TWMS J. App. and Eng. Math. V.12, N.1, 2022, pp. 357-366

SIGNED TOTAL DOUBLE ROMAN DOMINATION NUMBERS IN DIGRAPHS

J. AMJADI[1], F. POUR HOSSEINI[1], §

ABSTRACT. Let $D = (V, A)$ be a finite simple digraph. A signed total double Roman dominating function (STDRD-function) on the digraph D is a function $f : V(D) \to \{-1, 1, 2, 3\}$ satisfying the following conditions: (i) $\sum_{x \in N^-(v)} f(x) \geq 1$ for each $v \in V(D)$, where $N^-(v)$ consist of all in-neighbors of v, and (ii) if $f(v) = -1$, then the vertex v must have at least two in-neighbors assigned 2 under f or one in-neighbor assigned 3 under f, while if $f(v) = 1$, then the vertex v must have at least one in-neighbor assigned 2 or 3 under f. The weight of a STDRD-function f is the value $\sum_{x \in V(D)} f(x)$. The signed total double Roman domination number (STDRD-number) $\gamma^t_{sdR}(D)$ of a digraph D is the minimum weight of a STDRD-function on D. In this paper we study the STDRD-number of digraphs, and we present lower and upper bounds for $\gamma^t_{sdR}(D)$ in terms of the order, maximum degree and chromatic number of a digraph. In addition, we determine the STDRD-number of some classes of digraphs.

Keywords: signed total double Roman dominating function, signed total double Roman domination number, directed graph

AMS Subject Classification: 05C69

1. Introduction

Let G be a finite and simple graph with vertex set $V(G)$, and let $N_G(v) = N(v)$ be the open neighborhood of the vertex v. The concept of signed total domination number of an undirected graph was introduced by B. Zelinka in [1] and has been studied in [2, 3, 4, 5], and concept of signed total Roman domination number of a graph investigated by L. Volkmann in [6, 7]. Recently the concept of signed double Roman domination in graphs has been studied in [8, 9]. A *signed total double Roman dominating function* (STDRD-function) on a graph G is defined in [10] as a function $f : V(G) \longrightarrow \{-1, 1, 2, 3\}$ such that (i) every vertex v with $f(v) = -1$ is adjacent to least two vertices assigned 2 under f or to at least one vertex w with $f(w) = 3$, (ii) every vertex v with $f(v) = 1$ is adjacent to at least one vertex w with $f(w) \geq 2$ and (iii) $f(N(v)) = \sum_{x \in N(v)} f(x) \geq 1$ holds for each vertex $v \in V(G)$. The *signed total double Roman domination number* $\gamma^t_{sdR}(G)$ of

[1] Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran.
e-mail: j-amjadi@azaruniv.ac.ir; ORCID: https://orcid.org/0000-0001-9340-4773.
e-mail: fmporhosene1891@gmail.com; ORCID: https://orcid.org/0000-0001-7151-7697.
§ Manuscript received: February 08, 2020; accepted: June 06, 2020.

G is the minimum weight of a STDRD-function on G. A $\gamma^t_{sdR}(G)$-function is a STDRD-function on G of weight $\gamma^t_{sdR}(G)$. Following the ideas in [10, 11] and [12], we study the STDRD-functions on digraphs D.

Suppose D is a finite simple digraph with vertex set $V(D)$ and arc set $A(D)$ (briefly V and A). The *order* and the *size* of D are integers $n = n(D) = |V(D)|$ and $m = m(D) = |A(D)|$ respectively. If uv is an arc of D, then we also write $u \to v$, and we say that v is an *out-neighbor* of u and u is an *in-neighbor* of v and we also say that u *dominates* v. For each vertex v, the set of in-neighbors and out-neighbors of v are denoted by $N^-(v) = N^-_D(v)$ and $N^+(v) = N^+_D(v)$, respectively. Assume that $N^-_D[v] = N^-[v] = N^-(v) \cup \{v\}$ and $N^+_D[v] = N^+[v] = N^+(v) \cup \{v\}$. We write $d^+(v) = d^+_D(v)$ for the out-degree of a vertex v and $d^-(v) = d^-_D(v)$ for its in-degree. We denote the *minimum* and *maximum in-degree* and the *minimum* and *maximum out-degree* of D by $\delta^-(D) = \delta^-$, $\Delta^-(D) = \Delta^-$, $\delta^+(D) = \delta^+$ and $\Delta^+(D) = \Delta^+$, respectively. A digraph D is called r-out-regular if $\delta^+(D) = \Delta^+(D) = r$. In addition, suppose $\delta = \delta(D) = \min\{\delta^+(D), \delta^-(D)\}$ and $\Delta = \Delta(D) = \max\{\Delta^+(D), \Delta^-(D)\}$ is the *minimum* and *maximum degree* of D, respectively. A digraph D is called a *directed cycle* if its underlying graph is an n-cycle and $d^+_D(v) = d^-_D(v) = 1$ for every $v \in V(D)$. A digraph D is called *regular* or *r-regular* if $\delta(D) = \Delta(D) = r$. The *distance* $d_D(u, v)$ from a vertex u to a vertex v is the length of the shortest directed $u - v$ path in D. For every set $X \subseteq V(D)$, $D[X]$ is the subdigraph induced by X. For a real-valued function $f : V \longrightarrow \mathbb{R}$ the weight of f is $\omega(f) = \sum_{v\in V} f(v)$, and for $S \subseteq V$, we write $f(S) = \sum_{v\in S} f(v)$, so $\omega(f) = f(V)$. Consult West [13] for the notation and terminology which are not defined here.

