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CHARACTERIZATION OF SEMIGROUP BY ROUGH INTERVAL PYTHAGOREAN FUZZY SET

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Abstract. This paper expose a study on rough interval valued pythagorean fuzzy sets in semigroups. We characterize rough interval valued pythagorean fuzzy sets by an example. Characterize composition of two interval valued pythagorean fuzzy sets. Introduce rough interval valued pythagorean fuzzy left(right, bi-, interior-,(1,2)-)ideals in semigroups. Moreover we prove an interval valued pythagorean fuzzy set is an upper rough interval valued pythagorean fuzzy left(right) ideal of semigroup also we give an example for converse of this is not true. Lower and upper approximation of an interval valued pythagorean fuzzy ideal of semigroup is an interval valued pythagorean fuzzy ideal of semigroup.

Keywords: Pythogrean fuzzy set, Interval valued pythagorean fuzzy set, Rough set, Rough fuzzy set, Rough interval valued fuzzy set, Rough Pythagorean fuzzy ideals, Rough interval valued pythagorean fuzzy ideals in semigroups.

AMS Subject Classification: 03E72, 20M05, 20M12, 08A72, 34C41.

1. Introduction

One of the most interesting research areas for researchers has been fuzzy set theory and it was studied by Zadeh [11]. As a generalization of fuzzy set, Zadeh[10] studied interval valued fuzzy sets. This set deals with set of closed intervals of [0,1]. In 1982 Pawlak[5] introduced the concept of rough set theory. In recent era, many researchers studied the real life applications and theoretical research approach of this theory. Dubois and Prade[4] studied the concept of rough fuzzy sets and fuzzy rough sets by combining fuzzy sets and rough sets a fruitful way. Rough fuzzy sets are the fuzzy sets approximated in the crisp approximation space and fuzzy rough sets are the crisp sets approximated in the fuzzy approximation space. Subha et.al [8] studied interval valued rough fuzzy sets. In 1986 Atanassov [1] presented the intuitionistic fuzzy set as a generalization of fuzzy set. Ordinary intutionistic fuzzy set fails to handle the real life situations. Therefore, a

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more general set is needed for such situations. Recently, Yager [9] has introduced another class of nonstandard fuzzy sets called pythagorean fuzzy set. These fuzzy sets allow for the inclusion of imprecision and uncertainty in the specification of membership grades. Pythagorean fuzzy set is most powerful than intutionistic fuzzy set in dealing with the uncertainty, imprecision and vagueness. The motivation of introducing pythgorean fuzzy set, is that in the real life decision process, the sum of the membership and non membership satisfies a certain criteria, given by experts, may be more than 1 but their square sum is equal to or less than 1. Peng [7] studied interval valued pythogorean fuzzy set. Azmat et.al[2] introduced rough pythagorean fuzzy set.

2. Preliminaries

This section deals with basic concepts related to this work. Throughout this paper let us denote $\mathcal{S}$ as semigroup and $\Re$ as complete congruence relation on $\mathcal{S}$.

Definition 2.1. [3] *A mapping $\tilde{A} : \mathcal{S} \to D[0,1]$ is called an interval valued fuzzy subset of $\mathcal{S}$, where $\tilde{A}(s) = [A^-(s), A^+(s)]$ $s \in \mathcal{S}$, and A^- and A^+ are fuzzy subsets of $\mathcal{S}$ such that $A^-(s) \leq A^+(s)$, $s \in \mathcal{S}$. $D[0,1]$ denotes the set of closed subsets of $[0,1]$.*

Definition 2.2. [6] *The pair $(\mathcal{S}, \Re)$ is called an approximation space. Let A be any nonempty subset of $\mathcal{S}$. The sets $\mathcal{L}(A) = \{x \in \mathcal{S}/[x]_\Re \subseteq A\}$ and $\mathcal{U}(A) = \{x \in \mathcal{S}/[x]_\Re \cap A \neq \phi\}$ are called the lower and upper approximations of A. Then $\Re(A) = (\mathcal{L}(A), \mathcal{U}(A))$ is called rough set in $(\mathcal{S}, \Re)$ if and only if $\mathcal{L}(A) \neq \mathcal{U}(A)$.*

Definition 2.3. [4] *Let A be a fuzzy subset of $\mathcal{S}$. The fuzzy subsets of $\mathcal{S}$ defined by $\mathcal{U}(A)(x) = \bigvee_{a \in [x]_\Re} A(a)$ and $\mathcal{L}(A)(x) = \bigwedge_{a \in [x]_\Re} A(a)$ are called respectively, the upper and lower approximations of the fuzzy set A. $\Re(A) = (\mathcal{L}(A), \mathcal{U}(A))$ is called a rough fuzzy set of A with respect to $\Re$ if $\mathcal{L}(A) \neq \mathcal{U}(A)$.*

Definition 2.4. [8] *Let $\tilde{A}$ be an interval valued fuzzy subset of $\mathcal{S}$. Let $\mathcal{L}(\tilde{A})$ and $\mathcal{U}(\tilde{A})$ be the interval-valued fuzzy subset of $\mathcal{S}$ defined by,*

$$\mathcal{L}(\tilde{A})(x) = [\wedge A^-(y), \wedge A^+(y); y \in [x]_\Re]$$

$$\mathcal{U}(\tilde{A})(x) = [\vee A^-(y), \vee A^+(y); y \in [x]_\Re]$$

Then $\Re(\tilde{A}) = (\mathcal{L}(\tilde{A}), \mathcal{U}(\tilde{A}))$ is called an interval valued rough fuzzy subset of $\mathcal{S}$ if $\mathcal{L}(\tilde{A}) \neq \mathcal{U}(\tilde{A})$.

Definition 2.5. [9] *A pythagorean fuzzy set of $\mathcal{S}$ is defined as follows $I = \{\langle p, \xi_I, \omega_I / p \in \mathcal{S} \rangle\}$ where $\xi_I : \mathcal{S} \to [0,1]$ represents the degree of membership and $\omega_I : \mathcal{S} \to [0,1]$ represents the degree of non-membership function respectively, with the condition that $0 \leq (\xi_I)^2 + (\omega_I)^2 \leq 1$.*

Definition 2.6. [7] *An interval valued pythagoren fuzzy set of $\mathcal{S}$ is defined as follows $\tilde{I} = \left\{\left\langle p, \xi_{\tilde{I}}, \omega_{\tilde{I}} / p \in \mathcal{S} \right\rangle\right\}$ where $\xi_{\tilde{I}} : \mathcal{S} \to [0,1]$ represents the degree of membership and $\omega_{\tilde{I}} : S \to [0,1]$ represents the degree of non-membership function respectively, with the condition that $0 \leq \sup\left(\xi_{\tilde{I}}\right)^2 + \sup\left(\omega_{\tilde{I}}\right)^2 \leq 1$.*

Definition 2.7. [9] *Let $\wp = \{\langle p_1, \xi_\wp(s_1), \omega_\wp(p_1) / p_1 \in \mathcal{S} \rangle\}$ be an pythagorean fuzzy set of $\mathcal{S}$. A rough pythagorean fuzzy set is defined as $\Re(\wp) = (\mathcal{L}(\wp), \mathcal{U}(\wp))$ where $\mathcal{L}(\wp) = \{\langle p_1, \mathcal{L}(\xi_\wp), \mathcal{L}(\omega_\wp) \rangle, p_1 \in \mathcal{S}\}$ here $\mathcal{L}(\xi_\wp)(p_1) = \bigwedge_{p_2 \in [p_1]_\Re} \xi_\wp(p_2)$ and $\mathcal{L}\Re(\omega_\wp)(s_1) = \bigvee_{p_2 \in [p_1]_\Re} \omega_\wp(p_2)$*

with the condition that, $0 \leq \left\{(\mathcal{L}(\xi_{\wp}(p_1)))^2 + (\mathcal{L}(\omega_{\wp}(p_1)))^2\right\} \leq 1$ *also,*
$\mathcal{U}(\wp) = \{\langle p_1, \mathcal{U}(\xi_{\wp}), \mathcal{U}(\omega_{\wp})\rangle, p_1 \in \mathcal{S}\}$ *here* $\mathcal{U}(\xi_{\wp})(p_1) = \bigvee_{p_2 \in [p_1]_{\Re}} \xi_{\wp}(p_2)$ *and*
$\mathcal{U}(\omega_{\wp})(p_1) = \bigwedge_{p_2 \in [p_1]_{\Re}} \omega_{\wp}(p_2)$ *with the condition that,*
$0 \leq \left\{(\mathcal{U}(\xi_{\wp}(p_1)))^2 + (\mathcal{U}(\omega_{\wp}(p_1)))^2\right\} \leq 1.$

3. Rough Interval Pythagorean Fuzzy ($\mathcal{RP}_{iv}\mathcal{F}$) Sets

In this section we introduce the notion of $\mathcal{RP}_{iv}\mathcal{F}$ sets. Also discuss some properties of this set.

