ON GENERALIZED INTERVAL VALUED FUZZY QUASI-IDEALS OF SEMIGROUPS

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ABSTRACT. In this paper, we give the concepts of generalized interval valued fuzzy subsemigroups, which are generalizations of the notion of interval valued fuzzy subsemigroups and of $(\overline{\alpha}, \overline{\beta})$-interval valued fuzzy subsemigroups, where $\overline{\alpha} \prec \overline{\beta}$. Also we prove some of related properties of generalized interval valued fuzzy quasi-ideals on semigroups. In the main results, we characterize regular and intra-regular semigroups in terms of generalized interval valued fuzzy ideal and quasi-ideals.

Keywords: Generalized interval valued fuzzy subsemigroup, intra-regular semigroup regular semigroup.

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1. INTRODUCTION

An interval valued fuzzy subset of T is a function $\overline{f} : T \to C[0,1]$, where $C[0,1]$ is the set of all closed subintervals of $[0,1]$. The theory of interval valued fuzzy subsets has been discovered by Zadeh in 1975, [20]. Many papers on interval valued fuzzy subsets have appeared in many areas such as medical science [7], image processing [5], decision making [21] etc. In 1994, Biswas [6] initiated interval valued fuzzy subsets in algebraic structure. In 2006, Narayanan and Manikantan [12] studied and discussed the concept of interval valued fuzzy subsemigroups and kinds of interval value fuzzy ideals on semigroups. In 2012, Huret et al. [10] gave the concepts of interval valued fuzzy quasi-ideals on semigroups and they studied of its properties. In 2013, Singaram and Kandasamy [15] characterized regular and intra-regular semigroup in terms of interval valued fuzzy ideals. Later in 2014, Abdullah et al. [3] extended the concepts of interval valued fuzzy ideals to $(\overline{\alpha}, \overline{\beta})$-interval valued fuzzy ideals where $\overline{\alpha} \prec \overline{\beta}$, and they characterized regular semigroups in terms of $(\overline{\alpha}, \overline{\beta})$-interval valued fuzzy ideals. Moreover, the concepts of interval valued fuzzy subsets has been studied in other research and fields such as interval valued intuitionistic fuzzy ideals on LA-semigroups [16], interval valued intuitionistic $(\overline{S}, \overline{T})$-fuzzy ideals in ternary semigroups [17], interval valued $(\overline{\theta}, \overline{\delta})$-fuzzy KU-ideals of KU-algebras [4], etc.

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In this article, we provide some works as follows: In Section 2, we give basic definitions and some results which will be used in next sections. In Section 3, we give the concepts of generalized interval valued fuzzy subsemigroups which are generalizations of the concepts of $(\overline{\alpha}, \overline{\beta})$-interval valued fuzzy subsemigroups defined by [3]. In Section 4, we discuss some interesting properties of generalized interval valued fuzzy quasi-ideals on semigroups. In Section 5, we characterize regular and intra-regular semigroups in terms of generalized interval valued fuzzy quasi-ideals.

2. PRELIMINARIES

In this topic, we review some definitions and results which are used in the next section.

By a subsemigroup of a semigroup S we mean a non-empty subset K of S such that $K^2 \subseteq K$. A non-empty subset K of a semigroup S is called a *left (right) ideal* of S if $SK \subseteq K$ ($KS \subseteq K$). K is called an *ideal* if it is both a left ideal and a right ideal. A non-empty subset K of a semigroup S is called a *generalized bi-ideal* of S if $KSK \subseteq K$. A subsemigroup K of a semigroup S is called a *bi-ideal* of S if $KSK \subseteq K$. A non-empty subset K of S is called a *quasi-ideal* of S if $KS \cap SK \subseteq K$. We note here that every bi-ideal of a semigroup S is a generalized bi-ideal of S and every quasi-ideal of S is a bi-ideal of S. [11].

Definition 2.1. [19] A *fuzzy subset* f of a non-empty set T is a function $T \to [0, 1]$.

For any $\eta_1, \eta_2 \in [0, 1]$, we have

$$\eta_1 \vee \eta_2 = \max\{\eta_1, \eta_2\} \quad \text{and} \quad \eta_1 \wedge \eta_2 = \min\{\eta_1, \eta_2\}.$$

More generally, if $\{\eta_i : i \in \mathcal{J}\}$ is a collection of fuzzy sets of T, then

$$\bigvee_{i \in \mathcal{J}} \eta_i := \sup_{i \in \mathcal{J}}\{\eta_i\} \quad \text{and} \quad \bigwedge_{i \in \mathcal{J}} \eta_i := \inf_{i \in \mathcal{J}}\{\eta_i\}.$$

The set of all closed subintervals of $[0, 1]$ is denoted by $C[0, 1]$, that is

$$C[0, 1] = \{\overline{\eta} = [\eta^-, \eta^+] \mid 0 \le \eta^- \le \eta^+ \le 1\}.$$

Note that $[\eta, \eta] = \{\eta\}$ for all $\eta \in [0, 1]$. For $\eta = 0$ or 1 we shall denote $[0, 0]$ by $\overline{0}$ and $[1, 1]$ by $\overline{1}$.

Definition 2.2. [18] For each interval $\overline{\eta} = [\eta^-, \eta^+]$ and $\overline{\vartheta} = [\vartheta^-, \vartheta^+]$ in $C[0, 1]$, define the operations $\preceq, =, \curlywedge$ and $\curlyvee$ as follows:

    (1) $\overline{\eta} \preceq \overline{\vartheta}$ if and only if $\eta^- \le \vartheta^-$ and $\eta^+ \le \vartheta^+$,
    (2) $\overline{\eta} = \overline{\vartheta}$ if and only if $\eta^- = \vartheta^-$ and $\eta^+ = \vartheta^+$,
    (3) $\overline{\eta} \curlywedge \overline{\vartheta} = [(\eta^- \wedge \vartheta^-), (\eta^+ \wedge \vartheta^+)]$,
    (4) $\overline{\eta} \curlyvee \overline{\vartheta} = [(\eta^- \vee \vartheta^-), (\eta^+ \vee \vartheta^+)]$.

We write $\overline{\eta} \succeq \overline{\vartheta}$ whenever $\overline{\vartheta} \preceq \overline{\eta}$.

Proposition 2.3. [8] *For $\overline{\eta}, \overline{\vartheta}, \overline{\omega} \in C[0, 1]$, the following properties are true:*

    (1) $\overline{\eta} \curlywedge \overline{\eta} = \overline{\eta}$ *and* $\overline{\eta} \curlyvee \overline{\eta} = \overline{\eta}$,
    (2) $\overline{\eta} \curlywedge \overline{\vartheta} = \overline{\vartheta} \curlywedge \overline{\eta}$ *and* $\overline{\eta} \curlyvee \overline{\vartheta} = \overline{\vartheta} \curlyvee \overline{\eta}$,
    (3) $(\overline{\eta} \curlywedge \overline{\vartheta}) \curlywedge \overline{\omega} = \overline{\eta} \curlywedge (\overline{\vartheta} \curlywedge \overline{\omega})$ *and* $(\overline{\eta} \curlyvee \overline{\vartheta}) \curlyvee \overline{\omega} = \overline{\eta} \curlyvee (\overline{\vartheta} \curlyvee \overline{\omega})$,
    (4) $(\overline{\eta} \curlywedge \overline{\vartheta}) \curlyvee \overline{\omega} = (\overline{\eta} \curlyvee \overline{\omega}) \curlywedge (\overline{\vartheta} \curlyvee \overline{r})$ *and* $(\overline{\eta} \curlyvee \overline{\vartheta}) \curlywedge \overline{\omega} = (\overline{\eta} \curlywedge \overline{\omega}) \curlyvee (\overline{\vartheta} \curlywedge \overline{\omega})$,
    (5) *If* $\overline{\eta} \preceq \overline{\vartheta}$, *then* $\overline{\eta} \curlywedge \overline{\omega} \preceq \overline{\vartheta} \curlywedge \overline{\omega}$ *and* $\overline{\eta} \curlyvee \overline{\omega} \preceq \overline{\vartheta} \curlyvee \overline{\omega}$.