A vertex subset S of a digraph D is called a *dominating set* of D if every vertex not in S is adjacent from at least one vertex in S. The minimum cardinality of a dominating set in D is the *dominating number* $\gamma(D)$. A dominating set S of D is called a *total dominating set* (TD-set) of D if the subdigraph of D induced by S has no isolated vertices. The *total domination number* of D, denoted by $\gamma_t(D)$, is the minimum cardinality of a total dominating set of D. A *signed total dominating function on a digraph D* is defined in [14] as a function $f : V(D) \longrightarrow \{-1, 1\}$ such that $\sum_{x\in N^-(v)} f(x) \geq 1$ for every $v \in D$. The minimum cardinality of a signed total dominating function is the *signed total domination number* $\gamma_{st}(D)$. The concept of signed total roman domination number in digraphs was introduced by Volkmann in [15] and has been studied in [16, 17].

A *signed total double Roman dominating function* (STDRD-function) on D is a function $f : V(D) \longrightarrow \{-1, 1, 2, 3\}$ such that (i) $\sum_{x\in N^-(x)} f(x) \geq 1$ for every $v \in V(D)$, (ii) every vertex u for which $f(u) = -1$ has at least one in-neighbor z with $f(z) = 3$ or at least two in-neighbor v, w for which $f(v) = f(w) = 2$, (iii) every vertex v with $f(v) = 1$ has at least one in-neighbor z with $f(z) \geq 2$. The weight of a STDRD-function f on a digraph D is $\omega(f) = \sum_{v\in V(D)} f(v)$. The *signed total double Roman domination number (STDRD-number)* $\gamma^t_{sdR}(D)$ is the minimum weight of a STDRD-function on D. The signed total double Roman domination number exists when $\delta^- \geq 1$. Thus we assume throughout this paper that $\delta^-(D) \geq 1$.

In this paper we study the STDRD-number of digraphs, and we establish lower and upper bounds for $\gamma^t_{sdR}(D)$ in terms of the order, maximum degree and chromatic number of a directed graph. In addition, we determine the STDRD-number of some classes of digraphs.

The *associated digraph* $D(G)$ of a graph G is defined as a digraph obtained from G if each edge e of G is replaced by two oppositely oriented arcs with the same ends as e. Since $N^-_{D(G)}[v] = N_G[v]$ for each vertex $v \in V(G) = V(D(G))$, we have the next result.

Observation 1.1. *If $D(G)$ is the associated digraph of a graph G, then*

$$\gamma^t_{sdR}(D(G)) = \gamma^t_{sdR}(G).$$

Proposition 1.1. Let u be a vertex of indegree one in D and let f be a STDRD-function on D. Then the following holds.

(1) f assigns a positive value to the vertex of $N^-_D(u)$.
(2) If $f(u) = 1$, then f assigns the weight at least two to the vertex of $N^-_D(u)$.

Proof. (1) Since $f(N^-_D(u)) \geq 1$ and $|N^-_D(u)| = 1$, the results follows.
(2) Since $f(u) = 1$, the vertex u must have at least one in-neighbor assigned 2 or 3 under f and hence f assigns the weight at least two to the vertex of $N^-_D(u)$. □

Corollary 1.1. *If C_n is the directed cycle on n vertices, then $\gamma^t_{sdR}(C_n) = \lceil \frac{3n}{2} \rceil$.*

Proof. Let $C_n = v_1v_2 \dots v_nv_1$ where the arc goes from v_i into v_{i+1} for $1 \leq i \leq n-1$ and from v_n into v_1. First we show that $\gamma^t_{sdR}(C_n) \geq \lceil \frac{3n}{2} \rceil$. Let f be a $\gamma^t_{sdR}(C_n)$-function. By Proposition 1.1 (2), we have

$$2\gamma^t_{sdR}(C_n) = 2\sum_{i=1}^{n} f(v_i) = (f(v_n) + f(v_1)) + \sum_{i=1}^{n-1}(f(v_i) + f(v_{i+1})) \geq 3n$$

and this implies that $\gamma^t_{sdR}(C_n) \geq \lceil \frac{3n}{2} \rceil$.