Definition 3.1. *Let* $\tilde{\wp} = \{\langle \alpha_1, \xi_{\tilde{\wp}}(\alpha_1), \kappa_{\tilde{\wp}}(\alpha_1)/\alpha_1 \in \mathcal{S}\rangle\}$ *be an* $\mathcal{P}_{iv}\mathcal{F}$ *set of* $\mathcal{S}$*. The lower and upper-approximations of* $\mathcal{P}_{iv}\mathcal{F}$ *is defined as follows:*
$\mathcal{L}(\tilde{\wp}) = \{\langle \alpha_1, \mathcal{L}(\xi_{\tilde{\wp}}), \mathcal{L}(\kappa_{\tilde{\wp}})\rangle, \alpha_1 \in \mathcal{S}\}$ *and* $\mathcal{U}(\tilde{\wp}) = \{\langle \alpha_1, \mathcal{U}(\xi_{\tilde{\wp}}), \mathcal{U}(\kappa_{\tilde{\wp}})\rangle, \alpha_1 \in \mathcal{S}\}$ *where*
$\mathcal{L}(\xi_{\tilde{\wp}})(\alpha_1) = \bigwedge_{\alpha_2 \in [\alpha_1]_{\Re}} \xi_{\tilde{\wp}}(\alpha_2)$, $\mathcal{L}(\kappa_{\tilde{\wp}})(\alpha_1) = \bigvee_{\alpha_2 \in [\alpha_1]_{\Re}} \kappa_{\tilde{\wp}}(\alpha_2)$ *and*
$\mathcal{U}(\xi_{\tilde{\wp}})(\alpha_1) = \bigvee_{\alpha_2 \in [\alpha_1]_{\Re}} \xi_{\tilde{\wp}}(\alpha_2)$, $\mathcal{U}(\kappa_{\tilde{\wp}})(\alpha_1) = \bigwedge_{\alpha_2 \in [\alpha_1]_{\Re}} \kappa_{\tilde{\wp}}(\alpha_2)$ *with the condition that*
$0 \leq \sup\{\mathcal{L}(\xi_{\tilde{\wp}}(\alpha_1))\}^2 + \sup\{\mathcal{L}(\kappa_{\tilde{\wp}}(\alpha_1))\}^2 \leq 1$ *and*
$0 \leq \sup\{\mathcal{U}(\xi_{\tilde{\wp}}(\alpha_1))\}^2 + \sup\{\mathcal{U}(\kappa_{\tilde{\wp}}(\alpha_1))\}^2 \leq 1.$
The pair $\Re(\tilde{\wp}) = (\mathcal{L}(\tilde{\wp}), \mathcal{U}(\tilde{\wp}))$ *is called the* $\mathcal{RP}_{iv}\mathcal{F}$ *set of* $\mathcal{S}$*.*

Example 3.1. *Consider a semigroup* $\mathcal{S} = \{0, 1, 2, 3\}$ *with the binary operation* $'\bullet'$

•	0	1	2	3
0	0	2	2	3
1	2	1	2	3
2	2	2	2	3
3	3	3	3	3

Let $(\Re, \mathcal{S})$ *be an approximation space. Let* $\mathcal{S}/\Re = \{\{0\}, \{1\}, \{2, 3\}\}$ *be the set of equivalence classes of* $\mathcal{S}$*. Let* $\tilde{\wp}$ *be an* $\mathcal{P}_{iv}\mathcal{F}$ *set defined by*

$$\tilde{\wp} = \begin{cases} \langle 0, [0.3, 0.4], [0.5, 0.7]\rangle \\ \langle 1, [0.5, 0.6], [0.6, 0.7]\rangle \\ \langle 2, [0.1, 0.2], [0.7, 0.8]\rangle \\ \langle 3, [0.5, 0.8], [0.4, 0.6]\rangle \end{cases}$$

then lower and upper-approximations of $\tilde{\wp}$ *are*

$$\mathcal{L}(\tilde{\wp}) = \begin{cases} \langle 0, [0.3, 0.4], [0.5, 0.7]\rangle \\ \langle 1, [0.5, 0.6], [0.6, 0.7]\rangle \\ \langle 2, [0.1, 0.2], [0.7, 0.8]\rangle \\ \langle 3, [0.1, 0.2], [0.7, 0.8]\rangle \end{cases}$$

also

$$\mathcal{U}(\tilde{\wp}) = \begin{cases} \langle 0, [0.3, 0.4], [0.5, 0.7]\rangle \\ \langle 1, [0.5, 0.6], [0.6, 0.7]\rangle \\ \langle 2, [0.5, 0.8], [0.4, 0.6]\rangle \\ \langle 3, [0.5, 0.8], [0.4, 0.6]\rangle \end{cases}$$

Theorem 3.1. *Let* $\tilde{\wp}_1$ *and* $\tilde{\wp}_2$ *be two* $\mathcal{P}_{iv}\mathcal{F}$ *set of* $\mathcal{S}$ *then*
$\mathcal{U}(\tilde{\wp}_1) \circ \mathcal{U}(\tilde{\wp}_2) \subseteq \mathcal{U}(\tilde{\wp}_1 \circ \tilde{\wp}_2).$

Proof. Since $\mathcal{U}(\tilde{\wp}_1) = \langle \mathcal{U}(\xi_{\tilde{\wp}_1}), \mathcal{U}(\kappa_{\tilde{\wp}_1}) \rangle$ and $\mathcal{U}(\tilde{\wp}_2) = \langle \mathcal{U}(\xi_{\tilde{\wp}_2}), \mathcal{U}(\kappa_{\tilde{\wp}_2}) \rangle$.
Then, $\mathcal{U}(\tilde{\wp}_1) \circ \mathcal{U}(\tilde{\wp}_2) = \langle \mathcal{U}(\xi_{\tilde{\wp}_1}) \circ \mathcal{U}(\xi_{\tilde{\wp}_2}), \mathcal{U}(\kappa_{\tilde{\wp}_1}) \circ \mathcal{U}(\kappa_{\tilde{\wp}_2}) \rangle$ and
$\mathcal{U}(\tilde{\wp}_1 \circ \tilde{\wp}_2) = \langle \mathcal{U}(\xi_{\tilde{\wp}_1} \circ \xi_{\tilde{\wp}_2}), \mathcal{U}(\kappa_{\tilde{\wp}_1} \circ \kappa_{\tilde{\wp}_2}) \rangle$. We have to prove
$\mathcal{U}(\xi_{\tilde{\wp}_1}) \circ \mathcal{U}(\xi_{\tilde{\wp}_2})(\alpha_j) \leq \mathcal{U}(\xi_{\tilde{\wp}_1} \circ \xi_{\tilde{\wp}_2})(\alpha_j)$ and
$\mathcal{U}(\kappa_{\tilde{\wp}_1}) \circ \mathcal{U}(\kappa_{\tilde{\wp}_2})(\alpha_j) \geq \mathcal{U}(\kappa_{\tilde{\wp}_1} \circ \kappa_{\tilde{\wp}_2})(\alpha_j)$