Definition 2.4. [15] For each interval $\overline{\eta}_i = [\eta_i^-, \eta_i^+] \in C[0,1]$, $i \in \mathcal{J}$ where $\mathcal{J}$ is an index set, define

$$\underset{i \in \mathcal{J}}{\curlywedge} \overline{\eta}_i = [\underset{i \in \mathcal{J}}{\wedge} \eta_i^-, \underset{i \in \mathcal{J}}{\wedge} \eta_i^+] \quad \text{and} \quad \underset{i \in \mathcal{J}}{\curlyvee} \overline{\eta}_i = [\underset{i \in \mathcal{J}}{\vee} \eta_i^-, \underset{i \in \mathcal{J}}{\vee} \eta_i^+].$$

Definition 2.5. [18] Let T be a non-empty set. Then the function $\overline{f} : T \to C[0,1]$ is called an *interval valued fuzzy subset* (shortly, IVF subset) of T.

Definition 2.6. [15] For a non-empty subset K of a set T, the *interval valued characteristic function* of K is defined to be a function $\overline{\lambda}_K : T \to C[0,1]$ by

$$\overline{\lambda}_K(u) = \begin{cases} \overline{1} & \text{if} \quad u \in K, \\ \overline{0} & \text{if} \quad u \notin K \end{cases}$$

for all $u \in T$.

Definition 2.7. [8] For two IVF subsets $\overline{f}$ and $\overline{g}$ in a semigroup S, define
(1) $\overline{f} \sqsubseteq \overline{g} \Leftrightarrow \overline{f}(u) \preceq \overline{g}(u)$ for all $u \in S$,
(2) $\overline{f} = \overline{g} \Leftrightarrow \overline{f} \sqsubseteq \overline{g}$ and $\overline{g} \sqsubseteq \overline{f}$,
(3) $(\overline{f} \sqcap \overline{g})(u) = \overline{f}(u) \curlywedge \overline{g}(u)$ for all $u \in S$.

Definition 2.8. [15] For two IVF subsets $\overline{f}$ and $\overline{g}$ in a semigroup S, define the product $\overline{f} \circ \overline{g}$ as follows : for all $u \in S$,

$$(\overline{f} \circ \overline{g})(u) = \begin{cases} \underset{(x,y) \in F_u}{\curlyvee} \{\overline{f}(x) \curlywedge \overline{g}(y)\} & \text{if} \quad F_u \neq \emptyset, \\ \overline{0} & \text{if} \quad F_u = \emptyset, \end{cases}$$

$F_u := \{(x,y) \in S \times S \mid u = xy\}$.

The following definitions are types of IVF subsemigroups on semigroup.

Definition 2.9. [15, 12, 10] An IVF subset $\overline{f}$ of a semigroup S is said to be
(1) an *IVF subsemigroup* of S if $\overline{f}(uv) \succeq \overline{f}(u) \curlywedge \overline{f}(v)$ for all $u, v \in S$,
(2) an *IVF left (right) ideal* of S if $\overline{f}(uv) \succeq \overline{f}(v)$ $(\overline{f}(uv) \succeq \overline{f}(u))$ for all $u, v \in S$,
(3) an *IVF ideal* of S if it is both an IVF left ideal and an IVF right ideal of S,
(4) an *IVF generalized bi-ideal* of S if $\overline{f}(uvw) \succeq \overline{f}(u) \curlywedge \overline{f}(w)$ for all $u, v, w \in S$,
(5) an *IVF bi-ideal* of S if $\overline{f}$ is an IVF subsemigroup of S and $\overline{f}(uvw) \succeq \overline{f}(u) \curlywedge \overline{f}(w)$ for all $u, v, w \in S$,
(6) an *IVF quasi-ideal* of S if $(\overline{S} \circ \overline{f})(u) \curlywedge (\overline{f} \circ \overline{S})(x) \preceq \overline{f}(u)$ for all $u \in S$ where $\overline{S}$ is an IVF subset of S mapping every element of S on $\overline{1}$.

The concept of an $(\overline{\alpha}, \overline{\beta})$-IVF subsemigroup, $\overline{\alpha} \prec \overline{\beta}$, defined by Abdullah et al. [3] as follows:

Definition 2.10. [3] An IVF subset $\overline{f}$ of a semigroup S is said to be an $(\overline{\alpha}, \overline{\beta})$-*IVF subsemigroup* of S, where $\overline{\alpha} \prec \overline{\beta}$ and $\overline{\alpha}, \overline{\beta} \in C[0,1]$ if $\overline{f}(uv) \curlyvee \overline{\alpha} \succeq \overline{f}(u) \curlywedge \overline{f}(v) \curlywedge \overline{\beta}$ for all $u, v \in S$.

3. Generalized Interval valued fuzzy subsemigroup

From the definition 2.10 defined by [3], they used the condition $\overline{\alpha} \prec \overline{\beta}$ which are arbitrary elements in $C[0,1]$. In this topic, we shall give concepts of generalized interval valued fuzzy subsemigroups which are generalizations of the concepts of $(\overline{\alpha}, \overline{\beta})$-interval valued fuzzy subsemigroups defined by [3]. Later, we shall discuss important properties for reference in the next topic.

Definition 3.1. An IVF subset $\overline{f}$ of a semigroup S is said to be a $(\overline{s}, \overline{t})$-*IVF subsemigroup* of S, where $\overline{s}, \overline{t} \in C[0,1]$ if $\overline{f}(uv) \curlyvee \overline{s} \succeq \overline{f}(u) \curlywedge \overline{f}(v) \curlywedge \overline{t}$ for all $u, v \in S$.

We note here that every IVF subsemigroup is a $(\overline{0}, \overline{1})$-IVF subsemigroup of a semigroup.

Now we give an example satisfy Definition 3.1 but not satisfy Definition 2.9(1) and Definition 2.10.

Example 3.2. Let S be a semigroup defined by the following table:

·	a	b	c
a	a	b	c
b	b	b	c
c	c	c	b

Define IVF subset $\overline{f}$ of S as follows:

$$\overline{f}(u) = \begin{cases} [0.1, 0.5] & \text{for} \quad u = a, \\ [0.2, 0.4] & \text{for} \quad u = b, \\ [0.3, 0.5] & \text{for} \quad u = c \end{cases}$$

and let $\overline{s} = [0.4, 0.6]$, $\overline{t} = [0.3, 0.7]$. Then $\overline{f}$ is a $(\overline{s}, \overline{t})$-IVF subsemigroup of S. But the $\overline{f}$ is not an IVF subsemigroup of S, because $\overline{f}(c^2) = \overline{f}(b) = [0.2, 0.4] \not\succeq [0.3, 0.5] = \overline{f}(c) \curlywedge \overline{f}(c)$. And if $\overline{\alpha} = [0.1, 0.2]$ and $\overline{\beta} = [0.3, 0.4]$, then $\overline{f}(c^2) = \overline{f}(b) = [0.2, 0.4] \not\succeq [0.3, 0.4] = \overline{f}(c) \curlywedge \overline{f}(c) \curlywedge \overline{\beta}$. Thus $\overline{f}$ is not an $(\overline{\alpha}, \overline{\beta})$-IVF subsemigroup of S.

