TWMS J. App. and Eng. Math. V.12, N.2, 2022, pp. 536-546

EULERIAN AND HAMILTONIAN PROPERTIES OF GALLAI AND ANTI-GALLAI MIDDLE GRAPHS

S. GOYAL[1], D. JAIN[2], §

Abstract. The Gallai middle graph $\Gamma_M(G)$ of a graph $G = (V, E)$ is the graph whose vertex set is $V \cup E$ and two edges $e_i, e_j \in E$ are adjacent in $\Gamma_M(G)$, if they are adjacent edges of G and do not lie on a same triangle in G, or if $e_i = uv \in E$ then e_i is adjacent to u and v in $\Gamma_M(G)$. The anti-Gallai middle graph $\Delta_M(G)$ of a graph $G = (V, E)$ is the graph whose vertex set is $V \cup E$ and two edges $e_i, e_j \in E$ are adjacent in $\Delta_M(G)$ if they are adjacent in G and lie on a same triangle in G, or if $e_i = uv \in E$ then e_i is adjacent to u and v in $\Delta_M(G)$. In this paper, we investigate Eulerian and Hamiltonian properties of Gallai and anti-Gallai middle graphs.

Keywords: Euler graph, Hamiltonian graph, Gallai middle graph, anti-Gallai middle graph.

AMS Subject Classification: 05C45, 05C76.

1. Introduction

A graph $G = (V, E)$ is an ordered pair of set of vertices and edges, where edges are unordered pair of vertices. Also G is said to be a (p, q) graph if $|V| = p$ and $|E| = q$. Two vertices (edges) are said to be adjacent if they have a common edge (vertex). If a vertex v lies on an edge e, then they are said to be incident to each other. The degree $d(v)$ of a vertex $v \in V$ is the number of edges incident at v. A complete graph is the graph in which every vertex is adjacent to every another vertex. It is dented by K_n, where n is number of vertices. A regular graph is the graph in which every vertex of the graph has same degree. A walk is an alternating sequence of vertices and edges of G, whose starting and ending point is a vertex. A path in a graph G is a walk with no repeated vertex. A graph G is said to be connected if there exists a path between every pair of vertices of G. Let $G = (V, E)$ be a graph with $|V| = p$, then the adjacency matrix $A(G)$ of G is defined as $A(G) = [a_{ij}]_{p\times p}$, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

[1] Department of Mathematics & Statistics, Banasthali University, Banasthali-304022, Rajasthan, India.
e-mail: shanugoyalnewai@gmail.com; ORCID: https://orcid.org/0000-0003-0030-8550.
[2] Department of Mathematics, Govt. Polytechnic College, Bundi-323001, Rajasthan, India.
e-mail: dilip18pri@gmail.com; ORCID: https://orcid.org/0000-0001-6301-1362.
§ Manuscript received: March 03, 2020; accepted: April 26, 2020.

If A is an square matrix of order n, then the trace of A is denoted by $tr(A)$, is the sum of all of the entries in the main diagonal, where main diagonal of A consists of the entries $a_{11}, a_{22}, ..., a_{nn}$ (entries whose row number is the same as their column number).

A graph G is called Euler graph if there exists a closed walk in G with no repeated edge and all the edges are traversed exactly once. A closed path is called a cycle. A cycle is said to be spanning cycle if it contains all the vertices of the graph. A graph G is said to be Hamiltonian if it contains a spanning cycle. Vertices and edges of G are called elements of G.

Definition 1.1. *The line graph $L(G)$ of a graph G is defined as the graph whose vertices are the edges of G, with two vertices are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G.*

The line graphs were first studied by Whitney [15]. Several properties of line graph is studied in the literature [1], [2], [10].

Definition 1.2. *The middle graph $T_1(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$, two vertices in $T_1(G)$ are adjacent if and only if they are adjacent edges in G, or one is a vertex and another is an incident edge in G.*

A structural characterization and various properties of middle graphs were presented by Sampathkumar & Chikkodimath [11], [12], [13]. In the literature, middle graphs are also known as semi-total line graphs. Hamada & Yoshimura [7] have presented a characterization of middle graphs in terms of line graphs and also investigated traversability and connectivity properties of middle graphs.