Now we show that $\gamma^t_{sdR}(C_n) \leq \lceil \frac{3n}{2} \rceil$. Define $f : V(D) \to \{-1, 1, 2, 3\}$ by $f(v_{2i+1}) = 2$ and $f(v_{2i+2}) = 1$ for each $0 \leq i \leq \frac{n-2}{2}$ when n is even, and by $f(v_n) = 2$, $f(v_{2i+1}) = 2$ and $f(v_{2i+2}) = 1$ for each $0 \leq i \leq \frac{n-3}{2}$ when n is odd. Clearly f is a STDRDF of C_n of weight $\lceil \frac{3n}{2} \rceil$ yielding $\gamma^t_{sdR}(C_n) \leq \lceil \frac{3n}{2} \rceil$. Thus $\gamma^t_{sdR}(C_n) = \lceil \frac{3n}{2} \rceil$. □

In [10], the authors determine the STDRD-number of some classes of graphs including complete graphs, complete bipartite graphs and cycle.

Theorem A. If $n \neq 4$, then $\gamma^t_{sdR}(K_n) = 3$ and $\gamma^t_{sdR}(K_4) = 4$.

Theorem B. *If $n \neq 5$, then $\gamma^t_{sdR}(C_n) = n$ and $\gamma^t_{sdR}(C_5) = 6$.*

Theorem C. For $1 \leq m \leq n$,

$$\gamma^t_{sdR}(K_{m,n}) = \begin{cases} 4, & (m = n = 2, 4), (m = 2, n = 4), \textit{ or } (m = 1, n \geq 2) \\ 2, & (m = 3, n \neq 4) \textit{ or } m \geq 5 \\ 3 & \text{otherwise.} \end{cases}$$

Let K^*_n, C^*_n and $K^*_{m,n}$ are the associated digraphs of K_n, C_n and $K_{m,n}$, respectively. Using Observation 1.1 and Theorems A, B and C we obtain next result.

Corollary 1.2. (1) *If $n \neq 4$, then $\gamma^t_{sdR}(K^*_n) = 3$ and $\gamma^t_{sdR}(K^*_4) = 4$.*
(2) *If $n \neq 5$, then $\gamma^t_{sdR}(C^*_n) = n$ and $\gamma^t_{sdR}(C^*_5) = 6$.*
(3) For $1 \leq m \leq n$,

$$\gamma^t_{sdR}(K^*_{m,n}) = \begin{cases} 4, & (m = n = 2, 4), (m = 2, n = 4), \textit{ or } (m = 1, n \geq 2) \\ 2, & (m = 3, n \neq 4) \textit{ or } m \geq 5 \\ 3 & \text{otherwise.} \end{cases}$$

The proof of the following result can be found in Szekeres-Wilf [18].

Theorem D. For any graph G,

$$\chi(G) \leq 1 + \max\{\delta(H) \mid H \text{ is a subgraph of } G\}.$$

2. Basic Properties

In this section we investigate basic properties of the STDRD-functions and the STDRD-numbers of digraphs. The definitions immediately lead to our first proposition.

Proposition 2.1. *For any STDRD-function $f = (V_{-1}, V_1, V_2, V_3)$ on a digraph D of order n,*

(a) $|V_{-1}| + |V_1| + |V_2| + |V_3| = n$;
(b) $\omega(f) = |V_1| + 2|V_2| + 3|V_3| - |V_{-1}|$;
(c) $V_1 \cup V_2 \cup V_3$ *is a total dominating set of D. In particular, $|V_1 \cup V_2 \cup V_3| \geq \gamma_t(D)$ where $\gamma_t(D)$ is the total domination number of D.*

Proposition 2.2. *If $f = (V_{-1}, V_1, V_2, V_3)$ is a STDRD-function on a digraph D of order n with maximum out-degree Δ^+ and minimum out-degree δ^+, then*

(i) $(3\Delta^+ - 1)|V_3| + (2\Delta^+ - 1)|V_2| + (\Delta^+ - 1)|V_1| \geq (\delta^+ + 1)|V_{-1}|$;
(ii) $(3\Delta^+ + \delta^+)|V_3| + (2\Delta^+ + \delta^+)|V_2| + (\Delta^+ + \delta^+)|V_1| \geq n(\delta^+ + 1)$;
(iii) $(\Delta^+ + \delta^+)\omega(f) \geq n(\delta^+ - \Delta^+ + 2) + (\delta^+ - \Delta^+)(2|V_3| + |V_2|)$;
(iv) $\omega(f) \geq n(\delta^+ - 3\Delta^+ + 2)/(3\Delta^+ + \delta^+) + |V_2| + 2|V_3|$.

