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REGULARIZED TRACE FORMULA FOR STURM-LIOUVILLE PROBLEM WITH RETARDED ARGUMENT AND QUADRATICALLY EIGENPARAMETER-DEPENDENT BOUNDARY CONDITION

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Abstract. In this paper, a regularized trace formula for a discontinuous Sturm-Liouville equation with retarded argument is obtained for the case in which the spectral parameter occurs linearly in the equation and one of the boundary conditions and quadratically in the other one. The contour integration method is used to obtain that trace formula.

Keywords: Differential equation with retarded argument, trace formula, asymptotics of eigenvalues.

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1. Introduction

The regularized sum of eigenvalues is a sum with terms obtained by subtracting from the part of their asymptotics that makes the sum diverge. The trace and regularized trace formulas are used in the inverse problems of spectral analysis of differential operators, since they provide quantitative information of the operator. The first formula for the regularized trace was obtained in [11], where they considered the differential operator

$$-u''(x) + q(x)\,u(x) = \lambda u(x), \quad 0 < x < \pi,$$

with the boundary conditions

$$u'(0) = u'(\pi) = 0,$$

where the function $q(x)$ is the continuous differentiability. Let λ_n denote the eigenvalues of this operator and they obtained the following regularized trace formula for the operator:

$$\sum_{n=0}^{\infty}\left(\lambda_n - n^2 - \frac{1}{\pi}\int_0^{\pi} q(x)\,dx\right) = \frac{q(0)+q(\pi)}{4} - \frac{1}{2\pi}\int_0^{\pi} q(x)\,dx.$$

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Then, Levitan [17] suggested one more method for computing regularized traces of the Sturm-Liouville problem:

$$-u''(x)+q(x)u(x)=\lambda u(x), \quad 0<x<\pi,$$
$$u'(0)-hu(0)=0, \qquad u'(\pi)+Hu(\pi)=0,$$

and obtained the following regularized trace formula for the problem:

$$\sum_{n=0}^{\infty}\left(\lambda_n-n^2-c\right)=\frac{q(0)+q(\pi)}{4}-\frac{1}{2\pi}\int_0^{\pi}q(x)\,dx-\frac{h+H}{\pi}-\frac{h^2+H^2}{2},$$

where h and H are real numbers, $\lambda_n=n^2+c+O\left(\frac{1}{n^2}\right)$, $c=\frac{2}{\pi}\left(h+H+\frac{1}{2}\int_0^{\pi}q(x)\,dx\right)$.

These investigations were continued in many directions by applying different problems such as continuous Sturm-Liouville problems (see [1-3, 5, 12, 18, 28, 29, 31]), discontinuous Sturm-Liouville problems (see [13, 15, 26, 30]) and some Dirac systems (see [24, 32]). Also some surveies of the theory of regularized traces can be found in [10, 16, 23].

In recently, there has been an increasing interest in the study of differential equations with retarded argument (see [4, 6, 8, 9, 19, 25]). This type equations provide a mathematical model for physical system in which the rate of change of the system depends upon its past history (see [20]). Also several works dealing with the regularized trace for differential equations with retarded argument were [7, 14, 21, 22, 27]. In [27], the author obtained regularized sums from eigenvalues, oscillations of eigenfunctions and the solution of inverse nodal problem for a Sturm-Liouville problem which has one discontinuity point and none of the boundary conditions contain an eigenparameter. In [14], the author obtained a regularized trace formula for a discontinuous Sturm-Liouville problem with retarded argument and eigenparameter-dependent boundary conditions.

To derive trace formula the contour integration method and Levitan's method have been used in [7, 14, 27, 29, 30-31] and in [12, 13, 15, 21], respectively. The contour integration method is related to the argument principle and the residue calculation in the complex analysis. Levitan's method is based on Hadamard's theorem asserts that entire functions can be represented by an infinite product involving their zeros.