The following definitions of types generalized IVF ideals on semigroups.

Definition 3.3. Let $\overline{s}, \overline{t} \in C[0,1]$. An IVF subset $\overline{f}$ of a semigroup S is called

(1) a $(\overline{s}, \overline{t})$-*IVF left (right) ideal* of S if $\overline{f}(uv) \curlyvee \overline{s} \succeq \overline{f}(v) \curlywedge \overline{t}$ $(\overline{f}(uv) \curlyvee \overline{s} \succeq \overline{f}(u) \curlywedge \overline{t})$ for all $u, v \in S$,
(2) a $(\overline{s}, \overline{t})$-*IVF ideal* of S if it is both a $(\overline{s}, \overline{t})$-IVF left ideal and a $(\overline{s}, \overline{t})$-IVF right ideal of S,
(3) a $(\overline{s}, \overline{t})$-*IVF generalized bi-ideal* of S if $\overline{f}(uvw) \curlyvee \overline{s} \succeq \overline{f}(u) \curlywedge \overline{f}(w) \curlywedge \overline{t}$ for all $u, v, w \in S$,
(4) a $(\overline{s}, \overline{t})$-*IVF bi-ideal* of S if $\overline{f}$ is a $(\overline{s}, \overline{t})$-IVF subsemigroup of S and $\overline{f}(uvw) \curlyvee \overline{s} \succeq \overline{f}(u) \curlywedge \overline{f}(w) \curlywedge \overline{t}$ for all $u, v, w \in S$,

Remark 3.4. It is clear that every $(\overline{s}, \overline{t})$-IVF ideal of S is a $(\overline{s}, \overline{t})$-IVF bi-ideal of S and every $(\overline{s}, \overline{t})$-IVF bi-ideal of S is a $(\overline{s}, \overline{t})$-IVF generalized bi-ideal of S.

The following theorem can be easily proved.

Theorem 3.5. *Let S be a semigroup. Then the following statements holds:*

(1) *If K is a subsemigroup of a semigroup S, then the interval valued characteristic function $\overline{\lambda}_K$ is a $(\overline{s}, \overline{t})$-IVF subsemigroup of S for all $\overline{s}, \overline{t} \in C[0,1]$,*
(2) *If K is a left (right) ideal of a semigroup S, then the characteristic function $\overline{\lambda}_K$ is a $(\overline{s}, \overline{t})$-IVF left (right) ideal of S for all $\overline{s}, \overline{t} \in C[0,1]$,*
(3) *Let K be a non-empty subset of a semigroup S and let $\overline{s}, \overline{t} \in C[0,1]$ such that $\overline{s} \preceq \overline{t}$. Then K is a subsemigroup (left ideal, right ideal) of S if and only if $\overline{\lambda}_K$ is a $(\overline{s}, \overline{t})$-IVF subsemigroup (left ideal, right ideal) of S.*

4. Generalized Interval valued fuzzy quasi-ideal in semigroups

In this topic, we give the concept of generalized IVF quasi-ideals in semigroups which are generalizations of the concepts of $(\overline{\alpha}, \overline{\beta})$-IVF quasi-ideals defined by [3]. Also we study properties of generalized IVF quasi-ideals on semigroups.

Definition 4.1. An IVF subset $\overline{f}$ of a semigroup S is called a $(\overline{s}, \overline{t})$- IVF quasi-ideal of S, where $\overline{s}, \overline{t} \in C[0,1]$ if $\overline{f}(u) \curlyvee \overline{s} \succeq (\overline{S} \circ \overline{f})(u) \curlywedge (\overline{f} \circ \overline{S})(u) \curlywedge \overline{t}$ for all $u \in S$.

The following example is a $(\overline{s}, \overline{t})$-IVF quasi-ideal of a semigroup.

Example 4.2. Let S be semigroup given by the following table.

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	b
d	a	a	b	c

Define IVF subset $\overline{f}$ of S as follows:

$$\overline{f}(u) = \begin{cases} [0.7, 0.8] & \text{for} \quad u = a, \\ [0.4, 0.5] & \text{for} \quad u = b, \\ [0.6, 0.7] & \text{for} \quad u = c, \\ \overline{0} & \text{for} \quad u = d \end{cases}$$

and let $\overline{s} = [0.2, 0.5]$, $\overline{t} = [0.1, 0.6]$. We can check that $\overline{f}$ is a $(\overline{s}, \overline{t})$-IVF quasi-ideal.

The following theorem we present that every $(\overline{s}, \overline{t})$-IVF quasi ideal is a $(\overline{s}, \overline{t})$-IVF subsemigroup.

Theorem 4.3. *Every $(\overline{s}, \overline{t})$-IVF quasi-ideal of a semigroup S is a $(\overline{s}, \overline{t})$-IVF subsemigroup of S.*

Proof. Assume that $\overline{f}$ is a $(\overline{s}, \overline{t})$-IVF quasi-ideal of S and let $u, v \in S$. Then

$$\begin{aligned} \overline{f}(uv) \curlyvee \overline{s} \; & \succeq \; (\overline{S} \circ \overline{f})(uv) \curlywedge (\overline{f} \circ \overline{S})(uv) \curlywedge \overline{t} \\ & = \; \underset{(a,b) \in F_{uv}}{\curlyvee} \{\overline{f}(a) \curlywedge \overline{S}(b)\} \curlywedge \underset{(p,q) \in F_{uv}}{\curlyvee} \{\overline{S}(p) \curlywedge \overline{f}(q)\} \curlywedge \overline{t} \\ & \succeq \; (\overline{f}(u) \curlywedge \overline{S}(v)) \curlywedge (\overline{S}(u) \curlywedge \overline{f}(v)) \curlywedge \overline{t} \\ & = \; (\overline{f}(u) \curlywedge \overline{1}) \curlywedge (\overline{1} \curlywedge \overline{f}(v)) \curlywedge \overline{t} = \overline{f}(u) \curlywedge \overline{f}(v) \curlywedge \overline{t}. \end{aligned}$$

Thus, $\overline{f}(uv) \curlyvee \overline{s} \succeq \overline{f}(u) \curlywedge \overline{f}(v) \curlywedge \overline{t}$. Hence $\overline{f}$ is a $(\overline{s}, \overline{t})$-IVF subsemigroup of S. $\square$

The following theorem we will study relation of a $(\overline{s}, \overline{t})$-IVF quasi-ideal and a $(\overline{s}, \overline{t})$-generalized bi-ideal on semigroup.