Definition 1.3. *The Gallai graph $\Gamma(G)$ of a graph G is the graph in which $V(\Gamma(G)) = E(G)$ and two distinct edges of G are adjacent in $\Gamma(G)$ if they are adjacent in G, but do not span a triangle in G.*

Definition 1.4. *The anti-Gallai graph $\Delta(G)$ of a graph G is the graph in which $V(\Delta(G)) = E(G)$ and two distinct edges of G are adjacent in $\Delta(G)$ if they are adjacent in G and lie on a same triangle in G.*

These constructions were used by Gallai [4] in his investigation of comparability graphs; the notion was suggested by Sun [14]. Sun used the Gallai graphs to describe a nice class of perfect graphs. Gallai graphs are also used in polynomial time algorithm to recognize $k_{1,3}$-free perfect graphs by Chvatal & Sbihi [3]. Several properties of Gallai and anti-Gallai graphs are discussed in [8], [9]. Eulerian and Hamiltonian properties of Gallai and anti-Gallai total graphs are given by garg *et al.* in [5].

Motivated from the operators Gallai graph, anti-Gallai graph and middle graph, we introduce two new operators Gallai middle graph $\Gamma_M(G)$ and anti-Gallai middle graph $\Delta_M(G)$ of a graph G as follows:

Definition 1.5. *The Gallai middle graph $\Gamma_M(G)$ of a graph $G = (V, E)$ is the graph whose vertex set is $V \cup E$ and two edges $e_i, e_j \in E$ are adjacent in $\Gamma_M(G)$, if they are adjacent edges of G and do not lie on a same triangle in G, or if $e = uv \in E$ then e is adjacent to u and v in $\Gamma_M(G)$.*

Definition 1.6. *The* anti-Gallai middle graph $\Delta_M(G)$ *of* G *is the graph whose vertex set is* $V \cup E$ *and two edges* $e_i, e_j \in E$ *are adjacent in* $\Delta_M(G)$ *if they are adjacent in* G *and lie on a same triangle in* G*, or if* $e = uv \in E$ *then* e *is adjacent to* u *and* v *in* $\Delta_M(G)$*.*

In this paper, we present Eulerian and Hamiltonian properties of Gallai and anti-Gallai middle graphs. The Gallai middle graph $\Gamma_M(G)$ and anti-Gallai middle graph $\Delta_M(G)$ of G are shown in Figure 1. Throughout the paper we consider all graphs are simple (namely, with no loops or multiple edges).