In this paper, we consider a discontinuous Sturm-Liouville problem which contains the spectral parameter linearly in one of the boundary conditions and quadratically in the other boundary condition, with retarded argument. Namely, the problem consists of the Sturm-Liouville equation with retarded argument (the same problem was considered in [4]):

$$y''(x)+q(x)y(x-\Delta(x))+\lambda^2y(x)=0,\ x\in\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right], \tag{1}$$

with boundary conditions:

$$\lambda y(0)+y'(0)=0, \tag{2}$$
$$\lambda^2y(\pi)+y'(\pi)=0, \tag{3}$$

and transmission conditions:

$$y\left(\frac{\pi}{2}-0\right)-\delta y\left(\frac{\pi}{2}+0\right)=0, \tag{4}$$
$$y'\left(\frac{\pi}{2}-0\right)-\delta y'\left(\frac{\pi}{2}+0\right)=0, \tag{5}$$

where the real-valued function $q(x)$ is continuous in $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$ and has a finite limit $q\left(\frac{\pi}{2}\pm 0\right)=\lim_{x\to\frac{\pi}{2}\pm 0}q(x)$; the real-valued function $\Delta(x)\geq 0$ is continuous in $\left[0,\frac{\pi}{2}\right)\cup$

$\left(\frac{\pi}{2},\pi\right]$ and has a finite limit $\Delta\left(\frac{\pi}{2}\pm 0\right)=\lim_{x\to\frac{\pi}{2}\pm 0}\Delta(x)$; $x-\Delta(x)\geq 0$, if $x\in\left[0,\frac{\pi}{2}\right)$, $x-\Delta(x)\geq\frac{\pi}{2}$, if $x\in\left(\frac{\pi}{2},\pi\right]$; λ is an eigenparameter; $\delta\neq 0$ is an arbitrary real number.

Firstly, we obtain the asymptotic formula of the characteristic function $\omega(\lambda)$ whose zeros are eigenvalues of the problem (1)-(5). Some spectral properties were studied through in [4], so we mention them briefly. Then, we obtain our main results that are formulas for asymptotic representation of eigenvalues and a formula of the regularized trace of the problem (1)-(5).

2. Preliminaries

Let $\phi_1(x,\lambda)$ be a solution of (1) on $\left[0,\frac{\pi}{2}\right]$, satisfying the initial conditions

$$\phi_1(0,\lambda)=1,\ \phi_1'(0,\lambda)=-\lambda. \tag{6}$$

By using this solution, we shall define the solution $\phi_2(x,\lambda)$ of (1) on $\left[\frac{\pi}{2},\pi\right]$ by the initial conditions

$$\phi_2\left(\frac{\pi}{2},\lambda\right)=\delta^{-1}\phi_1\left(\frac{\pi}{2},\lambda\right),\qquad \phi_2'\left(\frac{\pi}{2},\lambda\right)=\delta^{-1}\phi_1'\left(\frac{\pi}{2},\lambda\right). \tag{7}$$

Consequently, the function $\phi(x,\lambda)$ is defined on $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$ by the equality

$$\phi(x,\lambda)=\begin{cases}\phi_1(x,\lambda), & x\in\left[0,\frac{\pi}{2}\right),\\ \phi_2(x,\lambda), & x\in\left(\frac{\pi}{2},\pi\right],\end{cases}$$

is a solution of (1), which satisfies the boundary conditions (2) and the transmission conditions (4) and (5).

Then the following integral equations hold:

$$\phi_1(x,\lambda)=\cos(\lambda x)-\sin(\lambda x)-\frac{1}{\lambda}\int_0^x q(\tau)\sin(\lambda(x-\tau))\phi_1(\tau-\Delta(\tau),\lambda)\,d\tau, \tag{8}$$

and

$$\begin{aligned}\phi_2(x,\lambda)=&\ \delta^{-1}\phi_1\left(\tfrac{\pi}{2},\lambda\right)\cos\left(\lambda\left(x-\tfrac{\pi}{2}\right)\right)+\frac{1}{\lambda}\delta^{-1}\phi_1'\left(\tfrac{\pi}{2},\lambda\right)\sin\left(\lambda\left(x-\tfrac{\pi}{2}\right)\right)\\ &-\frac{1}{\lambda}\int_{\frac{\pi}{2}}^{x} q(\tau)\sin(\lambda(x-\tau))\phi_2(\tau-\Delta(\tau),\lambda)\,d\tau.\end{aligned} \tag{9}$$