Proof. (i) Proposition 2.1 (a) implies that

$$\begin{aligned}
|V_{-1}| + |V_1| + |V_2| + |V_3| &= n \\
&\leq \sum_{v \in V(D)} \sum_{x \in N^-(v)} f(x) \\
&= \sum_{v \in V(D)} d_D^+(v) f(v) \\
&= \sum_{v \in V_3} 3d_D^+(v) + \sum_{v \in V_2} 2d_D^+(v) \\
&\quad + \sum_{v \in V_1} d_D^+(v) - \sum_{v \in V_{-1}} d_D^+(v) \\
&\leq 3\Delta^+|V_3| + 2\Delta^+|V_2| \\
&\quad +\Delta^+|V_1| - \delta^+|V_{-1}|.
\end{aligned}$$

This inequality chain leads to the desired bound.

(ii) Using Proposition 2.1 (a) and Part (i), we arrive at (ii).

(iii) This part can be obtained from Proposition 2.1 and Part (ii) as follows

$$\begin{aligned}
(\Delta^+ + \delta^+)\omega(f) &= (\Delta^+ + \delta^+)(4|V_3| + 3|V_2| + 2|V_1| - n) \\
&\geq 2n(\delta^+ + 1) - 2\Delta^+(2|V_3| + |V_2|) + (\Delta^+ + \delta^+)(2|V_3| + |V_2| - n) \\
&= n(\delta^+ - \Delta^+ + 2) + (\delta^+ - \Delta^+)(2|V_3| + |V_2|).
\end{aligned}$$

(iv) The inequality chain in the proof of Part (i) and Proposition 2.1 (a) implies

$$\begin{aligned}
n &\leq 3\Delta^+|V_1 \cup V_2 \cup V_3| - \delta^+|V_{-1}| \\
&= 3\Delta^+|V_1 \cup V_2 \cup V_3| - \delta^+(n - |V_1 \cup V_2 \cup V_3|) \\
&= (3\Delta^+ + \delta^+)|V_3 \cup V_2 \cup V_1| - n\delta^+
\end{aligned}$$

and so

$$|V_1 \cup V_2 \cup V_3| \geq \frac{n(\delta^+ + 1)}{3\Delta^+ + \delta^+}.$$

Applying above inequality and Proposition 2.1, we get

$$\begin{aligned}\omega(f) &= 2|V_1 \cup V_2 \cup V_3| - n + |V_2| + 2|V_3| \\ &\geq \frac{2n(\delta^+ + 1)}{3\Delta^+ + \delta^+} - n + |V_2| + 2|V_3| \\ &= \frac{n(\delta^+ - 3\Delta^+ + 2)}{3\Delta^+ + \delta^+} + |V_2| + 2|V_3|\end{aligned}$$

and the proof is complete. □

Corollary 2.1. For any r-out-regular digraph D of order n with $r \geq 1$, $\gamma^t_{sdR}(D) \geq n/r$.

Applying Corollary 2.1 and Observation 1.1, we obtain the next known result.

Corollary 2.2. (Ahangar et al. [10]) For any r-regular graph G of order n with $r \geq 1$, $\gamma^t_{sdR}(G) \geq n/r$.

If D is not out-regular, then we can get the next lower bound on the STDRD-number.

Corollary 2.3. If D is a digraph of order n with minimum out-degree δ^+, maximum out-degree Δ^+ and $\delta^+ < \Delta^+$, then

$$\gamma^t_{sdR}(D) \geq \left\lceil \frac{3\delta^+ - 3\Delta^+ + 4}{3\Delta^+ + \delta^+} \right\rceil n.$$

Proof. Multiplying both sides of the inequality in Proposition 2.2 (iv) by $\Delta^+ - \delta^+$ and adding the resulting inequality to the inequality in Proposition 2.2 (iii) leads to the desired result. □

Since $\Delta^+(D(G)) = \Delta(G)$ and $\delta^+(D(G)) = \delta(G)$, Observation 1.1 and Corollary 2.3 leads to the next known result.

Corollary 2.4. [10] If G is a non-regular graph of order n, minimum degree $\delta \geq 1$ and maximum degree Δ, then

$$\gamma^t_{sdR}(G) \geq \left\lceil \frac{(3\delta - 3\Delta + 4)n}{3\Delta + \delta} \right\rceil.$$

3. Bounds on the signed double Roman domination number

We start with a simple but sharp upper bound on the STDRD-number of a digraph.

Proposition 3.1. For any non-empty digraph D of order n with minimum in-degree $\delta^- \geq 1$, $\gamma^t_{sdR}(D) \leq 2n$.

Proof. Obviously the function f defined on D by $f(x) = 2$ for each $x \in V(D)$, is a STDRD-function on D yielding $\gamma^t_{sdR}(D) \leq 2n$. □

The bound in Proposition 3.1 can be improved if $\delta^-(D) \geq 2$.