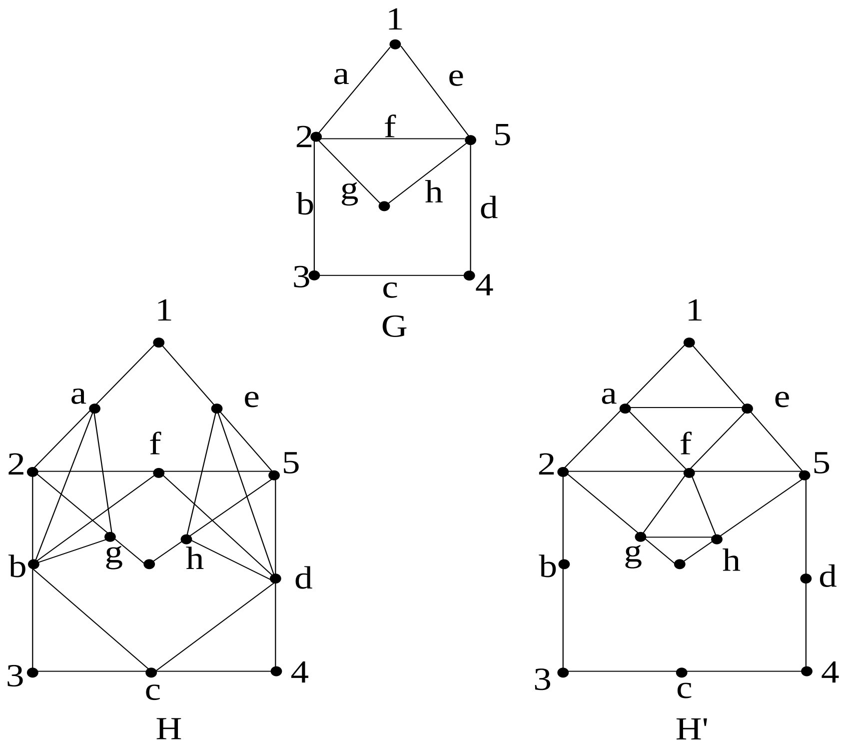

FIGURE 1. A graph G, its Gallai middle graph $H = \Gamma_M(G)$ and anti-Gallai middle graph $H' = \Delta_M(G)$

2. Eulerian Gallai middle graphs

The degree of a vertex $v' \in V(\Gamma_M(G))$ is denoted as $d_\Gamma(v')$.

Proposition 2.1. *Let* $G = (V, E)$ *be a graph.*

(*i*) *If* v *is a vertex of* G *and* v' *is the corresponding vertex of* $\Gamma_M(G)$*, then* $d_\Gamma(v') = d(v)$*,*

(*ii*) *If* $e = uv$ *is an edge of* G *and* e' *is the corresponding vertex of* $\Gamma_M(G)$*, then* $d_\Gamma(e') = d(u) + d(v) - 2t$*, where* t *is the number of triangles in* G *containing the edge* e*.*

Proof. (*i*) By definition, there is a bijective mapping from the edges incident to v in G to the vertices adjacent to v' in $\Gamma_M(G)$. Thus, $d_\Gamma(v') = d(v)$."

(*ii*) If $e = uv$ is an edge of G, then e' is adjacent to all the edges adjacent to e, but do not those edges which lie on a same triangle with e in G. It implies that they contribute the degree $(d(u) - 1) + (d(v) - 1) - 2t$ (because if e is the edge of a triangle, then it is not adjacent to those two edges of G which lie on a same triangle with e in G) and e' is also adjacent in $\Gamma_M(G)$ to the vertices to which it is incident in G. Therefore, $d_\Gamma(e') = d(u) + d(v) - 2t$.

□

Lemma 2.1. [6] *The number of triangles in a graph G is equal to $tr(A^3)/6$, where A is the adjacency matrix of G.*

Proposition 2.2. *Let G be a (p, q) graph, then the number of edges in the Gallai middle graph $\Gamma_M(G)$ is equal to $q+\frac{1}{2}\sum_{i=1}^{p}(d(v_i))^2-3(tr(A^3)/6)$, where A is the adjacency matrix and v_i is a vertex of G.*

Proof. Let $G = (V, E)$ be a (p, q) graph and $v_1, v_2, \ldots, v_i, \ldots, v_p$ be vertices of G. Then total degree of vertices of $\Gamma_M(G)$ is equal to (sum of degree of the vertices of G)+sum of the degree of the vertices corresponding to the edges of G. Let $E'(G)$ be the set of edges which do not lie on a triangle in G and $|E'(G)| = q_1$. Also let $E''(G)$ be the set of edges which lie on a triangle in G and $|E''(G)| = q_2$. Now if $e = v_iv_j \in E'(G)$, then degree of the corresponding vertex e' in $\Gamma_T(G)$ is equal to $d(v_i)+d(v_j)$, so the total degree of the vertices in $\Gamma_M(G)$ corresponding to such edges of G is $\sum\limits_{v_iv_j\in E'(G)}^{q_1}(d(v_i)+d(v_j))$. If $e = v_iv_j \in E''(G)$, then degree of the corresponding vertex e' in $\Gamma_M(G)$ is equal to $d(v_i) + d(v_j) - 2t_{ij}$, where t_{ij} is the number of triangles on which the edge v_iv_j lies, so the total degree of the vertices in $\Gamma_M(G)$ corresponding to such edges of G is $\sum\limits_{v_iv_j\in E''(G)}^{q_2}(d(v_i) + d(v_j) - 2t_{ij})$. Then by handshake lemma on G and $\Gamma_M(G)$ we have,
total degree of $\Gamma_M(G)$