Theorem 4.4. *Every $(\overline{s}, \overline{t})$-IVF quasi-ideal of a semigroup S is a $(\overline{s}, \overline{t})$-IVF generalized bi-ideal of S.*

Proof. Assume that $\overline{f}$ is a $(\overline{s}, \overline{t})$-IVF quasi-ideal of S and let $u, v, w \in S$. Then

$$\begin{aligned} \overline{f}(uvw) \curlyvee \overline{s} \; & \succeq \; (\overline{f} \circ \overline{S})(uvw) \curlywedge (\overline{S} \circ \overline{f})(uvw) \curlywedge \overline{t} \\ & = \; \underset{(a,b) \in F_{uvw}}{\curlyvee} \{\overline{f}(a) \curlywedge \overline{S}(b)\} \curlywedge \underset{(i,j) \in F_{uvw}}{\curlyvee} \{\overline{S}(i) \curlywedge \overline{f}(j)\} \curlywedge \overline{t} \\ & \succeq \; \overline{f}(u) \curlywedge \overline{S}(vw) \curlywedge \overline{S}(uv) \curlywedge \overline{f}(w) \curlywedge \overline{t} \\ & = \; (\overline{f}(u) \curlywedge \overline{1}) \curlywedge (\overline{1} \curlywedge \overline{f}(w)) \curlywedge \overline{t} = \overline{f}(u) \curlywedge \overline{f}(w) \curlywedge \overline{t}. \end{aligned}$$

Thus we obtain that $\overline{f}(uvw) \curlyvee \overline{s} \succeq \overline{f}(u) \curlywedge \overline{f}(w) \curlywedge \overline{t}$.
Hence $\overline{f}$ is a $(\overline{s}, \overline{t})$-IVF generalized bi-ideal of S. $\square$

The following example is a $(\overline{s}, \overline{t})$-IVF generalized bi-ideal of a semigroup S. But the converse of Theorem 4.4 is not true.

Example 4.5. Consider a semigroup $(S, \cdot)$ defined by the following table:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Define IVF subset $\overline{f}$ of S as follows:

$$\overline{f}(u) = \begin{cases} [0.8, 0.9], & \text{for} \quad u = a, \\ \overline{0} & \text{for} \quad u = b, \\ \overline{0.7} & \text{for} \quad u = c, \\ [0.4, 0.5] & \text{for} \quad u = d \end{cases}$$

and let $\overline{s} = [0.1, 0.8]$, $\overline{t} = [0.2, 0.7]$. By routine calculation, $\overline{f}$ is a $(\overline{s}, \overline{t})$-IVF generalized bi-ideal. But $\overline{f}$ is not a $(\overline{s}, \overline{t})$-IVF quasi-ideal, since $\overline{f}(b) \curlyvee \overline{s} = [0.1, 0.8] \not\succeq [0.2, 0.5] = \underset{(d,c) \in F_b}{\curlyvee} \{\overline{f}(d) \curlywedge \overline{\mathcal{S}}(c)\} \curlywedge \underset{(d,c) \in F_b}{\curlyvee} \{\overline{\mathcal{S}}(d) \curlywedge \overline{f}(c)\} \curlywedge \overline{t} = (\overline{\mathcal{S}} \circ \overline{f})(b) \curlywedge (\overline{f} \circ \overline{\mathcal{S}})(b) \curlywedge \overline{t}$.

The following result is an immediate consequence of Theorem 4.3 and Theorem 4.4.

Theorem 4.6. *Every $(\overline{s}, \overline{t})$-IVF quasi-ideal of a semigroup S is a $(\overline{s}, \overline{t})$-IVF bi-ideal of S.*

The following theorem we will study relation of a $(\overline{s}, \overline{t})$-IVF ideal and a $(\overline{s}, \overline{t})$-IVF quasi-ideal on semigroup.

Theorem 4.7. *Every $(\overline{s}, \overline{t})$-IVF left (right) ideal of a semigroup S is a $(\overline{s}, \overline{t})$-IVF quasi-ideal of S.*

Proof. Suppose that $\overline{f}$ is a $(\overline{s}, \overline{t})$-IVF left ideal of S and let $u \in S$.
If $F_u = \emptyset$, then it is easy to verify that $(\overline{S} \circ \overline{f})(u) \curlywedge \overline{t} \preceq \overline{f}(u) \curlyvee \overline{s}$.
If $F_u \neq \emptyset$, then

$$\begin{aligned} (\overline{S} \circ \overline{f})(u) \curlywedge \overline{t} &= \underset{(i,j) \in F_u}{\curlyvee} \{\overline{S}(i) \curlywedge \overline{f}(j)\} \curlywedge \overline{t} = \underset{(i,j) \in F_u}{\curlyvee} \{\overline{S}(i) \curlywedge \overline{f}(j) \curlywedge \overline{t}\} \\ &= \underset{(i,j) \in F_u}{\curlyvee} \{\overline{1} \curlywedge \overline{f}(j) \curlywedge \overline{t}\} = \underset{(i,j) \in F_u}{\curlyvee} \{\overline{f}(j)\} \curlywedge \overline{t} \\ &\preceq \underset{(i,j) \in F_u}{\curlyvee} \{\overline{f}(ij)\} \curlyvee \overline{s} \preceq \overline{f}(u) \curlyvee \overline{s}. \end{aligned}$$

Thus, we get that $(\overline{S} \circ \overline{f})(u) \curlywedge \overline{t} \preceq \overline{f}(u) \curlyvee \overline{s}$ and hence $\overline{f}(u) \curlyvee \overline{s} \succeq (\overline{S} \circ \overline{f})(u) \succeq (\overline{S} \circ \overline{f})(u) \curlywedge (\overline{f} \circ \overline{S})(u) \curlywedge \overline{t}$. Therefore $\overline{f}$ is a $(\overline{s}, \overline{t})$-IVF quasi-ideal of S. $\qquad\square$

The next theorem present that the intersection of two $(\overline{s}, \overline{t})$-IVF quasi-ideals of a semigroup is also a $(\overline{s}, \overline{t})$-IVF quasi-ideal of a semigroup. The proof is straightforward.

Theorem 4.8. *If $\overline{f}$ and $\overline{g}$ are two $(\overline{s}, \overline{t})$-IVF quasi-ideals of a semigroup, then $\overline{f} \sqcap \overline{g}$ is also a $(\overline{s}, \overline{t})$-IVF quasi-ideal.*

In the following theorems, we give a relationship between quasi-ideals of a semigroup S and the interval valued characteristic function.

Theorem 4.9. *If K is a quasi-ideal of a semigroup S, then the characteristic function $\overline{\lambda}_K$ is a $(\overline{s}, \overline{t})$-IVF quasi-ideal of S.*

Proof. Let $u \in S$ and let $\overline{s}, \overline{t} \in C[0,1]$.

If $u \in K$ or $F_u = \emptyset$, it is clear that $\overline{\lambda}_K(u) \curlyvee \overline{s} \succeq (\overline{\lambda}_K \circ \overline{\mathcal{S}})(u) \curlywedge (\overline{\mathcal{S}} \circ \overline{\lambda}_K)(u) \curlywedge \overline{t}$.

Assume that $u \notin K$ and $F_u \neq \emptyset$. Let $A := \{(a,b) \mid u = ab$ and $(a \notin K$ or $b \notin K)\}$. Thus $A \subseteq F_u$.

On the other hand if $(i,j) \in F_u$, then $u = ij \notin K$ which implies that $ij \notin KS \cap KS$ because K is a quasi-ideal of S. Thus $i \notin K$ or $j \notin K$, and so $(i,j) \in A$. Hence $F_u \subseteq A$. Therefore $A = F_u$. That is, $F_u = \{(i,j) \mid i \notin K$ or $j \notin K\}$. Thus

$$
\begin{aligned}
(\overline{\lambda}_K \circ \overline{\mathcal{S}})(u) \curlywedge (\overline{\mathcal{S}} \circ \overline{\lambda}_K)(u) \curlywedge \overline{t} &= \underset{(i,j)\in F_u}{\curlyvee}\{\overline{\lambda}_K(i) \curlywedge \overline{\mathcal{S}}(j)\} \curlywedge \underset{(i,j)\in F_u}{\curlyvee}\{\overline{\mathcal{S}}(i) \curlywedge \overline{\lambda}_K(j)\} \curlywedge \overline{t} \\
&= \underset{(i,j)\in F_u}{\curlyvee}\{\overline{\lambda}_K(i) \curlywedge \overline{\lambda}_K(j)\} \curlywedge \overline{t} = \overline{0}.
\end{aligned}
$$

Therefore, $\overline{\lambda}_K(u) \curlyvee \overline{s} \succeq (\overline{\lambda}_K \circ \overline{\mathcal{S}})(u) \curlywedge (\overline{\mathcal{S}} \circ \overline{\lambda}_K)(u) \curlywedge \overline{t}$.
Hence $\overline{\lambda}_K$ is a $(\overline{s}, \overline{t})$-IVF quasi-ideal of S. $\square$

The converse of the Theorem 4.9 is true for $\overline{s} \preceq \overline{t}$.