$$
\begin{aligned}
(\mathcal{U}(\xi_{\tilde{\wp}_1}) \circ \mathcal{U}(\xi_{\tilde{\wp}_2}))(\alpha_j) &= \bigvee_{\alpha_j = \alpha_l \alpha_m} [\mathcal{U}(\xi_{\tilde{\wp}_1})(\alpha_l) \wedge \mathcal{U}(\xi_{\tilde{\wp}_2})(\alpha_m)] \\
&= \bigvee_{\alpha_j = \alpha_l \alpha_m} \left[\left(\bigvee_{\alpha_k \in [\alpha_l]_{\Re}} \xi_{\tilde{\wp}_1}(\alpha_k) \right) \wedge \left(\bigvee_{\alpha n \in [\alpha_m]_{\Re}} \xi_{\tilde{\wp}_2}(\alpha_n) \right) \right] \\
&= \bigvee_{\alpha_j = \alpha_l \alpha_m} \left[\bigvee_{\alpha_k \in [\alpha_l]_{\Re}, \alpha_n \in [\alpha_m]_{\Re}} (\xi_{\tilde{\wp}_1}(\alpha_k) \wedge \xi_{\tilde{\wp}_2}(\alpha_n)) \right] \\
&\leq \bigvee_{\alpha_j = \alpha_l \alpha_m} \left[\bigvee_{\alpha_k \alpha_n \in [\alpha_l \alpha_m]_{\Re}} (\xi_{\tilde{\wp}_1}(\alpha_k) \wedge \xi_{\tilde{\wp}_2}(\alpha_n)) \right] \\
&= \bigvee_{\alpha_k \alpha_n \in [\alpha_j]_{\Re}} (\xi_{\tilde{\wp}_1}(\alpha_k) \wedge \xi_{\tilde{\wp}_2}(\alpha_n)) \\
&= \bigvee_{\alpha_t \in [\alpha_j]_{\Re}, \alpha_t = \alpha_k \alpha_n} (\xi_{\tilde{\wp}_1}(\alpha_k) \wedge \xi_{\tilde{\wp}_2}(\alpha_n)) \\
&= \bigvee_{\alpha_t \in [\alpha_j]_{\Re}} \left[\bigvee_{\alpha_t = \alpha_k \alpha_n} (\xi_{\tilde{\wp}_1}(\alpha_k) \wedge \xi_{\tilde{\wp}_2}(\alpha_n)) \right] \\
&= \bigvee_{\alpha_t \in [\alpha_j]_{\Re}} [(\xi_{\tilde{\wp}_1} \circ \xi_{\tilde{\wp}_2})(\alpha_t)] \\
&= \mathcal{U}(\xi_{\tilde{\wp}_1} \circ \xi_{\tilde{\wp}_2})(\alpha_j)
\end{aligned}
$$

Further,

$$
\begin{aligned}
(\mathcal{U}(\kappa_{\tilde{\wp}_1}) \circ \mathcal{U}(\kappa_{\tilde{\wp}_2}))(\alpha_j) &= \bigwedge_{\alpha_j = \alpha_l \alpha_m} [\mathcal{U}(\kappa_{\tilde{\wp}_1})(\alpha_l) \vee \mathcal{U}(\kappa_{\tilde{\wp}_2})(\alpha_m)] \\
&= \bigwedge_{\alpha_j = \alpha_l \alpha_m} \left[\left(\bigwedge_{\alpha_k \in [\alpha_l]_{\Re}} \kappa_{\tilde{\wp}_1}(\alpha_k) \right) \vee \left(\bigwedge_{\alpha_n \in [\alpha_m]_{\Re}} \kappa_{\tilde{\wp}_2}(\alpha_n) \right) \right] \\
&= \bigwedge_{\alpha_j = \alpha_l \alpha_m} \left[\bigwedge_{\alpha_k \in [\alpha_l]_{\Re}, \alpha_n \in [\alpha_m]_{\Re}} (\kappa_{\tilde{\wp}_1}(\alpha_k) \vee \kappa_{\tilde{\wp}_2}(\alpha_n)) \right] \\
&\geq \bigwedge_{\alpha_j = \alpha_l \alpha_m} \left[\bigwedge_{\alpha_k \alpha_n \in [\alpha_l \alpha_m]_{\Re}} (\kappa_{\tilde{\wp}_1}(\alpha_k) \vee \kappa_{\tilde{\wp}_2}(\alpha_n)) \right] \\
&= \bigwedge_{\alpha_k \alpha_n \in [\alpha_j]_{\Re}} (\kappa_{\tilde{\wp}_1}(\alpha_k) \vee \kappa_{\tilde{\wp}_2}(\alpha_n)) \\
&= \bigwedge_{\alpha_t \in [\alpha_j]_{\Re}, \alpha_t = \alpha_k \alpha_n} (\kappa_{\tilde{\wp}_1}(\alpha_k) \vee \kappa_{\tilde{\wp}_2}(\alpha_n)) \\
&= \bigwedge_{\alpha_t \in [\alpha_j]_{\Re}} \left[\bigwedge_{\alpha_t = \alpha_k \alpha_n} (\kappa_{\tilde{\wp}_1}(\alpha_k) \vee \kappa_{\tilde{\wp}_2}(\alpha_n)) \right] \\
&= \bigwedge_{\alpha_t \in [\alpha_j]_{\Re}} [(\kappa_{\tilde{\wp}_1} \circ \kappa_{\tilde{\wp}_2})(\alpha_t)] \\
&= \mathcal{U}(\kappa_{\tilde{\wp}_1} \circ \kappa_{\tilde{\wp}_2})(\alpha_j).
\end{aligned}
$$

Hence $\mathcal{U}(\tilde{\wp}_1) \circ \mathcal{U}(\tilde{\wp}_2) \subseteq \mathcal{U}(\tilde{\wp}_1 \circ \tilde{\wp}_2)$. In a similar way we prove the following theorem. □

Theorem 3.2. *Let $\tilde{\wp}_1$ and $\tilde{\wp}_2$ be two $\mathcal{P}_{iv}\mathcal{F}$ sets of $\mathcal{S}$ then*
$\mathcal{L}(\tilde{\wp}_1) \circ \mathcal{L}(\tilde{\wp}_2) \subseteq \mathcal{L}(\tilde{\wp}_1 \circ \tilde{\wp}_2)$.

4. Rough Interval Pythagorean Fuzzy Ideal ($\mathcal{RP}_{iv}\mathcal{F}_I$) in semigroup

In this section we define rough interval pythagorean fuzzy left(right, bi-,interior,(1,2)-)ideals ($\mathcal{RP}_{iv}\mathcal{F}_{LI}(\mathcal{F}_{RI}, \mathcal{F}_{BI}, \mathcal{F}_{I_rI})$) of $\mathcal{S}$.
Some basic concepts:

- The composition of two $\mathcal{P}_{iv}\mathcal{F}$ set $\tilde{\wp}_1$ and $\tilde{\wp}_2$ is defined by $\tilde{\wp}_1 \circ \tilde{\wp}_2 = \langle \xi_{\tilde{\wp}_1} \circ \xi_{\tilde{\wp}_2}, \kappa_{\tilde{\wp}_1} \circ \kappa_{\tilde{\wp}_2} \rangle$ where $(\xi_{\tilde{\wp}_1} \circ \xi_{\tilde{\wp}_2})(\delta_1) = \bigvee_{\delta_1=\delta_2\delta_3} [\xi_{\tilde{\wp}_1}(\delta_2) \wedge \xi_{\tilde{\wp}_2}(\delta_3)]$ and $(\kappa_{\tilde{\wp}_1} \circ \kappa_{\tilde{\wp}_2})(\delta_1) = \bigwedge_{\delta_1=\delta_2\delta_3} [\kappa_{\tilde{\wp}_1}(\delta_2) \vee \kappa_{\tilde{\wp}_2}(\delta_3)]$ for all $\delta_1, \delta_2, \delta_3 \in \mathcal{S}$.