$$
\begin{aligned}
&= (2q) + \sum_{v_iv_j\in E'(G)}^{q_1}(d(v_i) + d(v_j)) + \sum_{v_iv_j\in E''(G)}^{q_2}(d(v_i) + d(v_j) - 2t_{ij}) \\
&= 2q + \sum_{v_iv_j\in E(G)}^{q}(d(v_i) + d(v_j)) - 2\sum_{v_iv_j\in E''(G)}^{q_2}(t_{ij}) \\
&= 2q + \sum_{i=1}^{p}((d(v_i))^2) - 2(3 \times \text{total no. of triangles in } G) \\
&= 2q + \sum_{i=1}^{p}(d(v_i))^2 - 6(\text{number of triangles in } G) \\
&= 2q + \sum_{i=1}^{p}(d(v_i))^2 - 6\left(\frac{tr(A^3)}{6}\right), \text{using Lemma 2.1.}
\end{aligned}
$$

Then by hand shake lemma (sum of the degree of the vertices is equal to twice the number of edges in G), the total number of edges in $\Gamma_M(G)$,

$$|E(\Gamma_M(G))| = q + \tfrac{1}{2}\sum_{i=1}^{p}(d(v_i))^2 - 3\left(\tfrac{tr(A^3)}{6}\right).$$

□

A graph is called l-triangular if each edge of G lies on l number of triangles in G.

Proposition 2.3. *The Gallai middle graph $\Gamma_M(G)$ of a graph G is $2l$-regular if and only if G is $2l$-regular and l-triangular.*

Proof. Let G be a $2l$-regular and l-triangular graph. Now we have to show that $\Gamma_M(G)$ is $2l$-regular. Since G is $2l$-regular, degree of each vertex is same. Proposition 2.1 implies that each corresponding vertex in $\Gamma_M(G)$ is of degree $2l$. Also it is given that G is l-triangular. It follows that every vertex corresponding to the edges of G has degree $d(u) + d(v) - 2l$ (by proposition 2.1), where u and v are the end vertices of the edge and this is equal to $2l$ (because G is $2l$-regular). Therefore, degree of each vertex of $\Gamma_M(G)$ is same. Hence, $\Gamma_M(G)$ is regular. Conversely, suppose that $\Gamma_M(G)$ is regular. Now we have to show that G is $2l$-regular and l-triangular. Suppose G is not $2l$-regular or not l-triangular. If G is not $2l$-regular, then degree of every vertex of $\Gamma_M(G)$ corresponding to the vertices of G is not same, which is a contradiction to our fact that $\Gamma_M(G)$ is regular. Hence G is $2l$-regular. Now if G is not l-triangular, then degree of every vertex of $\Gamma_M(G)$ corresponding to the edges of G is $d(u) + d(v) - 2t$ (by Proposition 2.1), where t is the number of triangles containing the edge, which is again a contradiction to our fact that $\Gamma_M(G)$ is regular. Thus, every edge of G lies on l triangles. Hence, G is $2l$-regular and l-triangular.

□

Proposition 2.4. *The Gallai middle graph $\Gamma_M(G)$ of G is connected if and only if G is connected.*

Proof. Necessity: Let G be connected, that means there is a path between each pair of vertices in G. Since $\Gamma_M(G)$ has a subdivision graph of G as a subgraph, $\exists$ a path between each pair of vertices (because G is connected). Hence $\Gamma_M(G)$ is connected.

Sufficiency: Suppose $\Gamma_M(G)$ is connected. Now we have to show that G is connected. Let on contrary, G be disconnected, then $\exists$ at least a pair of vertices which has no path between them. Let u, v be such two vertices, then u and v also have no path in $\Gamma_M(G)$. It follows that $\Gamma_M(G)$ is disconnected graph, a contradiction to the hypothesis. Hence the theorem.

□

For any integer $n \geq 1$ the n^{th} *Gallai middle graph* of G is defined recursively as, $\Gamma^n_M(G) = \Gamma_M(\Gamma^{n-1}_M(G))$, where $\Gamma^0_M(G) = G$.