Theorem 3.1. If D is a digraph of order n with minimum in-degree $\delta^- \geq 2$, then

$$\gamma^t_{sdR}(D) \leq 2n - 3\left\lceil \frac{\delta^- - 1}{2} \right\rceil + 1.$$

Proof. Assume that $t = \lceil \frac{\delta^- - 1}{2} \rceil$. It follows from

$$n \cdot \Delta^+(D) \geq \sum_{x \in V(D)} d^+(x) = \sum_{x \in V(D)} d^-(x) \geq n \cdot \delta^-(D),$$

that $\Delta^+(D) \geq \delta^-(D) \geq t$. Suppose $u \in V(D)$ is a vertex with out-degree $\Delta^+(D)$, and let $B = \{w_1, w_2, \ldots, w_t\}$ be a set of t out-neighbor of u. Define the function f on $V(D)$ by $f(u) = 3, f(x) = -1$ for $x \in B$ and $f(x) = 2$ for $x \in V(D) - (B \cup \{u\})$. Then for each vertex $z \in V(D)$ we have

$$f(N^-(z)) \geq -t + 2(\delta^- - t) = 2\delta^- - 3t \geq 1$$

and so f is a STDRD-function with weight $3 - t + 2(n - t - 1) = 2n - 3t + 1$. This implies that $\gamma^t_{sdR}(D) \leq 2n - 3\lceil \frac{\delta^- - 1}{2} \rceil + 1$. □

Next we present a bound on the signed total double Roman domination number in terms of the order and signed total domination number of digraphs.

Theorem 3.2. *For any digraph D of order $n \geq 2$ with $\delta^- \geq 1$,*

$$\gamma^t_{sdR}(D) \leq n + 2\gamma_{st}(D).$$

Proof. Let f be a $\gamma_{st}(D)$-function and let $P = \{v | f(v) = 1\}$ and $M = \{v | f(v) = -1\}$. Clearly, $|P| = \frac{n+\gamma_{st}(D)}{2}$ and $|M| = \frac{n-\gamma_{st}(D)}{2}$. Define $g : V(D) \longrightarrow \{-1, 1, 2, 3\}$ by $g(v) = 3$ for $v \in P$ and $g(v) = -1$ for $v \in M$. It is easy to see that g is a STDRD-function of D and hence $\gamma^t_{sdR}(D) \leq 3|P| - |M| = n + 2\gamma_{st}(D)$. □

Proposition 3.2. For any digraph D of order n,

$$\gamma^t_{sdR}(D) \geq 1 + \Delta^-(D) - n.$$

Moreover, this bound is sharp.

Proof. Assume $v \in V(D)$ is a vertex with in-degree $\Delta^-(D)$, and f is a $\gamma^t_{sdR}(D)$-function. By definition we have

$$\begin{aligned} \gamma^t_{sdR}(D) &= \sum_{x \in N^-(v)} f(x) + \sum_{x \in V(D) \setminus N^-(v)} f(x) \\ &\geq 1 + \sum_{x \in V(D) \setminus N^-(v)} f(x) \\ &\geq 1 - (n - (\Delta^-(D))) \\ &= 1 + \Delta^-(D) - n \end{aligned}$$

as desired.

To show the sharpness, let n, t be integers such that $n \geq 3$ and $2t + 3 \leq n - 1$, and let $K_{1,n-1}$ be a star centered at u with leaves $u_1, u_2, \ldots, u_{n-1}$. Assume D_t is a digraph obtained from $K_{1,n-1}$ by orienting the arcs from u into leaves and adding arcs (u_2, u), (u_i, u_1) for $2 \leq i \leq 2t + 3$. Define g on $V(D_t)$ by $g(u) = 3$ and $g(u_2) = \ldots = g(u_t) = g(u_{t+1}) = 1$ and $g(x) = -1$ otherwise. One can see that g is a STDRD-function on D_t with weight $4 + 2t - n = 1 + \Delta^-(D_t) - n$ implying that $\gamma^t_{sdR}(D_t) \leq 1 + \Delta^-(D_t) - n$ and so $\gamma^t_{sdR}(D_t) = 1 + \Delta^-(D_t) - n$. Therefore the bound of Proposition 3.2 is sharp for $\Delta^-(D)$ odd.

Assume now that $t \geq 2$ be an integer with $2t + 4 \leq n - 1$ and let D_{2t} be the digraph obtained from $K_{1,n-1}$ by orienting the arcs from u into leaves and adding arcs (u_2, u), (u_i, u_1) for $2 \leq i \leq 2t + 4$. Define h on $V(D_{2t})$ by $h(u) = 3, h(u_2) = 2$ and $h(u_3) = h(u_4) = \ldots = h(u_{t+1}) = 1$ and $h(x) = -1$ otherwise. Clearly h is a STDRD-function on D_{2t} with weight $5 + 2t - n = 1 + \Delta^-(D_{2t}) - n$. It follows that $\gamma^t_{sdR}(D_{2t}) = 1 + \Delta^-(D_{2t}) - n$ by Proposition 3.2. Hence the bound of Proposition 3.2 is sharp for $\Delta^-(D)$ even too. □

Theorem 3.3. *For any digraph D of order $n \geq 3$,*

$$\gamma^t_{sdR}(D) \geq 6 - n.$$

Furthermore, this bound is sharp.