An $\mathcal{P}_{iv}\mathcal{F}$ set $\tilde{\wp}$ is said to be

- $\mathcal{P}_{iv}\mathcal{F}$ subsemi-group if $\xi_{\tilde{\wp}}(\delta_1\delta_2) \geq min\{\xi_{\tilde{\wp}}(\delta_1), \xi_{\tilde{\wp}}(\delta_2)\}$ and $\kappa_{\tilde{\wp}}(\delta_1\delta_2) \leq max\{\kappa_{\tilde{\wp}}(\delta_1), \kappa_{\tilde{\wp}}(\delta_2)\}$
- $\mathcal{P}_{iv}\mathcal{F}_I$ of $\mathcal{S}$ if $\xi_{\tilde{\wp}}(\delta_1\delta_2) \geq \xi_{\tilde{\wp}}(\delta_1) \vee \xi_{\tilde{\wp}}(\delta_2)$ and $\kappa_{\tilde{\wp}}(\delta_1\delta_2) \leq \kappa_{\tilde{\wp}}(\delta_1) \wedge \kappa_{\tilde{\wp}}(\delta_2)$ for all $\delta_1, \delta_2, \delta_3 \in \mathcal{S}$

An $\mathcal{P}_{iv}\mathcal{F}$ subsemi-group $\tilde{\wp}$ is said to be

- $\mathcal{P}_{iv}\mathcal{F}_{BI}$ of $\mathcal{S}$ if $\xi_{\tilde{\wp}}(\delta_1\delta_2\delta_3) \geq \xi_{\tilde{\wp}}(\delta_1) \vee \xi_{\tilde{\wp}}(\delta_3)$ and $\kappa_{\tilde{\wp}}(\delta_1\delta_2\delta_3) \leq \kappa_{\tilde{\wp}}(\delta_1) \wedge \kappa_{\tilde{\wp}}(\delta_3)$.
- $\mathcal{P}_{iv}\mathcal{F}_{II}$ of $\mathcal{S}$ if it satisfy $\xi_{\tilde{\wp}}(\delta_1\delta_2\delta_3) \geq \xi_{\tilde{\wp}}(\delta_2)$ and $\kappa_{\tilde{\wp}}(\delta_1\delta_2\delta_3) \leq \kappa_{\tilde{\wp}}(\delta_2)$.
- $\mathcal{P}_{iv}\mathcal{F}_{(1,2)}$ of $\mathcal{S}$ if $\xi_{\tilde{\wp}}(\delta_1\delta(\delta_2\delta_3)) \geq \xi_{\tilde{\wp}}(\delta_1) \wedge \xi_{\tilde{\wp}}(\delta_2) \wedge \xi_{\tilde{\wp}}(\delta_3)$ and $\kappa_{\tilde{\wp}}(\delta_1\delta(\delta_2\delta_3)) \leq \kappa_{\tilde{\wp}}(\delta_1) \vee \kappa_{\tilde{\wp}}(\delta_2) \vee \kappa_{\tilde{\wp}}(\delta_3)$ for all $\delta_1, \delta_2, \delta_3, \delta \in \mathcal{S}$.

Definition 4.1. *If lower (upper)-approximation of $\mathcal{P}_{iv}\mathcal{F}$ subsemi-group of $\mathcal{S}$ is an $\mathcal{P}_{iv}\mathcal{F}$ subsemi-group of $\mathcal{S}$ then it is known as lower(upper)-$\mathcal{RP}_{iv}\mathcal{F}$ subsemi-group of $\mathcal{S}$.*

Definition 4.2. *If an $\mathcal{P}_{iv}\mathcal{F}$ subsemi-group of $\mathcal{S}$ is both lower and upper-$\mathcal{RP}_{iv}\mathcal{F}$ subsemi-group of $\mathcal{S}$ then it is called $\mathcal{RP}_{iv}\mathcal{F}$ subsemi-group of $\mathcal{S}$.*

Theorem 4.1. *Let $\tilde{\wp}$ be an $\mathcal{P}_{iv}\mathcal{F}$ subsemi-group of $\mathcal{S}$. Then upper(lower)-approximation of $\tilde{\wp}$ is an $\mathcal{P}_{iv}\mathcal{F}$ subsemi-group of $\mathcal{S}$,*

Proof. Since $\tilde{\wp}$ is an $\mathcal{P}_{iv}\mathcal{F}$ subsemi-group of $\mathcal{S}$. Our aim is to prove $\mathcal{U}(\tilde{\wp}) = (\mathcal{U}(\xi_{\tilde{\wp}}), \mathcal{U}(\kappa_{\tilde{\wp}}))$ an $\mathcal{P}_{iv}\mathcal{F}$ subsemi-group of $\mathcal{S}$. For that let us consider, $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3 \in \mathcal{S}$.

$$\begin{aligned}
\mathcal{U}(\xi_{\tilde{\wp}})(\beta_1\beta_2) &= \bigvee_{\beta_3\in[\beta_1\beta_2]_{\Re}} \xi_{\tilde{\wp}}(\beta_3) \\
&\geq \bigvee_{\beta_3\in[\beta_1]_{\Re}[\beta_2]_{\Re}} \xi_{\tilde{\wp}}(\beta_3) \\
&= \bigvee_{\alpha_1\alpha_2\in[\beta_1]_{\Re}[\beta_2]_{\Re}} \xi_{\tilde{\wp}}(\alpha_1\alpha_2) \\
&\geq \bigvee_{\alpha_1\in[\beta_1]_{\Re}\alpha_2\in[\beta_2]_{\Re}} [\xi_{\tilde{\wp}}(\alpha_1) \wedge \xi_{\tilde{\wp}}(\alpha_2)] \\
&= \left[\bigvee_{\alpha_1\in[\beta_1]_{\Re}} \xi_{\tilde{\wp}}(\alpha_1)\right] \wedge \left[\bigvee_{\alpha_2\in[\beta_2]_{\Re}} \xi_{\tilde{\wp}}(\alpha_2)\right] \\
&\geq \mathcal{U}(\xi_{\tilde{\wp}})(\beta_1) \wedge \mathcal{U}(\xi_{\tilde{\wp}})(\beta_2)
\end{aligned}$$

Also,

$$\begin{aligned}
\mathcal{U}(\kappa_{\tilde{\wp}})(\beta_1\beta_2) &= \bigvee_{\beta_3\in[\beta_1\beta_2]_{\Re}} \kappa_{\tilde{\wp}}(\beta_3) \\
&\leq \bigvee_{\beta_3\in[\beta_1]_{\Re}[\beta_2]_{\Re}} \kappa_{\tilde{\wp}}(\beta_3) \\
&= \bigvee_{\alpha_1\alpha_2\in[\beta_1]_{\Re}[\beta_2]_{\Re}} \kappa_{\tilde{\wp}}(\alpha_1\alpha_2) \\
&\leq \bigvee_{\alpha_1\in[\beta_1]_{\Re}\alpha_2\in[\beta_2]_{\Re}} [\kappa_{\tilde{\wp}}(\alpha_1) \wedge \kappa_{\tilde{\wp}}(\alpha_2)] \\
&= \left[\bigvee_{\alpha_1\in[\beta_1]_{\Re}} \kappa_{\tilde{\wp}}(\alpha_1)\right] \wedge \left[\bigvee_{\alpha_2\in[\beta_2]_{\Re}} \kappa_{\tilde{\wp}}(\alpha_2)\right]
\end{aligned}$$

$\leq \mathcal{U}(\kappa_{\tilde{\wp}})(\beta_1) \wedge \mathcal{U}(\kappa_{\tilde{\wp}})(\beta_2)$

Consequently, $\tilde{\wp}$ is an upper-$\mathcal{RP}_{iv}\mathcal{F}$ subsemi-group of $\mathcal{S}$. Similarly we can prove for lower-approximation. □

Corollary 4.1. *If $\tilde{\wp}$ is an $\mathcal{P}_{iv}\mathcal{F}$ subsemigroup of $\mathcal{S}$. Then $\tilde{\wp}$ is an $\mathcal{RP}_{iv}\mathcal{F}$ subsemi-group of $\mathcal{S}$.*

Definition 4.3. *If lower(upper)-approximation of $\mathcal{P}_{iv}\mathcal{F}$ set is an $\mathcal{P}_{iv}\mathcal{F}_{LI}$ of $\mathcal{S}$ then it is called a lower(upper)-$\mathcal{RP}_{iv}\mathcal{F}_{LI}$ of $\mathcal{S}$.*
An $\mathcal{P}_{iv}\mathcal{F}$ set is both lower and upper-$\mathcal{RP}_{iv}\mathcal{F}_{LI}$ of $\mathcal{S}$ then it is called a $\mathcal{RP}_{iv}\mathcal{F}_{LI}$ of $\mathcal{S}$.

