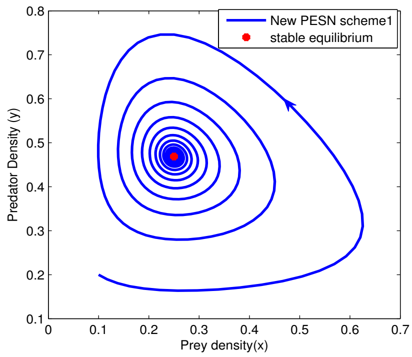


(c) $h = 0.2$, $x_0 = 0.3$, $y_0 = 7.5$

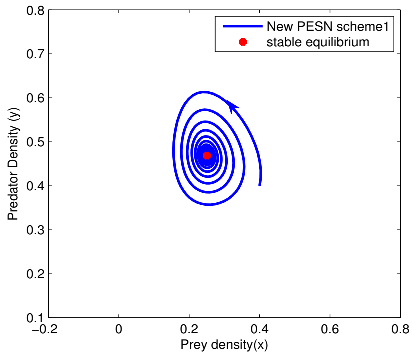
FIGURE 1. Numerical results obtained by the new method (for $q = 1.4$), ESN method (for $q = 3.1$), explicit Euler method and RK2 with $a = 2$, $b = 1$, $c = 0.5$ and $d = 6$.



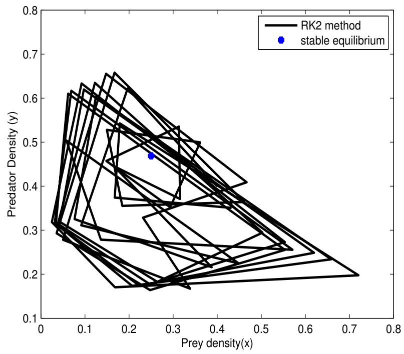
(a) $h = 1.3$, $x_0 = 0.4$, $y_0 = 0.4$



(b) $h = 2.1$, $x_0 = 0.1$, $y_0 = 0.2$



(c) $h = 4.6$, $x_0 = 0.4$, $y_0 = 0.4$



(d) $h = 4.6$, $x_0 = 0.4$, $y_0 = 0.4$

FIGURE 2. Numerical results with $a = 2$, $b = 1$, $c = 1$ and $d = 0.2$ and $q = 1.4$.

Scheme 2. We construct our second new NSFD scheme as follow:

$$\begin{aligned} \frac{x_{k+1} - x_k}{h} &= bx_k(1 - x_{k+1}) - ag(x_k)x_{k+1}y_{k+1}, \\ \frac{y_{k+1} - y_k}{h} &= g(x_k)x_{k+1}y_{k+1} - dy_{k+1}. \end{aligned} \tag{11}$$

However, before solving the system (5), we are going to make some remarks about implementation of (11). Suppose that the following predictor-corrector iteration are being used to solve the nonlinear equations (5).

$$x^c_{k+1} = \frac{(1+hb)x_k}{1+hbx_k+ahg(x_k)y^p_{k+1}}, \tag{12}$$

$$y^c_{k+1} = \frac{y_k + hg(x_k)x^c_{k+1}y^p_{k+1}}{1+hd}. \tag{13}$$

More precisely, suppose that $x_1 = x(0)$ and $y_1 = y(0)$, which is the initial value of (5). By using an explicit method

$$x^p_{k+1} = \frac{(1+hb)x_k}{1+hbx_k+ahg(x_k)y_k}, \tag{14}$$

$$y^p_{k+1} = \frac{(1+hx_kg(x_k))y_k}{1+hd}, \tag{15}$$

we make an initial guess for x_{k+1} and y_{k+1} [9]. These values are substituted in to (12). So, we can obtain an improved approximation x_{k+1}. This value together with y^p_{k+1} are substituted in (13) to get improved y_{k+1}. Then, the process will continue.