Theorem 4.10. [3] *Let K be a non-empty subset of a semigroup S and let $\overline{s}, \overline{t} \in C[0,1]$ such that $\overline{s} \preceq \overline{t}$. Then K is a quasi-ideal of S if and only if $\overline{\lambda}_K$ is a $(\overline{s}, \overline{t})$-IVF quasi-ideal of S.*

5. Characterization regular and intra-regular semigroups in terms $(\overline{s}, \overline{t})$-IVF quasi-ideals and $(\overline{s}, \overline{t})$-IVF left (right) ideals

In this topic, we shall characterizations of a semigroup which is both regular and intra-regular in terms of $(\overline{s}, \overline{t})$-IVF quasi-ideals and $(\overline{s}, \overline{t})$-IVF left (right) ideals.

For any IVF subset $\overline{f}$ of a semigroup S and $\overline{s}, \overline{t} \in C[0,1]$, define

$$
\overline{f}_{(\overline{s},\overline{t})}(u) := (\overline{f}(u) \curlywedge \overline{s}) \curlyvee \overline{t}
$$

for all $u \in S$.

For any IVF subsets $\overline{f}$ and $\overline{g}$ of a semigroup S and $\overline{s}, \overline{t} \in C[0,1]$, define the operation " $\underset{(\overline{s},\overline{t})}{\sqcap}$ " as follows:

$$
(\overline{f} \underset{(\overline{s},\overline{t})}{\sqcap} \overline{g})(u) := (\overline{f}(u) \curlywedge \overline{g}(u) \curlywedge \overline{s}) \curlyvee \overline{t}
$$

for all $u \in S$. And define the product $\overline{f} \underset{(\overline{s},\overline{t})}{\circ} \overline{g}$ as follows: for all $u \in S$,

$$
(\overline{f} \underset{(\overline{s},\overline{t})}{\circ} \overline{g})(u) := ((\overline{f} \circ \overline{g})(u) \curlywedge \overline{s}) \curlyvee \overline{t},
$$

where

$$
(\overline{f} \circ \overline{g})(u) = \begin{cases} \underset{(x,y)\in F_u}{\curlyvee}\{\overline{f}(x) \curlywedge \overline{g}(y)\} & \text{if} \quad F_u \neq \emptyset, \\ \overline{0} & \text{if} \quad F_u = \emptyset, \end{cases}
$$

where $F_u := \{(x,y) \in S \times S \mid u = xy\}$.

Some equivalent conditions are important properties for $(\overline{s}, \overline{t})$-IVF subsemigroups of semigroups.

Theorem 5.1. *An IVF subset $\overline{f}$ is a $(\overline{s}, \overline{t})$-IVF subsemigroup of a semigroup S if and only if $\overline{f} \underset{(\overline{t},\overline{s})}{\circ} \overline{f} \sqsubseteq \overline{f}_{(\overline{t},\overline{s})}$.*

Proof. ($\Rightarrow$) Assume that $\overline{f}$ is a $(\overline{s}, \overline{t})$-IVF subsemigroup of a semigroup S and let $u \in S$. Then, $(\overline{f} \underset{(\overline{t},\overline{s})}{\circ} \overline{f})(u) = ((\overline{f} \circ \overline{f})(u) \curlywedge \overline{t}) \curlyvee \overline{s}$.

If $F_u = \emptyset$, then it is easy to verify that, $(\overline{f} \underset{(\overline{t},\overline{s})}{\circ} \overline{f})(u) \preceq \overline{f}_{(\overline{t},\overline{s})}(u)$.

If $F_u \neq \emptyset$, then

$$
\begin{aligned}
(\overline{f} \underset{(\overline{t},\overline{s})}{\circ} \overline{f})(u) &= ((\overline{f} \circ \overline{f})(u) \curlywedge \overline{t}) \curlyvee \overline{s} = (\underset{(x,y)\in F_u}{\curlyvee} \{\overline{f}(x) \curlywedge \overline{f}(y)\} \curlywedge \overline{t}) \curlyvee \overline{s} \\
&= (\underset{(x,y)\in F_u}{\curlyvee} \{\overline{f}(x) \curlywedge \overline{f}(y) \curlywedge \overline{t}\} \curlywedge \overline{t}) \curlyvee \overline{s} \\
&\preceq (\underset{(x,y)\in F_u}{\curlyvee} \{\overline{f}(xy) \curlyvee \overline{s}\} \curlywedge \overline{t}) \curlyvee \overline{s} \\
&= ((\overline{f}(u) \curlyvee \overline{s}) \curlywedge \overline{t}) \curlyvee \overline{s} = ((\overline{f}(u) \curlyvee \overline{s}) \curlyvee \overline{s}) \curlywedge (\overline{t} \curlyvee \overline{s}) \\
&= (\overline{f}(u) \curlyvee \overline{s}) \curlywedge (\overline{t} \curlyvee \overline{s}) = (\overline{f}(u) \curlywedge \overline{t}) \curlyvee \overline{s} = \overline{f}_{(\overline{t},\overline{s})}(u).
\end{aligned}
$$

Thus, we obtain that $(\overline{f} \underset{(\overline{t},\overline{s})}{\circ} \overline{f})(u) \preceq \overline{f}_{(\overline{t},\overline{s})}(u)$. Hence, $\overline{f} \underset{(\overline{t},\overline{s})}{\circ} \overline{f} \sqsubseteq \overline{f}_{(\overline{t},\overline{s})}$.

($\Leftarrow$) Suppose $\overline{f} \underset{(\overline{t},\overline{s})}{\circ} \overline{f} \sqsubseteq \overline{f}_{(\overline{t},\overline{s})}$. Let $u, v \in S$ and $\overline{s}, \overline{t} \in C[0,1]$. Since $\overline{f} \underset{(\overline{t},\overline{s})}{\circ} \overline{f} \sqsubseteq \overline{f}_{(\overline{t},\overline{s})}$, we have $(\overline{f} \underset{(\overline{t},\overline{s})}{\circ} \overline{f})(uv) \preceq \overline{f}_{(\overline{t},\overline{s})}(uv)$. Thus,

$$
\begin{aligned}
\overline{f}(uv) \curlyvee \overline{s} &\succeq (\overline{f}(uv) \curlywedge \overline{t}) \curlyvee \overline{s} = \overline{f}_{(\overline{t},\overline{s})}(uv) \succeq (\overline{f} \underset{(\overline{t},\overline{s})}{\circ} \overline{f})(uv) \\
&= ((\overline{f} \circ \overline{f})(uv) \curlywedge \overline{t}) \curlyvee \overline{s} = (\underset{(i,j)\in F_{uv}}{\curlyvee} \{\overline{f}(i) \curlywedge \overline{f}(j)\} \curlywedge \overline{t}) \curlyvee \overline{s} \\
&\succeq (\overline{f}(u) \curlywedge \overline{f}(v) \curlywedge \overline{t}) \curlyvee \overline{s} \succeq \overline{f}(u) \curlywedge \overline{f}(v) \curlywedge \overline{t}.
\end{aligned}
$$

Hence, $\overline{f}(uv) \curlyvee \overline{s} \succeq \overline{f}(u) \curlywedge \overline{f}(v) \curlywedge \overline{t}$. Therefore $\overline{f}$ is a $(\overline{s}, \overline{t})$-IVF subsemigroup of S. $\qquad \square$

Remark 5.2. Since $\overline{\lambda}_K$ is an interval valued characteristic function we have

$$
(\overline{\lambda}_K)_{(\overline{s},\overline{t})}(u) = \begin{cases} \overline{s} \curlyvee \overline{t} & \text{if} \quad u \in K, \\ \overline{t} & \text{if} \quad u \notin K. \end{cases}
$$

This lemma is basic properties of a $(\overline{s}, \overline{t})$-interval valued characteristic function and the proofs of (1) and (2) are straightforward.