Theorem 4.2. *Let $\tilde{\wp}$ be an $\mathcal{P}_{iv}\mathcal{F}_{LI}(\mathcal{F}_{RI})$ of $\mathcal{S}$. Then $\tilde{\wp}$ is upper-$\mathcal{RP}_{iv}\mathcal{F}_{LI}(\mathcal{F}_{RI})$ of $\mathcal{S}$.*

Proof. Since $\tilde{\wp}$ be an $\mathcal{P}_{iv}\mathcal{F}_{LI}$ set of $\mathcal{S}$. To prove $\mathcal{U}(\tilde{\wp}) = (\mathcal{U}(\xi_{\tilde{\wp}}),\mathcal{U}(\kappa_{\tilde{\wp}}))$ is an $\mathcal{P}_{iv}\mathcal{F}_{LI}$. Let $\gamma_1, \gamma_2, \gamma_3, a_1, b_2 \in \mathcal{S}$.

$$\begin{aligned}
\mathcal{U}(\xi_{\tilde{\wp}})(\gamma_1\gamma_2) &= \bigvee_{\gamma_3 \in [\gamma_1\gamma_2]_{\Re}} \xi_{\tilde{\wp}}(\gamma_3) \\
&\geq \bigvee_{\gamma_3 \in [\gamma_1]_{\Re}[\gamma_2]_{\Re}} \xi_{\tilde{\wp}}(\gamma_3) \\
&= \bigvee_{a_1 b_2 \in [\gamma_1]_{\Re}[\gamma_2]_{\Re}} \xi_{\tilde{\wp}}(a_1 b_2) \\
&\geq \bigvee_{b_2 \in [\gamma_2]_{\Re}} \xi_{\tilde{\wp}}(b_2) \\
&\geq \mathcal{U}(\xi_{\tilde{\wp}})(\gamma_2)
\end{aligned}$$

Further,

$$\begin{aligned}
\mathcal{U}(\kappa_{\tilde{\wp}})(\gamma_1\gamma_2) &= \bigwedge_{\gamma_3 \in [\gamma_1\gamma_2]_{\Re}} \kappa_{\tilde{\wp}}(\gamma_3) \\
&\leq \bigwedge_{\gamma_3 \in [\gamma_1]_{\Re}[\gamma_2]_{\Re}} \kappa_{\tilde{\wp}}(\gamma_3) \\
&= \bigwedge_{a_1 b_2 \in [\gamma_1]_{\Re}[\gamma_2]_{\Re}} \kappa_{\tilde{\wp}}(a_1 b_2) \\
&\leq \bigwedge_{b_2 \in [\gamma_2]_{\Re}} \kappa_{\tilde{\wp}}(b_2) \\
&\leq \mathcal{U}(\kappa_{\tilde{\wp}})(\gamma_2)
\end{aligned}$$

Consequently, $\tilde{\wp}$ is an upper-$\mathcal{RP}_{iv}\mathcal{F}_{LI}$ of $\mathcal{S}$. By the same token we can prove for right ideals. □

The converse of Theorem 4.2 does not hold in general this is illustrated by the following example.

Example 4.1. *Consider a semigroup $\mathcal{S} = \{a, b, c, d\}$ with the multiplication table.*

•	a	b	c	d
a	a	b	b	d
b	b	b	b	d
c	b	b	b	d
d	d	d	d	d

Let $(\Re, \mathcal{S})$ be a complete congruence relation on $\mathcal{S}$, the equivalence classes of $\mathcal{S}$ are given by $\mathcal{S}/\Re = \{\{a\}, \{c\}, \{b, d\}\}$. Consider a $\mathcal{P}_{iv}\mathcal{F}$ set $\tilde{\wp}$ as follows,

$$\tilde{\wp} = \begin{cases} \langle a, [0.7, 0.8], [0.1, 0.2] \rangle \\ \langle b, [0.8, 0.9], [0.3, 0.4] \rangle \\ \langle c, [0.5, 0.6], [0.3, 0.4] \rangle \\ \langle d, [0.7, 0.8], [0.1, 0.2] \rangle \end{cases}$$

and we get

$$\overline{\Re}(\tilde{\wp}) = \begin{cases} \langle a, [0.7, 0.8], [0.1, 0.2] \rangle \\ \langle b, [0.8, 0.9], [0.1, 0.2] \rangle \\ \langle c, [0.5, 0.6], [0.3, 0.4] \rangle \\ \langle d, [0.8, 0.9], [0.1, 0.2] \rangle \end{cases}$$

Then $\overline{\Re}(\tilde{\wp})$ *is an* $\mathcal{P}_{iv}\mathcal{F}_I$ *of* $\mathcal{S}$ *but* $\tilde{\wp}$ *is not an* $\mathcal{P}_{iv}\mathcal{F}_I$ *of* $\mathcal{S}$*, because* $\xi_{\tilde{\wp}}(bd) = \xi_{\tilde{\wp}}(d) = [0.7, 0.8]$ *and* $\xi_{\tilde{\wp}}(b) \vee \xi_{\tilde{\wp}}(d) = [0.8, 0.9] \Rightarrow \xi_{\tilde{\wp}}(bd) \ngeq \xi_{\tilde{\wp}}(b) \vee \xi_{\tilde{\wp}}(d)$ *also* $\kappa_{\tilde{\wp}}(ac) = \kappa_{\tilde{\wp}}(b) = [0.3, 0.4]$ *and* $\kappa_{\tilde{\wp}}(a) \wedge \kappa_{\tilde{\wp}}(c) = [0.1, 0.2] \Rightarrow \kappa_{\tilde{\wp}}(ac) \nleq \kappa_{\tilde{\wp}}(d) \wedge \kappa_{\tilde{\wp}}(c)$.

Theorem 4.3. *Let* $\tilde{\wp}$ *be an* $\mathcal{P}_{iv}\mathcal{F}_{LI}(\mathcal{F}_{\mathcal{R}I})$ *of* $\mathcal{S}$*. Then* $\tilde{\wp}$ *is lower-*$\mathcal{R}\mathcal{P}_{iv}\mathcal{F}_{LI}(\mathcal{F}_{\mathcal{R}I})$ *of* $\mathcal{S}$*.*

Proof. To prove $\mathcal{L}(\tilde{\wp}) = (\mathcal{L}(\xi_{\tilde{\wp}}), \mathcal{L}(\kappa_{\tilde{\wp}}))$ is an $\mathcal{P}_{iv}\mathcal{F}_{LI}$ of $\mathcal{S}$. Let $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{S}$.

$$\begin{aligned} \mathcal{L}(\xi_{\tilde{\wp}})(\gamma_1\gamma_2) &= \bigwedge_{\gamma_3 \in [\gamma_1\gamma_2]_\Re} \xi_{\tilde{\wp}}(\gamma_3) \\ &= \bigwedge_{\gamma_3 \in [\gamma_1]_\Re[\gamma_2]_\Re} \xi_{\tilde{\wp}}(\gamma_3) \\ &= \bigwedge_{a_1b_2 \in [\gamma_1]_\Re[\gamma_2]_\Re} \xi_{\tilde{\wp}}(a_1b_2) \\ &\geq \bigvee_{b_2 \in [\gamma_2]_\Re} \xi_{\tilde{\wp}}(b_2) \\ &\geq \mathcal{L}(\xi_{\tilde{\wp}})(\gamma_2) \end{aligned}$$

Next,

$$\begin{aligned} \mathcal{L}(\kappa_{\tilde{\wp}})(\gamma_1\gamma_2) &= \bigvee_{\gamma_3 \in [\gamma_1\gamma_2]_\Re} \kappa_{\tilde{\wp}}(\gamma_3) \\ &= \bigvee_{\gamma_3 \in [\gamma_1]_\Re[\gamma_2]_\Re} \kappa_{\tilde{\wp}}(\gamma_3) \\ &= \bigvee_{a_1b_2 \in [\gamma_1]_\Re[\gamma_2]_\Re} \kappa_{\tilde{\wp}}(a_1b_2) \\ &\leq \bigwedge_{b_2 \in [\gamma_2]_\Re} \kappa_{\tilde{\wp}}(b_2) \\ &\leq \mathcal{L}(\kappa_{\tilde{\wp}})(\gamma_2) \end{aligned}$$

Hence, $\tilde{\wp}$ is an lower-$\mathcal{R}\mathcal{P}_{iv}\mathcal{F}_{LI}$ of $\mathcal{S}$. By the similar way we can prove for right ideals. □