Solving the equations (8) and (9) by the method of successive approximation, we obtain the following asymptotic formulas for $|\lambda|\to\infty$:

$$\begin{aligned}\phi_1(x,\lambda)=&\cos(\lambda x)-\sin(\lambda x)-\frac{1}{2\lambda}\int_0^x q(\tau)\sin(\lambda(x-\Delta(\tau)))\,d\tau\\ &-\frac{1}{2\lambda}\int_0^x q(\tau)\sin(\lambda(x-2\tau+\Delta(\tau)))\,d\tau-\frac{1}{2\lambda}\int_0^x q(\tau)\cos(\lambda(x-\Delta(\tau)))\,d\tau\\ &+\frac{1}{2\lambda}\int_0^x q(\tau)\cos(\lambda(x-2\tau+\Delta(\tau)))\,d\tau+O\left(\frac{e^{|\Im\lambda|x}}{\lambda^2}\right),\end{aligned} \tag{10}$$

and

$$\begin{aligned}\phi_2(x,\lambda)=&\ \delta^{-1}\cos(\lambda x)-\delta^{-1}\sin(\lambda x)-\frac{\delta^{-1}}{2\lambda}\int_0^x q(\tau)\sin(\lambda(x-\Delta(\tau)))\,d\tau\\ &-\frac{\delta^{-1}}{2\lambda}\int_0^x q(\tau)\sin(\lambda(x-2\tau+\Delta(\tau)))\,d\tau-\frac{\delta^{-1}}{2\lambda}\int_0^x q(\tau)\cos(\lambda(x-\Delta(\tau)))\,d\tau\\ &+\frac{\delta^{-1}}{2\lambda}\int_0^x q(\tau)\cos(\lambda(x-2\tau+\Delta(\tau)))\,d\tau+O\left(\frac{e^{|\Im\lambda|x}}{\lambda^2}\right).\end{aligned} \tag{11}$$

Differentiating (11) and (12) with respect to x, we get

$$\begin{aligned}\phi_1'(x,\lambda) &= -\lambda\sin(\lambda x) - \lambda\cos(\lambda x) - \frac{1}{2}\int_0^x q(\tau)\cos(\lambda(x-\Delta(\tau)))\,d\tau \\ &- \frac{1}{2}\int_0^x q(\tau)\cos(\lambda(x-2\tau+\Delta(\tau)))\,d\tau + \frac{1}{2}\int_0^x q(\tau)\sin(\lambda(x-\Delta(\tau)))\,d\tau \\ &- \frac{1}{2}\int_0^x q(\tau)\sin(\lambda(x-2\tau+\Delta(\tau)))\,d\tau + O\left(\frac{e^{|\Im\lambda|x}}{\lambda}\right),\end{aligned} \tag{12}$$

and

$$\begin{aligned}\phi_2'(x,\lambda) &= -\lambda\delta^{-1}\sin(\lambda x) - \lambda\delta^{-1}\cos(\lambda x) - \frac{\delta^{-1}}{2}\int_0^x q(\tau)\cos(\lambda(x-\Delta(\tau)))\,d\tau \\ &- \frac{\delta^{-1}}{2}\int_0^x q(\tau)\cos(\lambda(x-2\tau+\Delta(\tau)))\,d\tau + \frac{\delta^{-1}}{2}\int_0^x q(\tau)\sin(\lambda(x-\Delta(\tau)))\,d\tau \\ &- \frac{\delta^{-1}}{2}\int_0^x q(\tau)\sin(\lambda(x-2\tau+\Delta(\tau)))\,d\tau + O\left(\frac{e^{|\Im\lambda|x}}{\lambda}\right).\end{aligned} \tag{13}$$

The solution $\phi(x,\lambda)$ defined above is a nontrivial solution of (1) satisfying the boundary condition (2) and the transmission conditions (4) and (5). Putting $\phi(x,\lambda)$ into the boundary condition (3), we get the following characteristic equation

$$\omega(\lambda) := \lambda^2\phi(\pi,\lambda) + \phi'(\pi,\lambda) = 0. \tag{14}$$

Similar to [4, 8, 25], it can be proven that there are infinitely many eigenvalues λ_n^2 of the problem (1)-(5), which are coincide with the zeros of the characteristic function $\omega(\lambda)$.