Theorem 3.2. *The scheme* (12)–(15) *is unconditionally PESN.*

Proof. Since the constants a, b, d and the function g are positive then the system (12)-(15) is unconditionally positive and its equilibrium points are exactly the same equilibria E_0 and E_1 of the system (5). To study the elementary stability of scheme (12), we consider the following expressions:

$$\begin{aligned} F_2(x_k, y_k) &= \frac{(1+hb)x_k}{1+hbx_k+ahg(x_k)G_1(x_k,y_k)}, \\ G_2(x_k, y_k) &= \frac{y_k + hg(x_k)F_2(x_k,y_k)G_1(x_k,y_k)}{1+hd}, \end{aligned} \tag{16}$$

where

$$F_1(x_k, y_k) = x^p_{k+1}, \qquad G_1(x_k, y_k) = y^p_{k+1}.$$

As in the proof of the previous theorem, the Jacobian matrix J of the scheme (16) is $J(x_k, y_k) = [j_{mn}(x_k, y_k)]_{2\times 2}$, where

$$
\begin{aligned}
j_{11}(x_k,y_k) &= \frac{1+bh}{1+bhx_k+ahg(x_k)G_1(x_k,y_k)} \\
&\quad - \frac{hx_k\left(b+ag'(x_k)G_1(x_k,y_k)+ag(x_k)\frac{\partial G_1}{\partial x}(x_k,y_k)(1+bh)\right)}{(1+bhx_k+ahg(x_k)G_1(x_k,y_k))^2}, \\
j_{12}(x_k,y_k) &= -\frac{hx_k\left(ag(x_k)\frac{\partial G_1}{\partial y}(x_k,y_k)(1+bh)\right)}{(1+bhx_k+ahg(x_k)G_1(x_k,y_k))^2}, \\
j_{21}(x_k,y_k) &= \frac{h}{1+hd}\Bigg(g'(x_k)F_2(x_k,y_k)G_1(x_k,y_k) \\
&\quad + g(x_k)\left(G_1(x_k,y_k)\frac{\partial F_2}{\partial x}(x_k,y_k)+F_2(x_k,y_k)\frac{\partial G_1}{\partial x}(x_k,y_k)\right)\Bigg), \\
j_{22}(x_k,y_k) &= \frac{1+hg(x_k)G_1(x_k,y_k)\frac{\partial F_2}{\partial y}(x_k,y_k)+hg(x_k)F_2(x_k,y_k)\frac{\partial G_1}{\partial y}(x_k,y_k)}{1+hd}.
\end{aligned}
$$

Substituting E_0 in $J(x_k, y_k)$ we derive

$$
J(0,0) = \begin{pmatrix} 1+hb & 0 \\ 0 & \dfrac{1}{1+hd} \end{pmatrix},
$$

from which

$$
\lambda_1 = 1+hb, \quad \lambda_2 = \frac{1}{1+hd}.
$$

Since $|\lambda_1| > 1$ therefore, the equilibrium point E_0 is always unstable. The Jacobian matrix is

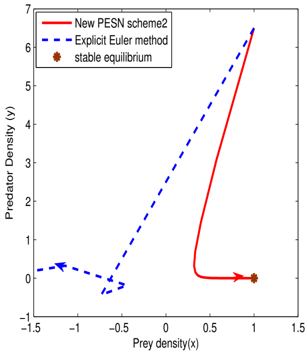
$$
J(1,0) = \begin{pmatrix} \dfrac{1}{1+hb} & -\dfrac{ahg(1)(1+g(1))}{(1+hb)(1+hd)} \\ 0 & \dfrac{1+hd+hg(1)+h^2g^2(1)}{1+2hd+h^2d^2} \end{pmatrix},
$$

and the eigenvalues of the matrix $J(1,0)$

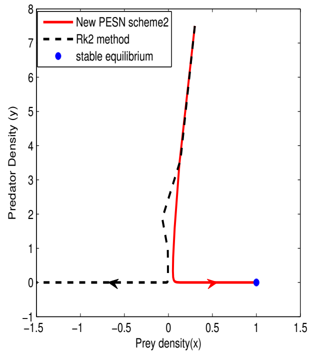
$$
\lambda_1 = \frac{1}{1+hb} \quad \text{and} \quad \lambda_2 = \frac{1+hd+hg(1)+h^2g^2(1)}{1+2hd+h^2d^2}.
$$