Example 4.2. *Let* $S = \{0, 1, 2, 3\}$ *be a semigroup with the following multiplication table*

•	0	1	2	3
0	0	0	0	3
1	0	0	1	3
2	0	1	2	3
3	3	3	3	3

Let $(\Re, \mathcal{S})$ *be an approximation space. The equivalence classes of* $\mathcal{S}$ *are given by* $\mathcal{S}/\Re = \{\{0, 1, 2\}, \{3\}\}$*. Consider a* $\mathcal{P}_{iv}\mathcal{F}$ *set* $\tilde{\wp}$ *as follows*

$$\tilde{\wp} = \begin{cases} \langle 0, [0.5, 0.7], [0.1, 0.2] \rangle \\ \langle 1, [0.7, 0.8], [0.3, 0.4] \rangle \\ \langle 2, [0.8, 0.9], [0.2, 0.3] \rangle \\ \langle 3, [0.8, 0.9], [0.1, 0.2] \rangle \end{cases}$$

then $\mathcal{L}(\tilde{\wp})$ *and* $\mathcal{U}(\tilde{\wp})$

$$\mathcal{L}(\tilde{\wp}) = \begin{cases} \langle 0, [0.5, 0.7], [0.3, 0.4] \rangle \\ \langle 1, [0.5, 0.7], [0.3, 0.4] \rangle \\ \langle 2, [0.5, 0.7], [0.3, 0.4] \rangle \\ \langle 3, [0.8, 0.9], [0.1, 0.2] \rangle \end{cases}$$

$$\mathcal{U}(\tilde{\wp}) = \begin{cases} \langle 0, [0.8, 0.9], [0.1, 0.2] \rangle \\ \langle 1, [0.8, 0.9], [0.1, 0.2] \rangle \\ \langle 2, [0.8, 0.9], [0.1, 0.2] \rangle \\ \langle 3, [0.8, 0.9], [0.1, 0.2] \rangle \end{cases}$$

$\mathcal{P}_{iv}\mathcal{F}_I$ of $\mathcal{S}$. Hence $\tilde{\wp}$ is $\mathcal{RP}_{iv}\mathcal{F}_I$.

Theorem 4.4. *If $\tilde{\wp}_1$ and $\tilde{\wp}_2$ is the $\mathcal{P}_{iv}\mathcal{F}_{\mathcal{R}I}$ and $\mathcal{P}_{iv}\mathcal{F}_{LI}$ of $\mathcal{S}$, respectively. Then $\mathcal{U}(\tilde{\wp}_1 \circ \tilde{\wp}_2) \subseteq \mathcal{U}(\tilde{\wp}_1) \cap \mathcal{U}(\tilde{\wp}_2)$.*

Proof. Since $\tilde{\wp}_1 = (\xi_{\tilde{\wp}_1}, \kappa_{\tilde{\wp}_1})$ is a $\mathcal{P}_{iv}\mathcal{F}_{\mathcal{R}I}$ and $\tilde{\wp}_2 = (\xi_{\tilde{\wp}_2}, \kappa_{\tilde{\wp}_2})$ is a $\mathcal{P}_{iv}\mathcal{F}_{LI}$ of $\mathcal{S}$.
Then, $\mathcal{U}(\tilde{\wp}_1 \circ \tilde{\wp}_2) = (\mathcal{U}(\xi_{\tilde{\wp}_1} \circ \xi_{\tilde{\wp}_2}), \mathcal{U}(\kappa_{\tilde{\wp}_1} \circ \kappa_{\tilde{\wp}_2}))$ and
$\mathcal{U}(\tilde{\wp}_1) \cap \mathcal{U}(\tilde{\wp}_2) = (\mathcal{U}(\xi_{\tilde{\wp}_1}) \cap \mathcal{U}(\xi_{\tilde{\wp}_2}), \mathcal{U}(\kappa_{\tilde{\wp}_1}) \cap \mathcal{U}(\kappa_{\tilde{\wp}_2}))$
We have to prove that,
$\mathcal{U}(\xi_{\tilde{\wp}_1} \circ \xi_{\tilde{\wp}_2}) \leq \mathcal{U}(\xi_{\tilde{\wp}_1}) \cap \mathcal{U}(\xi_{\tilde{\wp}_2})$, $\mathcal{U}(\kappa_{\tilde{\wp}_1} \circ \kappa_{\tilde{\wp}_2}) \geq \mathcal{U}(\kappa_{\tilde{\wp}_1}) \cap \mathcal{U}(\kappa_{\tilde{\wp}_2})$
Let us consider for all $\eta_1, \eta_2, \eta_3, \eta_4 \in \mathcal{S}$,

$$\begin{aligned}
\mathcal{U}(\xi_{\tilde{\wp}_1} \circ \xi_{\tilde{\wp}_2})(\eta_1) &= \bigvee_{\eta_2 \in [\eta_1]_{\Re}} (\xi_{\tilde{\wp}_1} \circ \xi_{\tilde{\wp}_2})(\eta_2) \\
&= \bigvee_{\eta_2 \in [\eta_1]_{\Re}} \left[\bigvee_{\eta_2 = \eta_3\eta_4} (\xi_{\tilde{\wp}_1}(\eta_3) \wedge \xi_{\tilde{\wp}_2}(\eta_4)) \right] \\
&\leq \bigvee_{\eta_2 \in [\eta_1]_{\Re}} \left[\bigvee_{\eta_2 = \eta_3\eta_4} (\xi_{\tilde{\wp}_1}(\eta_3\eta_4) \wedge \xi_{\tilde{\wp}_2}(\eta_3\eta_4)) \right] \\
&= \bigvee_{\eta_2 \in [\eta_1]_{\Re}} [\xi_{\tilde{\wp}_1}(\eta_2) \wedge \xi_{\tilde{\wp}_2}(\eta_2)] \\
&\leq \left[\bigvee_{\eta_2 \in [\eta_1]_{\Re}} \xi_{\tilde{\wp}_1}(\eta_2) \right] \wedge \left[\bigvee_{\eta_2 \in [\eta_1]_{\Re}} \xi_{\tilde{\wp}_2}(\eta_2) \right] \\
&= \mathcal{U}(\xi_{\tilde{\wp}_1})(\eta_1) \wedge \mathcal{U}(\xi_{\tilde{\wp}_2})(\eta_1) \\
&= (\mathcal{U}(\xi_{\tilde{\wp}_1}) \cap \mathcal{U}(\xi_{\tilde{\wp}_2}))(\eta_1)
\end{aligned}$$

Next,

$$\begin{aligned}
\mathcal{U}(\kappa_{\tilde{\wp}_1} \circ \kappa_{\tilde{\wp}_2})(\eta_1) &= \bigwedge_{\eta_2 \in [\eta_1]_{\Re}} (\kappa_{\tilde{\wp}_1} \circ \kappa_{\tilde{\wp}_2})(\eta_2) \\
&= \bigwedge_{\eta_2 \in [\eta_1]_{\Re}} \left[\bigwedge_{\eta_2 = \eta_3\eta_4} (\kappa_{\tilde{\wp}_1}(\eta_3) \vee \kappa_{\tilde{\wp}_2}(\eta_4)) \right] \\
&\geq \bigwedge_{\eta_2 \in [\eta_1]_{\Re}} \left[\bigwedge_{\eta_2 = \eta_3\eta_4} (\kappa_{\tilde{\wp}_1}(\eta_3\eta_4) \vee \kappa_{\tilde{\wp}_2}(\eta_3\eta_4)) \right] \\
&= \bigwedge_{\eta_2 \in [\eta_1]_{\Re}} [\kappa_{\tilde{\wp}_1}(\eta_2) \vee \kappa_{\tilde{\wp}_2}(\eta_2)] \\
&\geq \left[\bigwedge_{\eta_2 \in [\eta_1]_{\Re}} \kappa_{\tilde{\wp}_1}(\eta_2) \right] \vee \left[\bigwedge_{\eta_2 \in [\eta_1]_{\Re}} \kappa_{\tilde{\wp}_2}(\eta_2) \right] \\
&= \mathcal{U}(\kappa_{\tilde{\wp}_1})(\eta_1) \vee \mathcal{U}(\kappa_{\tilde{\wp}_2})(\eta_1) \\
&= (\mathcal{U}(\xi_{\tilde{\wp}_1}) \cup \mathcal{U}(\kappa_{\tilde{\wp}_2}))(\eta_1)
\end{aligned}$$