Putting the expressions (11) and (13) into (14), we obtain the following asymptotic formula for the characteristic function

$$\begin{aligned}\omega(\lambda) &= \lambda^2\delta^{-1}\{\cos(\lambda\pi) - \sin(\lambda\pi)\} - \lambda\delta^{-1}\Bigg\{\Bigg[1 - \frac{1}{2}\int_0^\pi q(\tau)\sin(\lambda\Delta(\tau))\,d\tau \\ &- \frac{1}{2}\int_0^\pi q(\tau)\sin(\lambda(2\tau-\Delta(\tau)))\,d\tau + \frac{1}{2}\int_0^\pi q(\tau)\cos(\lambda\Delta(\tau))\,d\tau \\ &- \frac{1}{2}\int_0^\pi q(\tau)\cos(\lambda(2\tau-\Delta(\tau)))\,d\tau\Bigg]\cos(\lambda\pi) + \Bigg[1 + \frac{1}{2}\int_0^\pi q(\tau)\cos(\lambda\Delta(\tau))\,d\tau \\ &+ \frac{1}{2}\int_0^\pi q(\tau)\cos(\lambda(2\tau-\Delta(\tau)))\,d\tau + \frac{1}{2}\int_0^\pi q(\tau)\sin(\lambda\Delta(\tau))\,d\tau \\ &- \frac{1}{2}\int_0^\pi q(\tau)\sin(\lambda(2\tau-\Delta(\tau)))\,d\tau\Bigg]\sin(\lambda\pi)\Bigg\} + O\left(e^{|\Im\lambda|\pi}\right).\end{aligned} \tag{15}$$

Let we define

$$\begin{aligned}A(\lambda,\Delta) &= \frac{1}{2}\int_0^\pi q(\tau)\sin(\lambda\Delta(\tau))\,d\tau, \\ B(\lambda,\Delta) &= \frac{1}{2}\int_0^\pi q(\tau)\sin(\lambda(2\tau-\Delta(\tau)))\,d\tau, \\ C(\lambda,\Delta) &= \frac{1}{2}\int_0^\pi q(\tau)\cos(\lambda\Delta(\tau))\,d\tau, \\ D(\lambda,\Delta) &= \frac{1}{2}\int_0^\pi q(\tau)\cos(\lambda(2\tau-\Delta(\tau)))\,d\tau.\end{aligned} \tag{16}$$

From (16), (15) can be written as

$$\begin{aligned}\omega(\lambda) &= \lambda^2\delta^{-1}\{\cos(\lambda\pi)-\sin(\lambda\pi)\} - \lambda\delta^{-1}\{[1-A(\lambda,\Delta)-B(\lambda,\Delta)\\ &+C(\lambda,\Delta)-D(\lambda,\Delta)]\cos(\lambda\pi)+[1+A(\lambda,\Delta)-B(\lambda,\Delta)+C(\lambda,\Delta)\\ &+D(\lambda,\Delta)]\sin(\lambda\pi)\}+O\left(e^{|\Im\lambda|\pi}\right).\end{aligned} \tag{17}$$

3. Main Results

In this section, we will obtain asymptotic formulas for the eigenvalues and the regularized trace formula for the problem (1)-(5).