Proof. Consider a $\gamma^t_{sdR}(D)$-function f. If $f(u) \geq 1$ for each $u \in V(D)$, then we have $\gamma^t_{sdR}(D) \geq n + 1 > 6 - n$ as desired. Hence we assume that $V_{-1} \neq \emptyset$. By definition there exist two vertices u and v in $V_1 \cup V_2 \cup V_3$ such that $f(u) + f(v) \geq 4$. This implies that $\gamma^t_{sdR}(D) \geq 4 - (n - 2) = 6 - n$ as desired.

To show the sharpness, we let $K_{1,n-1}$ $(n \geq 3)$ be a star centered at u with leaves $u_1, u_2, \ldots, u_{n-1}$. Assume D is a digraph obtained from $K_{1,n-1}$ by orienting the arcs from u into leaves and adding arcs (u_1, u). Define g on $V(D)$ by $g(u) = 3$ and $g(u_1) = 1$ and $g(x) = -1$ otherwise. One can see that g is a STDRD-function on D with weight $6 - n$ implying that $\gamma^t_{sdR}(D) = 6 - n$. □

Proposition 3.3. For any digraph D of order n with $\Delta^+(D) \geq 2$,

$$\gamma^t_{sdR}(D) \geq \frac{(2 - \Delta^+)n}{\Delta^+} + \frac{2\Delta^+ - 2}{\Delta^+}\gamma_t(D).$$

Proof. Let $f = (V_{-1}, V_1, V_2, V_3)$ be a $\gamma^t_{sdR}(D)$-function. Assume that $S = V_{-1} \cap N^+(V_3)$ and $T = V_{-1} \setminus S$. Since each vertex in V_3 dominate at most Δ^+ of S, we have $|S| \leq \Delta^+|V_3|$. Also, since each vertex in V_2 dominate at most Δ^+ vertices of T and since each vertex in T has at least two in-neighbors in V_2, we get $2|T| \leq |E(V_2, T)| \leq \Delta^+|V_2|$ implying that $|T| \leq \frac{\Delta^+}{2}|V_2|$. Thus $|V_{-1}| = |S| + |T| \leq \Delta^+|V_3| + \frac{\Delta^+}{2}|V_2|$. Thus

$$\begin{aligned}
\Delta^+\gamma^t_{sdR}(D) &= \Delta^+(|V_1| + 2|V_2| + 3|V_3| - |V_{-1}|) \\
&= \Delta^+(|V_1| + |V_2| + |V_3|) + 2\Delta^+|V_3| + \Delta^+|V_2| - \Delta^+|V_{-1}| \\
&\geq \Delta^+(|V_1| + |V_2| + |V_3|) + (2 - \Delta^+)|V_{-1}| \\
&= (2\Delta^+ - 2)(|V_1| + |V_2| + |V_3|) + (2 - \Delta^+)n \\
&\geq (2\Delta^+ - 2)\gamma_t(D) + (2 - \Delta^+)n, \quad (\text{ since } V_1 \cup V_2 \cup V_3 \text{ is a TD-set of } D)
\end{aligned}$$

and this leads to the desired bound. □

Next we present a lower bound in terms of the order and the (total) domination number.

Theorem 3.4. For any digraph D of order $n \geq 2$, $\gamma^t_{sdR}(D) \geq \gamma_t(D) + 2\gamma(D) - n$.

Proof. Consider a $\gamma^t_{sdR}(D)$-function $f = (V_{-1}, V_1, V_2, V_3)$. Note that $\gamma(D) \leq |V_2| + |V_3|$ because $V_2 \cup V_3$ dominates D, and $\gamma_t(D) \leq |V_1| + |V_2| + |V_3|$ since $V_1 \cup V_2 \cup V_3$ totally dominates D. Hence

$$\begin{aligned}
\gamma^t_{sdR}(D) &= |V_1| + 2|V_2| + 3|V_3| - |V_{-1}| \\
&\geq \gamma_t(D) + 2\gamma(D) - n + |V_1| \\
&\geq \gamma_t(D) + 2\gamma(D) - n.
\end{aligned}$$

□

For any digraph D, the *complement* $\overline{D}$ of D is the digraph with vertex set $V(D)$ such that for any two distinct vertices u and v, $(u, v) \in \overline{D}$ if and only if $(u, v) \notin D$. Next we present a lower bound on the sum $\gamma^t_{sdR}(D) + \gamma^t_{sdR}(\overline{D})$ for r-regular digraphs.