Corollary 2.5. *$\Gamma^n_M(G)$ of G is connected if and only if G is connected for all $n \geq 1$.*

Theorem 2.6. *The Gallai middle graph $\Gamma_M(G)$ of G is Eulerian if and only if G is Eulerian.*

Proof. Necessity: Let G be an Eulerian graph. Then G is connected and the degree of each vertex of G is even. Since G is connected, by the Proposition 2.4, $\Gamma_M(G)$ is also connected. Now by the Proposition 2.1(i), vertices of $\Gamma_M(G)$ corresponding to the vertices of G are of even degree. Also by Proposition 2.1(ii), vertices of $\Gamma_M(G)$ corresponding to the edges of G are of even degree. Thus, $\Gamma_M(G)$ is connected and all vertices are of even degree. Hence, $\Gamma_M(G)$ is Eulerian.

Sufficiency: Suppose $\Gamma_M(G)$ of a graph G is an Eulerian graph. It implies that $\Gamma_M(G)$ is connected and degree of each vertex of $\Gamma_M(G)$ is even. Since $\Gamma_M(G)$ is connected, then G is also connected by Proposition 2.4. Now we have to show that G is Eulerian. By the Proposition 2.1(i), $d(v) = d(v')$ for each vertex $v \in V(G)$ and v' is the corresponding vertex of v in $\Gamma_M(G)$. Thus, all the vertices of G are of even degree by our assumption.

Hence, G is Eulerian.

□

The Eulerian Gallai middle graph $\Gamma_M(G)$ of G is shown in Figure 2.

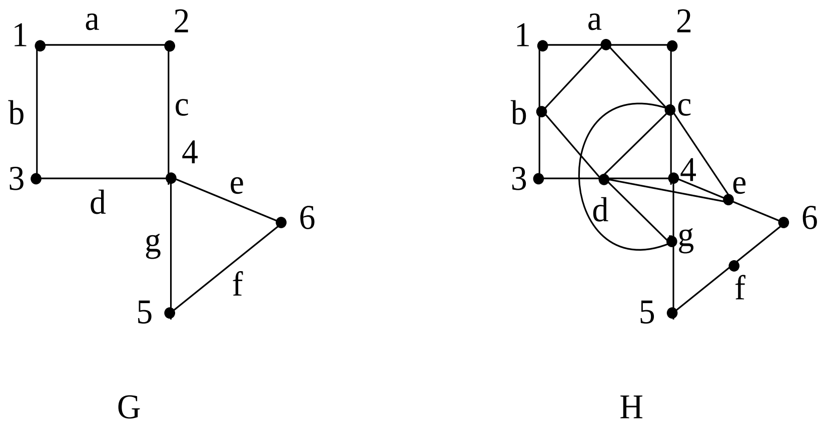

FIGURE 2. A graph G and its Eulerian Gallai middle graph $H = \Gamma_M(G)$

Corollary 2.7. *$\Gamma^n_M(G)$ of G is Eulerian if and only if G is Eulerian for all $n \geq 1$.*

3. Eulerian anti-Gallai middle graphs

The degree of a vertex $v' \in V(\Delta_M(G))$ is denoted as $d_\Delta(v')$.

Proposition 3.1. *Let $G = (V, E)$ be a graph.*

(*i*) *If v is a vertex of G and v' is the corresponding vertex of $\Delta_M(G)$, then $d_\Delta(v') = d(v)$,*

(*ii*) *If $e = uv$ is an edge of G and e' is the corresponding vertex of $\Delta_M(G)$, then $d_\Delta(e') = 2 + 2t$, where t denotes the number of triangles in G containing the edge e.*

Proof. (*i*) By definition, there is a bijective mapping from the edges incident to v in G to the vertices adjacent to v' in $\Delta_M(G)$. Thus, $d_\Delta(v') = d(v)$.