Theorem 3.1. *The eigenvalues of the problem (1)-(5) has the following asymptotic representation for sufficiently large* $|n|$:

$$\lambda_n = n+\frac{1}{4}-\frac{1}{\left(n+\frac{1}{4}\right)\pi}\{1-B(n,\Delta)+C(n,\Delta)\}+O\left(\frac{1}{n^2}\right), \tag{18}$$

where $B(\lambda,\Delta)$ *and* $C(\lambda,\Delta)$ *are given by (16).*

Proof. Let we define

$$\omega_0(\lambda) = \lambda^2\delta^{-1}\{\cos(\lambda\pi)-\sin(\lambda\pi)\}, \tag{19}$$

and denote by $\mu_n,\ n\in\mathbb{N}$, the zeros of (19) except that zero is multiplicity 2, then we have $\mu_{\pm 0}=0$ and $\mu_n = n+\frac{1}{4},\ n\in\mathbb{N}$. Denote by γ_n, the circle of radius ε, $0<\varepsilon<\frac{1}{2}$ with the centers at the points μ_n. Thus, on the contour γ_n, from (17) and (19), we have

$$\begin{aligned}\frac{\omega(\lambda)}{\omega_0(\lambda)} &= 1-\frac{1}{\lambda}\left\{[1-A(\lambda,\Delta)-B(\lambda,\Delta)+C(\lambda,\Delta)-D(\lambda,\Delta)]\frac{\cos(\lambda\pi)}{\cos(\lambda\pi)-\sin(\lambda\pi)}\right.\\ &\left.+[1+A(\lambda,\Delta)-B(\lambda,\Delta)+C(\lambda,\Delta)+D(\lambda,\Delta)]\frac{\sin(\lambda\pi)}{\cos(\lambda\pi)-\sin(\lambda\pi)}\right\}\\ &+O\left(\frac{1}{\lambda^2}\right).\end{aligned} \tag{20}$$

Expanding $\ln\frac{\omega(\lambda)}{\omega_0(\lambda)}$ by the Maclaurin formula, we obtain

$$\begin{aligned}\ln\frac{\omega(\lambda)}{\omega_0(\lambda)} &= -\frac{1}{\lambda}\left\{[1-A(\lambda,\Delta)-B(\lambda,\Delta)+C(\lambda,\Delta)-D(\lambda,\Delta)]\ \frac{\cos(\lambda\pi)}{\cos(\lambda\pi)-\sin(\lambda\pi)}\right.\\ &\left.+[1+A(\lambda,\Delta)-B(\lambda,\Delta)+C(\lambda,\Delta)+D(\lambda,\Delta)]\frac{\sin(\lambda\pi)}{\cos(\lambda\pi)-\sin(\lambda\pi)}\right\}\\ &-\frac{1}{2\lambda^2}\left\{[1-A(\lambda,\Delta)-B(\lambda,\Delta)+C(\lambda,\Delta)-D(\lambda,\Delta)]^2\frac{\cos^2(\lambda\pi)}{(\cos(\lambda\pi)-\sin(\lambda\pi))^2}\right.\\ &+[1+A(\lambda,\Delta)-B(\lambda,\Delta)+C(\lambda,\Delta)+D(\lambda,\Delta)]^2\frac{\sin^2(\lambda\pi)}{(\cos(\lambda\pi)-\sin(\lambda\pi))^2}\\ &+2[1-A(\lambda,\Delta)-B(\lambda,\Delta)+C(\lambda,\Delta)-D(\lambda,\Delta)][1+A(\lambda,\Delta)-B(\lambda,\Delta)\\ &\left.+C(\lambda,\Delta)+D(\lambda,\Delta)]\frac{\cos(\lambda\pi)\sin(\lambda\pi)}{(\cos(\lambda\pi)-\sin(\lambda\pi))^2}\right\}+O\left(\frac{1}{\lambda^3}\right).\end{aligned}$$