If the equilibrium E_1 of the system (5) is stable then $g(1) < d$ and therefore $|\lambda_1| < 1$ and $|\lambda_2| < 1$. Thus E_1 is an stable equilibrium point of scheme (16). Therefore, these results guarantee a dynamical consistency between the system (5) and the numerical scheme (16) around the equilibrium points E_0 and E_1. Hence, the new proposed scheme is PESN. □

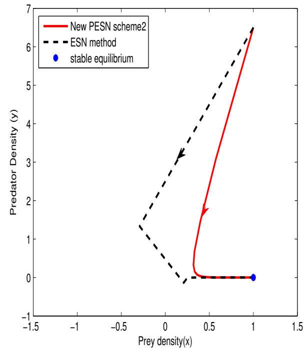
Figure 3, shows that the numerical results obtained from the new scheme, RK2, explicit Euler and ESN Euler methods for solving (5) with the same parameters used in the previous scheme. We observe that the new scheme preserve the stability of the equilibrium $(1, 0)$ and positivity of the solution, while the explicit Euler method and RK2 approximations to evolve away from the equilibrium $(1, 0)$ and produce the negative value, see Figure 3((a),(b)). Furthermore, approximation of the solution obtained by ESN method is negative, Figure 3(c).



(a) $h = 0.2$, $x_0 = 1$, $y_0 = 6.5$



(b) $h = 0.2$, $x_0 = 1$, $y_0 = 6.5$



(c) $h = 0.2$, $x_0 = 0.3$, $y_0 = 7.5$

FIGURE 3. Numerical results obtained by the new method (for $q = 1.4$), ESN method (for $q = 3.1$), explicit Euler method and RK2 with $a = 2$, $b = 1$, $c = 0.5$ and $d = 6$.

4. Conclusion

In this manuscript, we applied the nonstandard discretization methods to solve numerically the Rosenzweig-MacArthur predator-prey model with a Holling-type II functional response which has a finite number of hyperbolic equilibria. The new proposed schemes preserve the stability of all equilibria and the positivity of all solutions. Comparing with the standard numerical methods e.g. the explicit Euler method and the Runge-Kutta method, we do feel that our results indicate that a properly implemented version of our scheme should be useful for the numerical integration of mentioned predator-prey model. Our interest for future works is to construct more nonstandard schemes for the general case of biological systems.

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References

[1] Anguelov, R., Dumont, Y. and Lubuma, J., (2012), On Nonstandard Finite Difference Schemes in Biosciences, AIP Conf. Proc. 1487(1), pp. 212–223.
[2] Anguelov, R., and Lubuma, J. M. -S., (2001), Contributions to the mathematics of the nonstandard finite difference method and applications, Numer. Methods. Partial. Differ. Equ. 17 pp. 518–543.
[3] Anguelov, R. and Lubuma, J. M.-S., (2003), Nonstandard finite difference method by nonlocal approximation, Math. Comput. Simul., 61(36), pp. 465–475.
[4] Anguelov, R., Kama, P. and Lubuma, J. M.-S., (2005), On non-standard finite difference models of reaction-diffusion equations, J. Comput. Appl. Math., 175(1), pp. 11–29.
[5] Brauer, F. and Castillo-Chavez, C., (2001), Mathematical Models in Population Biology and Epidemiology, Springer, New York.
[6] Dimitrov, D. T. and Kojouharov, H. V., (2007), Nonstandard numerical methods for a class of predator-prey models with predator interference, Electron. J. Differ. Equ., 15, pp. 67–75.
[7] Dimitrov, D. T. and Kojouharov, H. V., (2008), Nonstandard finite difference methods for predator-prey mpdels with general functional response, Math. Comput. Simu., 78, pp. 1–11.
[8] Dimitrov, D. T. and Kojouharov, H. V., (2007), Stability-Preserving finite-difference methods for general multi-dimensional autonomous dynamical system, Int. J. Numer. Anal. Model., 4(2), pp. 282–292.
[9] Dimitrov, D. T. and Kojouharov, H. V., (2006), Positive and elementary stable nonstandard numerical methods with applications to predator-prey models, J. Comput. Appl. Math., 189, pp. 98–108.