Lemma 5.3. *Let K and L be non-empty subsets of a semigroup S and $\overline{s}, \overline{t} \in C[0,1]$. Then the following assertions hold.*

(1) $(\overline{\lambda}_K) \underset{(\overline{t},\overline{s})}{\sqcap} (\overline{\lambda}_L) = (\overline{\lambda}_{K \cap L})_{(\overline{t},\overline{s})}$.

(2) $(\overline{\lambda}_K) \underset{(\overline{t},\overline{s})}{\circ} (\overline{\lambda}_L) = (\overline{\lambda}_{KL})_{(\overline{t},\overline{s})}$.

The following Definition 5.4 and Lemma 5.5 will be used to prove in Theorem 5.6 and Theorem 5.7.

Definition 5.4. [11] A semigroup S is said to be *regular* if for each element $u \in S$, there exists an element $x \in S$ such that $u = uxu$.

Lemma 5.5. [11] *For a semigroup S, the following statements are equivalent.*

(1) *S is regular.*

(2) *For every quasi-ideal Q and every left ideal L of S, $Q \cap L \subseteq QL$.*

(3) *For every right ideal R every quasi-ideal Q and of S, $R \cap Q \subseteq RQ$.*

Theorem 5.6. *For a semigroup S, the following conditions are equivalent.*

(1) *S is regular.*

(2) *For every $(\overline{s},\overline{t})$-IVF quasi-ideal $\overline{f}$ and every $(\overline{s},\overline{t})$-IVF left ideal $\overline{g}$ of S, $\overline{f} \underset{(\overline{t},\overline{s})}{\sqcap} \overline{g} \sqsubseteq \overline{f} \underset{(\overline{t},\overline{s})}{\circ} \overline{g}$.*

(3) *For every $(\overline{s},\overline{t})$-IVF bi-ideal $\overline{f}$ and every $(\overline{s},\overline{t})$-IVF left ideal $\overline{g}$ of S, $\overline{f} \underset{(\overline{t},\overline{s})}{\sqcap} \overline{g} \sqsubseteq \overline{f} \underset{(\overline{t},\overline{s})}{\circ} \overline{g}$.*

(4) *For every $(\overline{s},\overline{t})$-IVF generalized bi-ideal $\overline{f}$ and every $(\overline{s},\overline{t})$-IVF left ideal $\overline{g}$ of S,*
  $\overline{f} \underset{(\overline{t},\overline{s})}{\sqcap} \overline{g} \sqsubseteq \overline{f} \underset{(\overline{t},\overline{s})}{\circ} \overline{g}$.

Proof. $(1) \Rightarrow (4)$ Suppose that S is regular and let $\overline{f}$ and $\overline{g}$ be a $(\overline{s},\overline{t})$-IVF generalized bi-ideal and a $(\overline{s},\overline{t})$-IVF left ideal of S respectively. Then, for any $u \in S$ there exists $x \in S$ such that $u = uxu$. By using Proposition 2.3 we get

$$
\begin{aligned}
(\overline{f} \underset{(\overline{t},\overline{s})}{\circ} \overline{g})(u) &= (\underset{(y,z)\in F_u}{\curlyvee} \{\overline{f}(y) \curlywedge \overline{g}(z)\} \curlywedge \overline{t}) \curlyvee \overline{s} = (\underset{(y,z)\in F_{uxu}}{\curlyvee} \{\overline{f}(y) \curlywedge \overline{g}(z)\} \curlywedge \overline{t}) \curlyvee \overline{s} \\
&\succeq ((\overline{f}(u) \curlywedge \overline{g}(xu)) \curlywedge \overline{t}) \curlyvee \overline{s} = (\overline{f}(u) \curlywedge (\overline{g}(xu) \curlyvee \overline{s}) \curlywedge \overline{t}) \curlyvee \overline{s} \\
&\succeq (\overline{f}(u) \curlywedge (\overline{g}(u) \curlywedge \overline{t}) \curlywedge \overline{t}) \curlyvee \overline{s} = (\overline{f}(u) \curlywedge \overline{g}(u) \curlywedge \overline{t}) \curlyvee \overline{s} \\
&= (\overline{f} \underset{(\overline{t},\overline{s})}{\sqcap} \overline{g})(u).
\end{aligned}
$$

Thus we have $(\overline{f} \underset{(\overline{t},\overline{s})}{\sqcap} \overline{g})(u) \preceq (\overline{f} \underset{(\overline{t},\overline{s})}{\circ} \overline{g})(u)$. Hence this shows that $\overline{f} \underset{(\overline{t},\overline{s})}{\sqcap} \overline{g} \sqsubseteq \overline{f} \underset{(\overline{t},\overline{s})}{\circ} \overline{g}$.

$(4) \Rightarrow (3) \Rightarrow (2)$ This is obvious because every $(\overline{s},\overline{t})$-IVF bi-ideal is a $(\overline{s},\overline{t})$-IVF generalized bi-ideal of S and every $(\overline{s},\overline{t})$-IVF quasi-ideal is a $(\overline{s},\overline{t})$-IVF bi-ideal of S.

$(2) \Rightarrow (1)$. Let Q and L be a quasi-ideal and a left ideal of S respectively. Then by Theorem 4.9 and 3.5, $\overline{\lambda}_Q$ and $\overline{\lambda}_L$ is a $(\overline{s},\overline{t})$-IVF quasi-ideal and a $(\overline{s},\overline{t})$-IVF left ideal of S respectively. By using supposition and Lemma 5.3, we get

$$
\overline{t} \curlyvee \overline{s} = (\overline{\lambda}_{Q\cap L})_{(\overline{t},\overline{s})}(u) = ((\overline{\lambda}_Q) \underset{(\overline{t},\overline{s})}{\sqcap} (\overline{\lambda}_L))(u) \sqsubseteq ((\overline{\lambda}_Q) \underset{(\overline{t},\overline{s})}{\circ} (\overline{\lambda}_L))(u) = (\overline{\lambda}_{QL})_{(\overline{t},\overline{s})}(u).
$$

Thus, $u \in QL$. Hence $Q \cap L \subseteq QL$. Consequently, S is regular, by Lemma 5.5. $\square$

Theorem 5.7. *For a semigroup S, the following conditions are equivalent.*

(1) *S is regular.*

(2) *For every $(\overline{s},\overline{t})$-IVF right ideal $\overline{f}$ and every $(\overline{s},\overline{t})$-IVF quasi-ideal $\overline{g}$ of S, $\overline{f} \underset{(\overline{t},\overline{s})}{\sqcap} \overline{g} \sqsubseteq$*
  $\overline{f} \underset{(\overline{t},\overline{s})}{\circ} \overline{g}$.