Theorem 3.5. Let D be an r-regular digraph of order n. Then

$$\gamma^t_{sdR}(D) + \gamma^t_{sdR}(\overline{D}) \geq \frac{4n}{n-1}.$$

If n is even, then $\gamma^t_{sdR}(D)+\gamma^t_{sdR}(\overline{D})\geq \frac{4(n-1)}{n-2}$.

Proof. Since D is r-regular, its complement $\overline{D}$ is $(n-r-1)$-regular. Corollary 2.1 implies that

$$\gamma^t_{sdR}(D)+\gamma^t_{sdR}(\overline{D})\geq n\Big(\frac{1}{r}+\frac{1}{n-r-1}\Big).$$

The conditions $r\geq 1, n-r-1\geq 1$ imply that $1\leq r\leq n-2$. Since the function $g(x)=\frac{1}{(x)}+\frac{1}{n-x-1}$ takes its minimum at $\frac{n-1}{2}$ for $1\leq x\leq n-2$, we obtain

$$\gamma^t_{sdR}(D)+\gamma^t_{sdR}(\overline{D})\geq n\Big(\frac{2}{n-1}+\frac{2}{n-1}\Big)=\frac{4n}{n-1},$$

and this is the desired bound. For even n, the function g takes its minimum at $r=x=(n-2)/2$ or $r=x=n/2$, because r is an integer and we have

$$\gamma^t_{sdR}(D)+\gamma^t_{sdR}(\overline{D})\geq n\Big(\frac{1}{r}+\frac{1}{n-r-1}\Big)\geq n\Big(\frac{2}{n}+\frac{2}{n-2}\Big)=\frac{4(n-1)}{n-2}.$$

and the proof is complete. □

4. A lower bound in terms of chromatic number

In this section we establish a sharp lower bounds on STDRD-number in terms of the order, the maximum degree and the chromatic number of D.

Theorem 4.1. *If D is a connected digraph of order $n\geq 3$ and k is a nonnegative integer such that $\delta^+(D)\geq k$, then*

$$\gamma^t_{sdR}(D)\geq \chi(G)+\left\lceil\frac{3}{2}(k-\Delta(G))\right\rceil+4-n$$

where G is the underlying graph of D.

Proof. Let $f=(V_{-1},V_1,V_2,V_3)$ be a $\gamma^t_{sdR}(D)$-function. Since $\delta^-(D)\geq 1$, we have $\Delta(G)\geq 2$. First let $\Delta(G)=2$. Then G is a path or a cycle and $k\leq 1$ because $\delta^-(D)\geq 1$. We claim that $k=1$. Let, to the contrary, $k=0$ and let $v\in V(D)$ be a vertex for which $d^+(x)=0$. Assume that P is a longest directed path that ends at v. If u is the first vertex of P, then obviously $d^-(u)=0$ which is a contradiction. Therefore $d^+(x)=d^-(x)=1$ for each $x\in V(D)$ and G is a cycle and D is directed cycle and the result follows from Theorem 3.3.

Assume that $\Delta(G)\geq 3$. We show that $k\leq\Delta(G)-2$. Suppose, to the contrary, that $k\geq\Delta(G)-1$. Since $k\leq d^+(x)$, $d^-(x)\geq 1$ and $d^-(x)+d^+(x)\leq\Delta(G)$ for each $x\in V(D)$, we have $d^-(x)=1$ for each $x\in V(D)$. But then $\Delta(G)-1\leq\frac{1}{n}\sum_{x\in V}d^+(x)=\frac{1}{n}\sum_{x\in V}d^-(x)=1$ and this leads to a contradiction. Hence $k\leq\Delta(G)-2$ and so $\mu=\frac{3\Delta(G)-3k-1}{4}\geq 1$.

For each $x\in V_{-1}$, we have

$$|E(V_{-1},x)|\leq 3|E(V_3,x)|+2|E(V_2,x)|+|E(V_1,x)|-1$$

and so

$$\begin{aligned}\Delta(G)\geq\deg(x) &= |E(V_{-1},x)|+|E(V_3,x)|+|E(V_2,x)|+|E(V_1,x)|+d^+(x)\\ &\geq |E(V_3,x)|+\frac{2|E(V_2,x)|}{3}+\frac{|E(V_1,x)|}{3}+|E(V_{-1},x)|+k\\ &\geq \frac{4|E(V_{-1},x)|}{3}+k+\frac{1}{3}\end{aligned}$$

which implies that $|E(V_{-1}, x)| \leq \frac{3\Delta(G)-3k-1}{4} = \mu$. Assume $H = D[V_{-1}]$ is the subdigraph induced by V_{-1} and let $H' = G[V_{-1}]$ be the underlying graph of H. Let H_1 be an induced subdigraph of H. Then $d^-(x) \leq |E(V_{-1}, x)| \leq \mu$ for each $x \in H_1$, and hence $\Sigma_{x\in V(H_1)} d^+(x) = \Sigma_{x\in V(H_1)} d^-(x) \leq \mu |V(H_1)|$. Hence there exists a vertex $x \in V(H_1)$ such that $d^+(x) \leq \mu$. It follows that $\delta(H_1') \leq 2\mu$, where H_1' is the underlying graph of H_1. We conclude from Proposition D that

$$\begin{aligned}\chi(H') &\leq 1 + \max\{\delta(H'') \mid H'' \text{ is a subgraph of } H'\} \\ &\leq 1 + 2\mu.\end{aligned}$$