[10] Dumont, Y., Chiroleu, F. and Domerg, C., (2008), On a temporal model for the Chikungunya disease: modeling theory and numerics, Math. Biosci., 213, pp. 80–91.
[11] Gonzalez-Parra, G., Arenas, A. J. and Cogollo, M. R., (2013), Numerical-analytical solutions of predator-prey models, WSEAS. Trans. Biol. Biomed., 10(2), pp. 79–87.
[12] Holling, C. S., (1965), The functional response of predators to prey density and its role inmimicry and population regulation, Mem. Entomol. Soc. Canada, 45, pp. 1–60.
[13] Horvath, Z., (1998), Positivity of Runge-Kutta and diagonally split Runge-Kutta methods, Appl. Numer. Math., 28(2-4), pp. 309–326.
[14] Lambert, J. D., (1991), Numerical methods for ordinary differential systems, John Wiley Sons, New York.
[15] Mehdizadeh Khalsaraei, M., (2010), An improvement on the positivity results for 2-stage explicit Runge-Kutta methods, J. Comput. Appl. Math., 235, pp. 137–143.
[16] Mehdizadeh Khalsaraei, M. and Shokri Jahandizi, R., (2014), Positivity-preserving nonstandard finite difference schemes for simulation of advection-diffusion reaction equations, Comput. Methods. Differ. Equ., 2(4), pp. 256–267.
[17] Mehdizadeh Khalsaraei, M., (2015), Positivity of an explicit Runge-Kutta method, Ain. Shams. Eng. J., 6(4), pp. 1217–1223.
[18] Mehdizadeh Khalsaraei, M. and Khodadoosti, F., (2016), Qualitatively stability of nonstandard 2-stage explicit Runge-Kutta methods of order two, Comp. Math. Math. Phys., 56(2), pp. 235–242.
[19] Mehdizadeh Khalsaraei, M. and Shokri Jahandizi, R., (2017), Efficient explicit nonstandard finite difference scheme with positivity- preserving property, GU J Sci., 30(1), pp. 259–268.
[20] Mehdizadeh Khalsaraei, M., Heydari, Sh., Davari Algoo, L. and Shokri Jahandizi, R., (2017), Positivity preserving and elementary stable nonstandard finite difference scheme for the predator-prey model, *Int. J. Math. Comput.*, 28(1), pp. 48–57.
[21] Mehdizadeh Khalsaraei, M., (2017), Nonstandard explicit third-order Runge-Kutta method with positivity property, *Int. J. Nonlinear Anal. Appl.*, 8(2), pp. 37–46.
[22] Mickens, R. E., (1994), Nonstandard Finite Difference Models of Differential Equations, World Scientific, Singapore.
[23] Mickens, R. E., (2003), A nonstandard finite-difference scheme for the Lotka-Volterra system, Appl. Numer. Math., 45, pp. 309–314.
[24] Murry, J. D., (1989), Mathematical Biology, Springer-Verlag.
[25] Nemati Saray, B. and Shahriari, M. (2016), Numerical solution of linear control systems using interpolation scaling functions, Comput. Methods Differ. Equ., 4(2), pp. 139–150.
[26] Pham Thi, N. N., Hundsdorfer, W. and Sommeijer, B. P., (2006), Positivity for explicit two-step methods in linear multistep and one-leg form, *BIT.*, 46, pp. 875–882.
[27] Wood, D. T., Kojouharov, H. V. and Dimitrov, D. T., (2017), Universal approaches to approximate biological systems with nonstandard finite difference methods, Math. Comput. Simul., 133, pp. 337-350.

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