(*ii*) If $e = uv$ is an edge of G, then e' is adjacent to all the edges adjacent to e and lie on a same triangle with e in G. It implies that they contribute the degree $2t$ (because if e is the edge of a triangle, then it is adjacent to those two edges of G which lie on a same triangle with e in G) and e' is also adjacent in $\Delta_M(G)$ to the vertices to which it is incident in G, therefore, $d_\Delta(e') = 2t + 2$.

□

Proposition 3.2. *Let $G = (V, E)$ be a graph. Then the number of edges in $\Delta_M(G)$ is equal to $2|E| + 3t$, where t is the number of triangles in G.*

Proof. Let $G = (V, E)$ be a graph and $|E| = q$. We know that $V(\Delta_M(G))= V \cup E$. If $e = uv$ is an edge of G, then eu and ev are the edges of $\Delta_M(G)$. So, every edge $e \in E$ contributes two edges in $\Delta_M(G)$. Also, two vertices in $\Delta_M(G)$, that correspond to 2 edges in G, are adjacent if the corresponding edges belong to the same triangle in G. It implies that every triangle in G contributes 3 edges in $\Delta_M(G)$. Thus, if there are t triangles in G, then $3t$ edges are there in $\Delta_M(G)$. Hence, the total number of edges in $\Delta_M(G)$ is $3t + 2q$.

□

Proposition 3.3. *If G has t triangles, then there are $4t$ triangles in $\Delta_M(G)$.*

Proof. Let $G = (V, E)$ be a graph with t number of triangles. If there is a triangle in G, then vertices corresponding to the edges of a triangle in G are adjacent in $\Delta_M(G)$. Thus, one triangle is formed in $\Delta_M(G)$ from a triangle of G. Since there are t triangles in G, t triangles are in $\Delta_M(G)$. Furthermore, two edges which are adjacent in $\Delta_M(G)$ has a common vertex v and both of them are adjacent to the vertex v in $\Delta_M(G)$. It follows that every edge of triangle in $\Delta_M(G)$ form a triangle with their common vertex. Therefore, from one triangle in G there are 4 triangles in $\Delta_M(G)$. Thus, there are $4t$ triangles in $\Delta_M(G)$.

□

Proposition 3.4. *The anti-Gallai total graph $\Delta_M(G)$ of a graph G is $2(l+1)$-regular if and only if G is l-triangular and $2(l+1)$-regular.*

Proof. Suppose G is l-triangular and $2(l+1)$-regular. By Proposition 3.1(i), the degree of each vertex of $\Delta_M(G)$ that corresponds to a vertex of G is also $2(l+1)$. Since G is l-triangular, Proposition 3.1(ii) implies that the degree of each vertex of $\Delta_M(G)$ that corresponds to an edge of G is $2(l+1)$. Thus, $\Delta_M(G)$ is $2(l+1)$-regular. Conversely, suppose $\Delta_M(G)$ is $2(l+1)$-regular. Assume G is not l-triangular or not $2(l+1)$-regular. If G is not l-triangular, then the degree of the vertices corresponding to the edges of G in $\Delta_M(G)$ is not same (by Proposition 3.1(ii)), which is a contradiction to our fact that $\Delta_M(G)$ is regular. Thus, G is l-triangular. Next, if G is not $2(l+1)$-regular, then the degree of every vertex in $\Delta_M(G)$ corresponding to the vertices of G is not same, which is a contradiction to our fact that $\Delta_M(G)$ is regular. Hence, G is l-triangular and $2(l+1)$-regular.

□

Proposition 3.5. *The anti-Gallai middle graph $\Delta_M(G)$ of G is connected if and only if G is connected.*

Proof. Similar to the argument for the proof of Proposition 2.4. □

For any integer $n \geq 1$ the n^{th} *anti-Gallai middle graph* of G is defined recursively as, $\Delta_M^n(G) = \Delta_M(\Delta_M^{n-1}(G))$, where $\Delta_M^0(G) = G$.

Corollary 3.6. *$\Delta_M^n(G)$ of G is connected if and only if G is connected for all $n \geq 1$.*

Theorem 3.7. *The anti-Gallai middle graph $\Delta_M(G)$ of G is Eulerian if and only if G is Eulerian.*

Proof. Similar to the argument for the proof of Theorem 2.6. □

The Eulerian anti-Gallai middle graph $\Delta_M(G)$ of G is shown in Figure 3.