(3) *For every $(\overline{s},\overline{t})$-IVF right ideal $\overline{f}$ and every $(\overline{s},\overline{t})$-IVF bi-ideal $\overline{g}$ of S, $\overline{f} \underset{(\overline{t},\overline{s})}{\sqcap} \overline{g} \sqsubseteq \overline{f} \underset{(\overline{t},\overline{s})}{\circ} \overline{g}$.*

(4) *For every $(\overline{s},\overline{t})$-IVF right ideal $\overline{f}$ and every $(\overline{s},\overline{t})$-IVF generalized bi-ideal $\overline{g}$ of S,*
  $\overline{f} \underset{(\overline{t},\overline{s})}{\sqcap} \overline{g} \sqsubseteq \overline{f} \underset{(\overline{t},\overline{s})}{\circ} \overline{g}$.

Proof. $(1) \Rightarrow (4)$ Suppose that S is regular and let $\overline{f}$ and $\overline{g}$ be a $(\overline{s},\overline{t})$-IVF right ideal and a $(\overline{s},\overline{t})$-IVF generalized bi-ideal of S respectively. Then, for any $u \in S$ there exists $x \in S$ such that $u = uxu$. By using Proposition 2.3 we get

$$
\begin{aligned}
(\overline{f} \underset{(\overline{t},\overline{s})}{\circ} \overline{g})(u) &= (\underset{(y,z)\in F_u}{\curlyvee} \{\overline{f}(y) \curlywedge \overline{g}(z)\} \curlywedge \overline{t}) \curlyvee \overline{s} = (\underset{(y,z)\in F_{uxu}}{\curlyvee} \{\overline{f}(y) \curlywedge \overline{g}(z)\} \curlywedge \overline{t}) \curlyvee \overline{s} \\
&\succeq ((\overline{f}(ux) \curlywedge \overline{g}(u)) \curlywedge \overline{t}) \curlyvee \overline{s} = ((\overline{f}(ux) \curlyvee \overline{s} \curlywedge \overline{g}(u)) \curlywedge \overline{t}) \curlyvee \overline{s} \\
&\succeq ((\overline{f}(ux) \curlywedge \overline{t} \curlywedge \overline{g}(u)) \curlywedge \overline{t}) \curlyvee \overline{s} = ((\overline{f}(ux) \curlywedge \overline{g}(u) \curlywedge \overline{t}) \curlywedge \overline{t}) \curlyvee \overline{s} \\
&= (\overline{f}(u) \curlywedge \overline{g}(u) \curlywedge \overline{t}) \curlyvee \overline{s} = (\overline{f} \underset{(\overline{t},\overline{s})}{\sqcap} \overline{g})(u).
\end{aligned}
$$

Thus we have $(\overline{f} \underset{(\overline{t},\overline{s})}{\sqcap} \overline{g})(u) \preceq (\overline{f} \underset{(\overline{t},\overline{s})}{\circ} \overline{g})(u)$. Hence this shows that $\overline{f} \underset{(\overline{t},\overline{s})}{\sqcap} \overline{g} \sqsubseteq \overline{f} \underset{(\overline{t},\overline{s})}{\circ} \overline{g}$.

$(4) \Rightarrow (3) \Rightarrow (2)$ This is obvious because every $(\overline{s},\overline{t})$-IVF bi-ideal is a $(\overline{s},\overline{t})$-IVF generalized bi-ideal of S and every $(\overline{s},\overline{t})$-IVF quasi-ideal is a $(\overline{s},\overline{t})$-IVF bi-ideal of S.

$(2) \Rightarrow (1)$. Let R and Q be a right ideal and a quasi-ideal of S respectively. Then by Theorem 3.5 and 4.9, $\overline{\lambda}_R$ and $\overline{\lambda}_Q$ is a $(\overline{s}, \overline{t})$-IVF right ideal and a $(\overline{s}, \overline{t})$-IVF quasi-ideal of S respectively. By using supposition and Lemma 5.3, we get

$$\overline{t} \curlyvee \overline{s} = (\overline{\lambda}_{R \cap Q})_{(\overline{t}, \overline{s})}(u) = ((\overline{\lambda}_R) \underset{(\overline{t}, \overline{s})}{\sqcap} (\overline{\lambda}_Q))(u) \sqsubseteq ((\overline{\lambda}_R) \underset{(\overline{t}, \overline{s})}{\circ} (\overline{\lambda}_Q))(u) = (\overline{\lambda}_{RQ})_{(\overline{t}, \overline{s})}(u).$$

Thus, $u \in RQ$. Hence $R \cap Q \subseteq RQ$. Consequently, S is regular, by Lemma 5.5. □

The following Definition 5.8 and Lemma 5.9 will be used to prove in Theorem 5.10.

Definition 5.8. [11] A semigroup S is said to be *intra-regular* if for each $u \in S$, there exist $a.b \in S$ such that $u = au^2b$.

Lemma 5.9. [11] *For a semigroup S, the following conditions are equivalent.*

(1) *S is regular and intra-regular.*
(2) *Every quasi-ideal of S is idempotent.*
(3) *Every bi-ideal of S is idempotent.*

Theorem 5.10. *For a semigroup S, the following conditions are equivalent.*

(1) *S is both regular and intra-regular.*
(2) *$\overline{f} \underset{(\overline{t}, \overline{s})}{\circ} \overline{f} = \overline{f}_{(\overline{s}, \overline{t})}$ for every $(\overline{s}, \overline{t})$-IVF quasi-ideal $\overline{f}$ of S.*
(3) *$\overline{f} \underset{(\overline{t}, \overline{s})}{\circ} \overline{f} = \overline{f}_{(\overline{s}, \overline{t})}$ for every $(\overline{s}, \overline{t})$-IVF bi-ideal $\overline{f}$ of S.*
(4) *$\overline{f} \underset{(\overline{t}, \overline{s})}{\sqcap} \overline{g} \sqsubseteq \overline{f} \underset{(\overline{t}, \overline{s})}{\circ} \overline{g}$ for every $(\overline{s}, \overline{t})$-IVF quasi-ideals $\overline{f}$ and $\overline{g}$ of S.*
(5) *$\overline{f} \underset{(\overline{t}, \overline{s})}{\sqcap} \overline{g} \sqsubseteq \overline{f} \underset{(\overline{t}, \overline{s})}{\circ} \overline{g}$ for every $(\overline{s}, \overline{t})$-IVF quasi-ideal $\overline{f}$ and every $(\overline{s}, \overline{t})$-IVF bi-ideal $\overline{g}$ of S.*
(6) *$\overline{f} \underset{(\overline{t}, \overline{s})}{\sqcap} \overline{g} \sqsubseteq \overline{f} \underset{(\overline{t}, \overline{s})}{\circ} \overline{g}$ for every $(\overline{s}, \overline{t})$-IVF bi-ideals $\overline{f}$ and $\overline{g}$ of S.*
(7) *$\overline{f} \underset{(\overline{t}, \overline{s})}{\sqcap} \overline{g} \sqsubseteq \overline{f} \underset{(\overline{t}, \overline{s})}{\circ} \overline{g}$ for every $(\overline{s}, \overline{t})$-IVF bi-ideal $\overline{f}$ and every $(\overline{s}, \overline{t})$-IVF generalized bi-ideal $\overline{g}$ of S.*
(8) *$\overline{f} \underset{(\overline{t}, \overline{s})}{\sqcap} \overline{g} \sqsubseteq \overline{f} \underset{(\overline{t}, \overline{s})}{\circ} \overline{g}$ for every $(\overline{s}, \overline{t})$-IVF generalized bi-ideal $\overline{f}$ and every $(\overline{s}, \overline{t})$-IVF quasi-ideal $\overline{g}$ of S.*
(9) *$\overline{f} \underset{(\overline{t}, \overline{s})}{\sqcap} \overline{g} \sqsubseteq \overline{f} \underset{(\overline{t}, \overline{s})}{\circ} \overline{g}$ for every $(\overline{s}, \overline{t})$-IVF generalized bi-ideal $\overline{f}$ and every $(\overline{s}, \overline{t})$-IVF bi-ideal $\overline{g}$ of S.*
(10) *$\overline{f} \underset{(\overline{t}, \overline{s})}{\sqcap} \overline{g} \sqsubseteq \overline{f} \underset{(\overline{t}, \overline{s})}{\circ} \overline{g}$ for every $(\overline{s}, \overline{t})$-IVF generalized bi-ideals $\overline{f}$ and $\overline{g}$ of S.*