Hence the proved. □

Theorem 4.5. *If $\tilde{\wp}_1$ and $\tilde{\wp}_2$ is the $\mathcal{P}_{iv}\mathcal{F}_{\mathcal{R}I}$ and $\mathcal{P}_{iv}\mathcal{F}_{LI}$ of $\mathcal{S}$ respectively. Then $\mathcal{L}(\tilde{\wp}_1 \circ \tilde{\wp}_2) \subseteq \mathcal{L}(\tilde{\wp}_1) \cap \mathcal{L}(\tilde{\wp}_2)$.*

Proof. Proof is similar to Theorem 4.4. □

Definition 4.4. *The lower and upper-approximation of $\mathcal{P}_{iv}\mathcal{F}$ set $\tilde{\wp}$ is $\mathcal{P}_{iv}\mathcal{F}_{\mathcal{B}I}$ of $\mathcal{S}$ then it is called an lower-$\mathcal{RP}_{iv}\mathcal{F}_{\mathcal{B}I}$ and upper-$\mathcal{RP}_{iv}\mathcal{F}_{\mathcal{B}I}$.*

An $\mathcal{P}_{iv}\mathcal{F}$ set $\tilde{\wp}$ is said to be an $\mathcal{RP}_{iv}\mathcal{F}_{\mathcal{B}I}$ of $\mathcal{S}$ if it is both lower-$\mathcal{RP}_{iv}\mathcal{F}_{\mathcal{B}I}$ and upper-$\mathcal{RP}_{iv}\mathcal{F}_{\mathcal{B}I}$.

Theorem 4.6. *If $\tilde{\wp}$ is an $\mathcal{P}_{iv}\mathcal{F}_{\mathcal{B}I}$ of $\mathcal{S}$. Then $\tilde{\wp}$ is upper-$\mathcal{RP}_{iv}\mathcal{F}_{\mathcal{B}I}$ of $\mathcal{S}$.*

Proof. Assume that $\tilde{\wp}$ is an $\mathcal{P}_{iv}\mathcal{F}_{\mathcal{B}I}$ of $\mathcal{S}$. Let $b_1, b_2, b_3, b_4 \in \mathcal{S}$. So,

$$\begin{aligned}
\mathcal{U}(\xi_{\tilde{\wp}})(b_1b_2b_3) &= \bigvee_{b_4\in[b_1b_2b_3]_{\Re}} \xi_{\tilde{\wp}}(b_4) \\
&\geq \bigvee_{b_4\in[b_1]_{\Re}[b_2]_{\Re}[b_3]_{\Re}} \xi_{\tilde{\wp}}(b_4) \\
&= \bigvee_{q_1s_2t_3\in[b_1]_{\Re}[b_2]_{\Re}[b_3]_{\Re}} \xi_{\tilde{\wp}}(q_1s_2t_3) \\
&= \bigvee_{q_1\in[b_1]_{\Re}s_2\in[b_2]_{\Re}t_3\in[b_3]_{\Re}} \xi_{\tilde{\wp}}(q_1s_2t_3) \\
&\geq \bigvee_{q_1\in[b_1]_{\Re}t_3\in[b_3]_{\Re}} \{\xi_{\tilde{\wp}}(q_1) \wedge \xi_{\tilde{\wp}}(t_3)\} \\
&= \left\{\bigvee_{q_1\in[b_1]_{\Re}} \xi_{\tilde{\wp}}(q_1)\right\} \wedge \left\{\bigvee_{t_3\in[b_3]_{\Re}} \xi_{\tilde{\wp}}(t_3)\right\} \\
&\geq \mathcal{U}(\xi_{\tilde{\wp}})(b_1) \wedge \mathcal{U}(\xi_{\tilde{\wp}})(b_3)
\end{aligned}$$

Further,

$$\begin{aligned}
\mathcal{U}(\kappa_{\tilde{\wp}})(b_1b_2b_3) &= \bigwedge_{b_4\in[b_1b_2b_3]_{\Re}} \kappa_{\tilde{\wp}}(b_4) \\
&\leq \bigwedge_{b_4\in[b_1]_{\Re}[b_2]_{\Re}[b_3]_{\Re}} \kappa_{\tilde{\wp}}(b_4) \\
&= \bigwedge_{q_1s_2t_3\in[b_1]_{\Re}[b_2]_{\Re}[b_3]_{\Re}} \kappa_{\tilde{\wp}}(q_1s_2t_3) \\
&= \bigwedge_{q_1\in[b_1]_{\Re}s_2\in[b_2]_{\Re}t_3\in[b_3]_{\Re}} \kappa_{\tilde{\wp}}(q_1s_2t_3) \\
&\leq \bigwedge_{q_1\in[b_1]_{\Re}t_3\in[b_3]_{\Re}} \{\kappa_{\tilde{\wp}}(q_1) \vee \kappa_{\tilde{\wp}}(t_3)\} \\
&= \left\{\bigwedge_{q_1\in[b_1]_{\Re}} \kappa_{\tilde{\wp}}(q_1)\right\} \vee \left\{\bigwedge_{t_3\in[b_3]_{\Re}} \kappa_{\tilde{\wp}}(t_3)\right\} \\
&\leq \mathcal{U}(\kappa_{\tilde{\wp}})(b_1) \vee \mathcal{U}(\kappa_{\tilde{\wp}})(b_3)
\end{aligned}$$

Hence $\mathcal{U}(\tilde{\wp}) = (\mathcal{U}(\xi_{\tilde{\wp}}), \mathcal{U}(\kappa_{\tilde{\wp}}))$ is $\mathcal{P}_{iv}\mathcal{F}_{\mathcal{B}I}$ of $\mathcal{S}$.
In a similar way we prove the following theorem. □

Theorem 4.7. *If $\tilde{\wp}$ is an $\mathcal{P}_{iv}\mathcal{F}_{\mathcal{B}I}$ of $\mathcal{S}$. Then $\tilde{\wp}$ is lower-$\mathcal{RP}_{iv}\mathcal{F}_{\mathcal{B}I}$ of $\mathcal{S}$.*

Corollary 4.2. *Let $\tilde{\wp}$ is an $\mathcal{P}_{iv}\mathcal{F}_{\mathcal{B}I}$ of $\mathcal{S}$. Then $\tilde{\wp}$ is $\mathcal{RP}_{iv}\mathcal{F}_{\mathcal{B}I}$ of $\mathcal{S}$.*

Definition 4.5. *The lower and upper-approximation of $\mathcal{P}_{iv}\mathcal{F}$ set $\tilde{\wp}$ is $\mathcal{P}_{iv}\mathcal{F}_{II}$ of $\mathcal{S}$ then it is called an lower-$\mathcal{RP}_{iv}\mathcal{F}_{II}$ and upper-$\mathcal{RP}_{iv}\mathcal{F}_{II}$.*

An $\mathcal{P}_{iv}\mathcal{F}$ set $\tilde{\wp}$ is said to be an $\mathcal{RP}_{iv}\mathcal{F}_{II}$ of $\mathcal{S}$ if it is both lower-$\mathcal{RP}_{iv}\mathcal{F}_{II}$ and upper-$\mathcal{RP}_{iv}\mathcal{F}_{II}$.