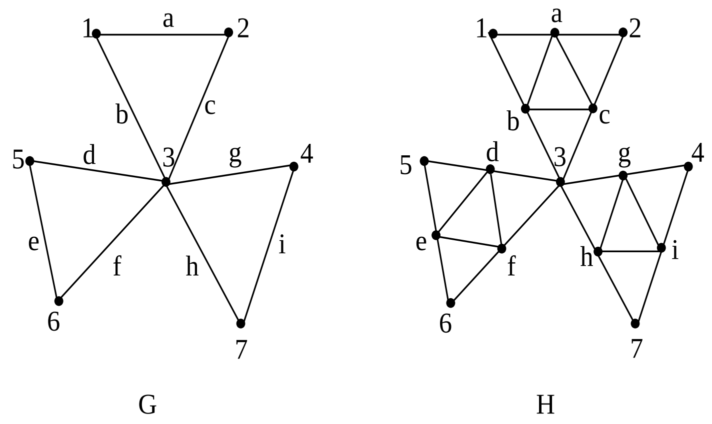

FIGURE 3. A graph G and its Eulerian anti-Gallai middle graph $H = \Delta_M(G)$

Corollary 3.8. *$\Delta_M^n(G)$ of G is Eulerian if and only if G is Eulerian for all $n \geq 1$.*

A graph G is called semi-Eulerian if and only if it has exactly two vertices of odd degree.

Proposition 3.9. *The anti-Gallai middle graph $\Delta_M(G)$ of G is semi-Eulerian if and only if G is semi-Eulerian.*

Proof. Let G be a semi-Eulerian graph. Now, we have to show that $\Delta_M(G)$ is semi-Eulerian. Since G is semi-Eulerian, it has exactly two vertices of odd degree. By Proposition 3.1, $\Delta_M(G)$ also has exactly two vertices of odd degree. It follows that $\Delta_M(G)$ is semi-Eulerian. Conversely, suppose $\Delta_M(G)$ is semi-Eulerian. Now, we have to show that G is semi-Eulerian. Since $\Delta_M(G)$ is semi-Eulerian, it has exactly two vertices of odd degree. These two vertices correspond to the vertices of G (by Proposition 3.1). Thus, G has exactly 2 vertices of odd degree. Hence, G is semi-Eulerian.

□

Corollary 3.10. *$\Delta_M^n(G)$ is semi-Eulerian if and only if G is semi-Eulerian for all $n \geq 1$.*

4. HAMILTONIAN GALLAI AND ANTI-GALLAI MIDDLE GRAPHS

In this section, we find some result on Hamiltonian property of Gallai and anti-Gallai middle graphs. Vertices and edges of G are called elements of G and set of all elements of a graph $G = (V, E)$ is $V \cup E$, where V and E are set of vertices and set of edges respectively.

Definition 4.1. *Two elements u and v of a graph G are said to be contact if one of the following holds:*

(i) u and v are adjacent edges.
(ii) one of u and v is a vertex and the other an incident edge.

For a (p, q) graph G, let $S = \{x_1, x_2, ..., x_{p+q}, x_1\}$ be a sequence of the $p + q$ elements of G.

Theorem 4.1. *The Gallai middle graph $\Gamma_M(G)$ of a non-trivial (p, q) graph G is Hamiltonian if and only if G contains a sequence $S = \{x_1, x_2, ..., x_{p+q}, x_1\}$ such that every two consecutive elements of S are contacts but not both are edges of an induced K_3 of G, where $x_i{}'s$ are element of G.*

Proof. Let G be a graph with a sequence S as stated. By definition, every two consecutive elements in S are adjacent vertices in $\Gamma_M(G)$. Thus, S corresponds to a Hamiltonian cycle in $\Gamma_M(G)$. The sufficiency holds.