Proof. $(1) \Rightarrow (10)$ Let $\overline{f}$ and $\overline{g}$ be $(\overline{s}, \overline{t})$-IVF generalized bi-ideals of S and let $u \in S$. Since S is both regular and intra-regular, there exist $x, y, z \in S$ such that $u = uxu$ and $u = yu^2z$. Thus, $u = uxu = uxuxu = ux(yu^2z)xu = (uxyu)(uzxu)$. Now by Proposition 2.3 we have

$$
\begin{aligned}
(\overline{f} \underset{(\overline{t}, \overline{s})}{\circ} \overline{g})(u) &= (\underset{(i,j) \in F_u}{\curlyvee} \{\overline{f}(i) \curlywedge \overline{g}(j)\} \curlywedge \overline{t}) \curlyvee \overline{s} = (\underset{(i,j) \in F_{(uxyu)(uzxu)}}{\curlyvee} \{\overline{f}(i) \curlywedge \overline{g}(j)\} \curlywedge \overline{t}) \curlyvee \overline{s} \\
&\succeq ((\overline{f}(uxyu) \curlywedge \overline{g}(uzxu)) \curlywedge \overline{t}) \curlyvee \overline{s} \\
&= ((\overline{f}(u(xy)u) \curlyvee \overline{s}) \curlywedge (\overline{g}(u(zx)u) \curlyvee \overline{s}) \curlywedge \overline{t}) \curlyvee \overline{s} \\
&\succeq ((\overline{f}(u) \curlywedge \overline{f}(u) \curlywedge \overline{t}) \curlywedge (\overline{g}(u) \curlywedge \overline{g}(u) \curlywedge \overline{t}) \curlywedge \overline{t}) \curlyvee \overline{s} \\
&= ((\overline{f}(u) \curlywedge \overline{t}) \curlywedge (\overline{g}(u) \curlywedge \overline{t}) \curlywedge \overline{t}) \curlyvee \overline{s} = (\overline{f}(u) \curlywedge \overline{g}(u) \curlywedge \overline{t}) \curlyvee \overline{s} = (\overline{f} \underset{(\overline{t}, \overline{s})}{\sqcap} \overline{g})(u).
\end{aligned}
$$

Hence we get that $(\overline{f} \underset{(\overline{t}, \overline{s})}{\sqcap} \overline{g})(u) \preceq (\overline{f} \underset{(\overline{t}, \overline{s})}{\circ} \overline{g})(u)$. Therefore, $\overline{f} \underset{(\overline{t}, \overline{s})}{\sqcap} \overline{g} \sqsubseteq \overline{f} \underset{(\overline{t}, \overline{s})}{\circ} \overline{g}$.

$(10) \Rightarrow (9) \Rightarrow (8) \Rightarrow (4)$ and $(10) \Rightarrow (7) \Rightarrow (6) \Rightarrow (5) \Rightarrow (4)$ This is obvious because every $(\overline{s}, \overline{t})$-IVF bi-ideal is a $(\overline{s}, \overline{t})$-IVF generalized bi-ideal of S and every $(\overline{s}, \overline{t})$-IVF quasi-ideal is a $(\overline{s}, \overline{t})$-IVF bi-ideal of S.

$(4) \Rightarrow (2)$ Take $\overline{f} = \overline{g}$ in (4), we get $\overline{f}_{(\overline{s}, \overline{t})} = \overline{f} \underset{(\overline{t}, \overline{s})}{\sqcap} \overline{f} \sqsubseteq \overline{f} \underset{(\overline{t}, \overline{s})}{\circ} \overline{f}$. Since every $(\overline{s}, \overline{t})$-IVF quasi-ideal of S is a $(\overline{s}, \overline{t})$-IVF subsemigroup of S and by Theorem 5.1, we have $\overline{f} \underset{(\overline{t}, \overline{s})}{\circ} \overline{f} \sqsubseteq \overline{f}_{(\overline{t}, \overline{s})}$. Thus $\overline{f} \underset{(\overline{t}, \overline{s})}{\circ} \overline{f} = \overline{f}_{(\overline{t}, \overline{s})}$.

$(6) \Rightarrow (3)$ Take $\overline{f} = \overline{g}$ in (6), we get $\overline{f}_{(\overline{s}, \overline{t})} = \overline{f} \underset{(\overline{t}, \overline{s})}{\sqcap} \overline{f} \sqsubseteq \overline{f} \underset{(\overline{t}, \overline{s})}{\circ} \overline{f}$. Since every $(\overline{s}, \overline{t})$-IVF quasi-ideal of S is a $(\overline{s}, \overline{t})$-IVF subsemigroup of S and by Theorem 5.1, we have $\overline{f} \underset{(\overline{t}, \overline{s})}{\circ} \overline{f} \sqsubseteq \overline{f}_{(\overline{t}, \overline{s})}$. Thus, we obtain that $\overline{f} \underset{(\overline{t}, \overline{s})}{\circ} \overline{f} = \overline{f}_{(\overline{t}, \overline{s})}$.

$(3) \Rightarrow (2)$ This is obvious because every $(\overline{s}, \overline{t})$-IVF quasi-ideal of S is a $(\overline{s}, \overline{t})$-IVF bi-ideal of S.

$(2) \Rightarrow (1)$ Let Q be a quasi-ideal of S. Then by Theorem 4.9, $\overline{\lambda}_Q$ is a $(\overline{s}, \overline{t})$-IVF quasi-ideal of S. By supposition and Lemma 5.3, we have

$$(\overline{\lambda}_{Q^2})_{(\overline{t}, \overline{s})}(u) = (\overline{\lambda}_Q) \underset{(\overline{t}, \overline{s})}{\circ} (\overline{\lambda}_Q)(u) = (\overline{\lambda}_Q)_{(\overline{t}, \overline{s})}(u) = \overline{t} \curlyvee \overline{s}.$$

Thus $Q^2 = Q$. Therefore by Lemma 5.9 that S is both regular and intra-regular. $\qquad \square$

6. Conclusion

In this article, we give concept of generalized interval valued fuzzy subsemigroups which are generalizations of $(\overline{\alpha}, \overline{\beta})$-interval valued fuzzy subsemigroups, where $\overline{\alpha} \prec \overline{\beta}$, defined by Abdullah et al. And we study properties of generalized IVF quasi-ideals on semigroups. This concept are useful for characterizing various subsemigroups. In this result, we have characterized regular and intra-regular semigroups in terms of generalized interval valued fuzzy subsemigroups. We hope that the study of regular and intra-regular semigroup in terms of generalized interval valued fuzzy subsemigroups are useful mathematical tools.

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