It is well know that the eigenvalues of the problem (1)-(5) from a sequence $\lambda_n = n+\frac{1}{4}+O\left(\frac{1}{n}\right)$. We continue making λ_n more precise. Using the residue theorem, we have

$$\begin{aligned}\lambda_n-\mu_n &= -\frac{1}{2\pi i}\int\limits_{\gamma_n}\ln\frac{\omega(\lambda)}{\omega_0(\lambda)}d\lambda\\ &= \frac{1}{2\pi i}\int\limits_{\gamma_n}\left[1-A(\lambda,\Delta)-B(\lambda,\Delta)+C(\lambda,\Delta)-D(\lambda,\Delta)\right]\frac{\cos(\lambda\pi)}{\lambda(\cos(\lambda\pi)-\sin(\lambda\pi))}d\lambda\\ &\quad+\frac{1}{2\pi i}\int\limits_{\gamma_n}\left[1+A(\lambda,\Delta)-B(\lambda,\Delta)+C(\lambda,\Delta)+D(\lambda,\Delta)\right]\frac{\sin(\lambda\pi)}{\lambda(\cos(\lambda\pi)-\sin(\lambda\pi))}d\lambda\\ &\quad+O\left(\frac{1}{n^2}\right)\\ &= -\frac{1}{\left(n+\frac{1}{4}\right)\pi}\left\{1-B(n,\Delta)+C(n,\Delta)\right\}+O\left(\frac{1}{n^2}\right),\end{aligned}$$

thus we have

$$\lambda_n = n+\frac{1}{4}-\frac{1}{\left(n+\frac{1}{4}\right)\pi}\left\{1-B(n,\Delta)+C(n,\Delta)\right\}+O\left(\frac{1}{n^2}\right).$$

The proof of Theorem 3.1 is complete. □

Theorem 3.2. *The following formula of the regularized trace for the problem (1)-(5) holds:*

$$\begin{aligned}\sum_{n=0}^{\infty}\Big\{&\lambda_n^2-\mu_n^2+\frac{2}{\pi}\left(1-B(n,\Delta)+C(n,\Delta)\right)-\frac{2}{\pi}\frac{1}{\left(n+\frac{1}{4}\right)}\left(1-B(n,\Delta)+C(n,\Delta)\right)\times\\ &\left(A(n,\Delta)+D(n,\Delta)\right)\Big\}\\ &=\left(1-A(0,\Delta)-B(0,\Delta)+C(0,\Delta)-D(0,\Delta)\right)^2,\end{aligned}\tag{21}$$

where $A(\lambda,\Delta)$, $B(\lambda,\Delta)$, $C(\lambda,\Delta)$ and $D(\lambda,\Delta)$ are given by (16).

Proof. Denote by Γ_{N_0}, the counterclockwise square contours EFGH with $E=\frac{3}{4}+N_0-\left(\frac{1}{2}+N_0\right)i$, $F=\frac{3}{4}+N_0+\left(\frac{1}{2}+N_0\right)i$, $G=-\frac{1}{4}-N_0+\left(\frac{1}{2}+N_0\right)i$, $H=-\frac{1}{4}-N_0-\left(\frac{1}{2}+N_0\right)i$ and N_0 is a naturel number. Asymptotic formula of λ_n implies that for all sufficiently large N_0, the numbers λ_n, with $|n|\le N_0$, are inside Γ_{N_0}, and the numbers λ_n, with $|n|>N_0$, are outside Γ_{N_0}. Obviously, μ_n do not lie on the contour Γ_{N_0}. It follows that