Since $3|V_3| + 2|V_2| + |V_1| > 3$, we have $3 - |V_{-1}| < \gamma^t_{sdR}(D)$. On the other hand, since $|V_1| + |V_2| + |V_3| = n - |V_{-1}| < n + \gamma^t_{sdR}(D) - 3$, we have

$$\begin{aligned}\chi(G) &\leq \chi(G[V_{-1}]) + \chi(G[V_1 \cup V_2 \cup V_3])\} \\ &\leq 2\mu + 1 + |V_1| + |V_2| + |V_3| \\ &< 2\mu + n + \gamma^t_{sdR}(D) - 3.\end{aligned}$$

Thus $\gamma^t_{sdR}(D) > \chi(G) + \frac{3(k-\Delta(G))+1}{2} + 3 - n$, as desired. □

References

[1] Zelinka, B., (2001), Signed total domination number of a graph, Czechoslovak Math. J., 51, pp. 225–229.

[2] Henning, M. A., (2004), Signed total dominations in graphs, Discrete Math., 278, pp. 109-125.

[3] Volkmann, L., (2018), Lower bounds on the signed (total) k-domination number, Commun. Comb. Optim., 3, (2), pp. 173–178.

[4] Wang, J.-X., (2006), Lower bounds of the signed total domination number in graphs, Journal Anhui University of Technology 23, pp. 478–480.

[5] Dehgardi, N. and Volkmann, L., (2020), Nonnegative signed total Roman domination in graphs, Commun. Comb. Optim., 5, (1), pp. 139–155.

[6] Volkmann, L., (2016), Signed total Roman domination in graphs, J. Comb. Optim., 32, (3), pp. 855-871.

[7] Volkmann, L., (2016), On the signed total Roman domination and domatic numbers of graphs, Discrete Apll. Math., 214, pp. 179–186.

[8] Abdollahzadeh Ahangar, H., Chellali, M. and Sheikholeslami, S. M., (2019), Signed double Roman domination in graphs, Discrete Appl. Math., 257, pp. 1–11.

[9] Abdollahzadeh Ahangarand, H., Chellali M. and Sheikholeslami, S. M., (2019), Signed double Roman domination of graphs, Filomat, 33, (1), pp. 121–134.

[10] Abdollahzadeh Ahangar, H., Khoeilar, R., Shahbazi L. and Sheikholeslami, S. M., Signed total double Roman domination, Ars Combin., (to appear)

[11] Abdollahzadeh Ahangar, H., Khoeilar, R., Shahbazi L. and Sheikholeslami, S. M., Bounds on signed total double Roman domination number in graphs, Commun. Comb. Optim., (to appear)

[12] Abdollahzadeh Ahangar, H., Khoeilar, R., Shahbazi L. and Sheikholeslami, S.M., (2020), Signed total double Roman k-domination in graphs, Discrete Math. Algorithms Appl. 12(1) ID: 2050009.

[13] West, D. B., (2000), Introduction to Graph Theory, Prentice-Hall, Inc.

[14] Sheikholeslami, S. M., (2011), Signed total domination numbers of directed graphs, Util. Math., 85, pp. 213–218.

[15] Volkmann, L., (2017), Signed total Roman domination in digraphs, Discuss. Math. Graph Theory, 37, (1), pp. 261-272.

[16] Dehgardi, N. and Volkmann, L., (2017), Signed total Roman k-domination in directed graphs, Commun. Comb. Optim., 1, (2), pp. 165–178.

[17] Amjadi, J., Twin signed total Roman domatic numbers in digraphs, Commun. Comb. Optim., (to appear)

[18] Szekeres, G. and Wilf, H. S., (1963), An inequality for the chromatic number of a graph, J. Combin. Theory, 4, pp. 1–3.

J. Amjadi received his Ph.D. degree in mathematics from Tabriz University, in 2005. He is currently an associate professor at Azarbaijan Shahid Madani University, Tabriz, Iran. His research interests include graph theory and algebraic graph theory.

F. Pour Hosseinin is currently pursuing the Ph.D. degree in mathematics at Azarbaijan Shahid Madani University, Tabriz, Iran. Her research interest covers graph theory.