Theorem 4.8. *If $\tilde{\wp}$ is $\mathcal{P}_{iv}\mathcal{F}_{II}$ of $\mathcal{S}$ then it is upper(lower)-$\mathcal{RP}_{iv}\mathcal{F}_{II}$ of $\mathcal{S}$.*

Proof. We've to prove $\mathcal{U}(\tilde{\wp}) = (\mathcal{U}(\xi_{\tilde{\wp}}), \mathcal{U}(\kappa_{\tilde{\wp}}))$ is an $\mathcal{P}_{iv}\mathcal{F}_{II}$. For that let us take $i_1, i_2, i_3, i_4 \in \mathcal{S}$

Consider,

$$\begin{aligned}
\mathcal{U}(\xi_{\tilde{\wp}})(i_1i_2i_3) &= \bigvee_{i_4\in[i_1i_2i_3]_{\Re}} \xi_{\tilde{\wp}}(i_4) \\
&\geq \bigvee_{i_4\in[i_1]_{\Re}[i_2]_{\Re}[i_3]_{\Re}} \xi_{\tilde{\wp}}(i_4)
\end{aligned}$$

$$= \bigvee_{a_1b_1c_1 \in [i_1]_{\Re}[i_2]_{\Re}[i_3]_{\Re}} \xi_{\tilde{\wp}}(a_1b_1c_1)$$
$$= \bigvee_{b_1 \in [i_2]_{\Re}} \xi_{\tilde{\wp}}(b_1)$$
$$\geq \mathcal{U}(\xi_{\tilde{\wp}})(i_2)$$

Further,

$$\mathcal{U}(\kappa_{\tilde{\wp}})(i_1i_2i_3) = \bigwedge_{i_4 \in [i_1i_2i_3]_{\Re}} \xi_{\tilde{\wp}}(i_4)$$
$$\leq \bigwedge_{i_4 \in [i_1]_{\Re}[i_2]_{\Re}[i_3]_{\Re}} \kappa_{\tilde{\wp}}(i_4)$$
$$= \bigwedge_{a_1b_1c_1 \in [i_1]_{\Re}[i_2]_{\Re}[i_3]_{\Re}} \kappa_{\tilde{\wp}}(a_1b_1c_1)$$
$$= \bigwedge_{b_1 \in [i_2]_{\Re}} \kappa_{\tilde{\wp}}(b_1)$$
$$\leq \mathcal{U}(\kappa_{\tilde{\wp}})(i_2)$$

Similarly we prove for lower approximation. □

Corollary 4.3. *Let $\tilde{\wp}$ be an $\mathcal{P}_{iv}\mathcal{F}_{II}$ of $\mathcal{S}$. Then $\tilde{\wp}$ is $\mathcal{R}\mathcal{P}_{iv}\mathcal{F}_{II}$ of$\mathcal{S}$.*

Definition 4.6. *The lower and upper-approximation of $\mathcal{P}_{iv}\mathcal{F}$ set $\tilde{\wp}$ is $\mathcal{P}_{iv}\mathcal{F}_{(1,2)I}$ of $\mathcal{S}$ then it is called a lower-$\mathcal{R}\mathcal{P}_{iv}\mathcal{F}_{(1,2)I}$ and upper-$\mathcal{R}\mathcal{P}_{iv}\mathcal{F}_{(1,2)I}$.*

An $\mathcal{P}_{iv}\mathcal{F}$ set $\tilde{\wp}$ is said to be a $\mathcal{R}\mathcal{P}_{iv}\mathcal{F}_{(1,2)I}$ of $\mathcal{S}$ if it is both lower-$\mathcal{R}\mathcal{P}_{iv}\mathcal{F}_{(1,2)I}$ and upper-$\mathcal{R}\mathcal{P}_{iv}\mathcal{F}_{(1,2)I}$.

Theorem 4.9. *If $\tilde{\wp}$ is an $\mathcal{P}_{iv}\mathcal{F}_{(1,2)I}$ of $\mathcal{S}$. Then $\tilde{\wp}$ is upper(lower)-$\mathcal{R}\mathcal{P}_{iv}\mathcal{F}_{(1,2)I}$ of $\mathcal{S}$.*

Proof. Let $\rho_1, \rho_2, \rho_3, \rho_0 \in \mathcal{S}$. To prove $\mathcal{U}(\tilde{\wp}) = (\mathcal{U}(\xi_{\tilde{\wp}}), \mathcal{U}(\kappa_{\tilde{\wp}}))$.

$$\mathcal{U}(\xi_{\tilde{\wp}})(\rho_1\rho_0(\rho_2\rho_3)) = \bigvee_{\rho_4 \in [\rho_1\rho_0(\rho_2\rho_3)]_{\Re}} \xi_{\tilde{\wp}}(\rho_4)$$
$$\geq \bigvee_{\rho_4 \in [\rho_1]_{\Re}[\rho_0]_{\Re}[(\rho_2\rho_3)]_{\Re}} \xi_{\tilde{\wp}}(\rho_4)$$
$$= \bigvee_{a_1b_2c_3d_4 \in [\rho_1]_{\Re}[\rho_0]_{\Re}[\rho_2]_{\Re}[\rho_3]_{\Re}} \xi_{\tilde{\wp}}(a_1b_2c_3d_4)$$
$$\geq \bigvee_{a_1c_3d_4 \in [\rho_1]_{\Re}[\rho_2]_{\Re}[\rho_3]_{\Re}} \{\xi_{\tilde{\wp}}(a_1) \wedge \xi_{\tilde{\wp}}(c_3) \wedge \xi_{\tilde{\wp}}(d_4)\}$$
$$= \left\{\bigvee_{a_1 \in [\rho_1]_{\Re}} \xi_{\tilde{\wp}}(a_1)\right\} \wedge \left\{\bigvee_{c_3 \in [\rho_2]_{\Re}} \xi_{\tilde{\wp}}(c_3)\right\} \wedge \left\{\bigvee_{d_4 \in [\rho_3]_{\Re}} \xi_{\tilde{\wp}}(d_4)\right\}$$
$$\geq \mathcal{U}(\xi_{\tilde{\wp}})(\rho_1) \wedge \mathcal{U}(\xi_{\tilde{\wp}})(\rho_2) \wedge \mathcal{U}(\xi_{\tilde{\wp}})(\rho_3)$$

Next,

$$\mathcal{U}(\kappa_{\tilde{\wp}})(\rho_1\rho_0(\rho_2\rho_3)) = \bigvee_{\rho_4 \in [\rho_1\rho_0(\rho_2\rho_3)]_{\Re}} \kappa_{\tilde{\wp}}(\rho_4)$$
$$\leq \bigwedge_{\rho_4 \in [\rho_1]_{\Re}[\rho_0]_{\Re}[(\rho_2\rho_3)]_{\Re}} \kappa_{\tilde{\wp}}(\rho_4)$$
$$= \bigwedge_{a_1b_2c_3d_4 \in [\rho_1]_{\Re}[\rho_0]_{\Re}[\rho_2]_{\Re}[\rho_3]_{\Re}} \kappa_{\tilde{\wp}}(a_1b_2c_3d_4)$$
$$\leq \bigwedge_{a_1c_3d_4 \in [\rho_1]_{\Re}[\rho_2]_{\Re}[\rho_3]_{\Re}} \{\kappa_{\tilde{\wp}}(a_1) \vee \kappa_{\tilde{\wp}}(c_3) \vee \kappa_{\tilde{\wp}}(d_4)\}$$
$$= \left\{\bigwedge_{a_1 \in [\rho_1]_{\Re}} \kappa_{\tilde{\wp}}(a_1)\right\} \vee \left\{\bigvee_{c_3 \in [\rho_2]_{\Re}} \kappa_{\tilde{\wp}}(c_3)\right\} \vee \left\{\bigvee_{d_4 \in [\rho_3]_{\Re}} \kappa_{\tilde{\wp}}(d_4)\right\}$$
$$\leq \mathcal{U}(\kappa_{\tilde{\wp}})(\rho_1) \vee \mathcal{U}(\kappa_{\tilde{\wp}})(\rho_2) \vee \mathcal{U}(\kappa_{\tilde{\wp}})(\rho_3)$$

Hence, $\mathcal{U}(\tilde{\wp}) = (\mathcal{U}(\xi_{\tilde{\wp}}), \mathcal{U}(\kappa_{\tilde{\wp}}))$ is a $\mathcal{P}_{iv}\mathcal{F}_{(1,2)I}$ of $\mathcal{S}$. Similarly we can prove for lower approximation. □

Corollary 4.4. *If $\tilde{\wp}$ is an $\mathcal{P}_{iv}\mathcal{F}_{(1,2)I}$ of $\mathcal{S}$. Then $\tilde{\wp}$ is $\mathcal{R}\mathcal{P}_{iv}\mathcal{F}_{(1,2)I}$ of $\mathcal{S}$.*

5. Conclusions

In this work, we have presented the notion of rough interval valued pythagorean fuzzy sets. Also we have discussed rough interval valued pythagorean fuzzy ideals in semigroups.

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