$$\begin{aligned}&\sum_{\Gamma_{N_0}}\left(\lambda_n^2-\mu_n^2\right) = -\frac{1}{2\pi i}\int\limits_{\Gamma_{N_0}}2\lambda\ln\frac{\omega(\lambda)}{\omega_0(\lambda)}d\lambda\\ &=\frac{1}{2\pi i}\int\limits_{\Gamma_{N_0}}2\left[1-A(\lambda,\Delta)-B(\lambda,\Delta)+C(\lambda,\Delta)-D(\lambda,\Delta)\right]\frac{\cos(\lambda\pi)}{\cos(\lambda\pi)-\sin(\lambda\pi)}d\lambda\\ &\quad+\frac{1}{2\pi i}\int\limits_{\Gamma_{N_0}}2\left[1+A(\lambda,\Delta)-B(\lambda,\Delta)+C(\lambda,\Delta)+D(\lambda,\Delta)\right]\frac{\sin(\lambda\pi)}{\cos(\lambda\pi)-\sin(\lambda\pi)}d\lambda\\ &\quad+\frac{1}{2\pi i}\int\limits_{\Gamma_{N_0}}\left[1-A(\lambda,\Delta)-B(\lambda,\Delta)+C(\lambda,\Delta)-D(\lambda,\Delta)\right]^2\frac{\cos^2(\lambda\pi)}{\lambda(\cos(\lambda\pi)-\sin(\lambda\pi))^2}d\lambda\\ &\quad+\frac{1}{2\pi i}\int\limits_{\Gamma_{N_0}}\left[1+A(\lambda,\Delta)-B(\lambda,\Delta)+C(\lambda,\Delta)+D(\lambda,\Delta)\right]^2\frac{\sin^2(\lambda\pi)}{\lambda(\cos(\lambda\pi)-\sin(\lambda\pi))^2}d\lambda\\ &\quad+\frac{1}{2\pi i}\int\limits_{\Gamma_{N_0}}2\left[1-A(\lambda,\Delta)-B(\lambda,\Delta)+C(\lambda,\Delta)-D(\lambda,\Delta)\right]\left[1+A(\lambda,\Delta)-B(\lambda,\Delta)\right.\\ &\quad\left.+C(\lambda,\Delta)+D(\lambda,\Delta)\right]\frac{\cos(\lambda\pi)\sin(\lambda\pi)}{\lambda(\cos(\lambda\pi)-\sin(\lambda\pi))}d\lambda+O\left(\frac{1}{N_0}\right).\end{aligned}$$

By residue calculations, we get

$$\begin{aligned}\sum_{n=0}^{N_0}\left(\lambda_n^2-\mu_n^2\right) &= -\frac{1}{\pi}\sum_{n=0}^{N_0}\left[1-A(n,\Delta)-B\left(n,\Delta\right)+C\left(n,\Delta\right)-D\left(n,\Delta\right)\right]\\ &-\frac{1}{\pi}\sum_{n=0}^{N_0}\left[1+A(n,\Delta)-B\left(n,\Delta\right)+C\left(n,\Delta\right)+D\left(n,\Delta\right)\right]\\ &-\frac{2}{\pi}\sum_{n=0}^{N_0}\left[1-A(n,\Delta)-B\left(n,\Delta\right)+C\left(n,\Delta\right)-D\left(n,\Delta\right)\right]^2\tfrac{1}{4n+1}\\ &+\frac{2}{\pi}\sum_{n=0}^{N_0}\left[1+A(n,\Delta)-B\left(n,\Delta\right)+C\left(n,\Delta\right)+D\left(n,\Delta\right)\right]^2\tfrac{1}{4n+1}\\ &+\left[1-A(0,\Delta)-B\left(0,\Delta\right)+C\left(0,\Delta\right)-D\left(0,\Delta\right)\right]^2+O\left(\frac{1}{N_0}\right),\end{aligned}$$

which implies that

$$\begin{aligned}&\sum_{n=0}^{N_0}\left\{\lambda_n^2-\mu_n^2+\frac{2}{\pi}\left(1-B\left(n,\Delta\right)+C\left(n,\Delta\right)\right)-\frac{2}{\pi}\tfrac{1}{\left(n+\frac{1}{4}\right)}\left(1-B\left(n,\Delta\right)+C\left(n,\Delta\right)\right)\times\right.\\ &\quad\left.\left(A\left(n,\Delta\right)+D\left(n,\Delta\right)\right)\right\}\\ &\quad=\left(1-A\left(0,\Delta\right)-B\left(0,\Delta\right)+C\left(0,\Delta\right)-D\left(0,\Delta\right)\right)^2+O\left(\frac{1}{N_0}\right).\end{aligned} \tag{22}$$

Passing limit as $N_0\to\infty$ in (22), we get the regularized trace formula (21). □

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