Let $\Gamma_M(G)$ be a Hamiltonian graph. It follows that it contains a Hamiltonian cycle,

$$C = (v_1, v_2, \ldots, v_{p+q}, v_1).$$

Let x_i be that element of G associated with the vertex v_i. Thus, we get a sequence, say $S = \{x_1, x_2, \ldots, x_{p+q-1}, x_{p+q}, x_1\}$ of elements of G. By the definition of $\Gamma_M(G)$, we know that two consecutive vertices correspond to two adjacent edges not belong to an induced K_3 or correspond to a vertex and its incident edge. This implies that two edges that belong to an induced K_3 and two vertices are not consecutive elements of S. Thus, C corresponds to a sequence S of elements of G such that every two consecutive elements are contacts but not both are edges of an induced K_3 of G. □

The Hamiltonian Gallai middle graph $\Gamma_M(G)$ of G is shown in Figure 4. A required sequence S of G is $\{1, a, 2, b, 3, h, 4, c, f, 6, g, 5, e, d, 1\}$.

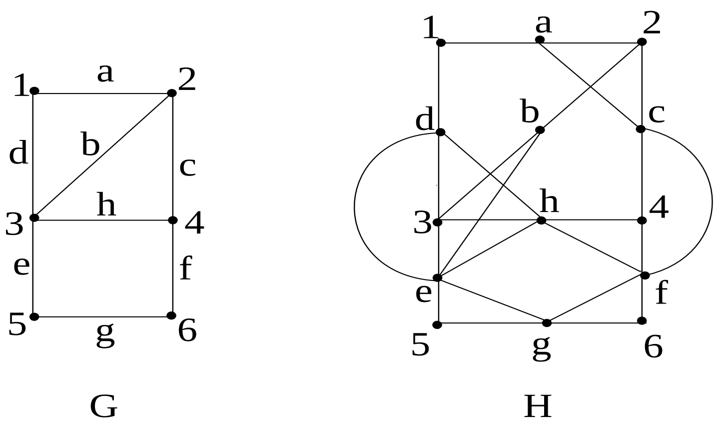

FIGURE 4. A graph G and its Hamiltonian Gallai middle graph $H = \Gamma_M(G)$

By a similar argument, we have the following theorem on Hamiltonian anti-Gallai middle graph.

Theorem 4.2. *The anti-Gallai middle graph $\Delta_M(G)$ of a non-trivial (p, q) graph G is Hamiltonian if and only if G contains a sequence $S = \{x_1, x_2, ..., x_{p+q}, x_1\}$ such that every two consecutive elements of S are contacts but not both are edges not belong to an induced K_3 of G, where $x_i{}'s$ are element of G.*

Proof. Similar to the argument for the proof of Theorem 4.1. □

The Hamiltonian anti-Gallai middle graph $\Delta_M(G)$ of G is shown in Figure 5. A required sequence S of G is : $\{1, a, e, 5, f, 2, c, h, 4, d, g, 3, b, 1\}$.

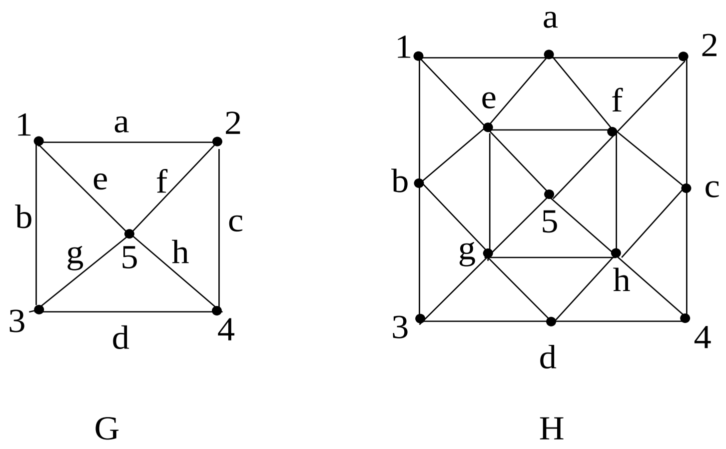

FIGURE 5. A graph G and its Hamiltonian anti-Gallai middle graph $H = \Delta_M(G)$

5. Conclusion

In this paper, we have introduced two graph operators, namely, Gallai middle graph and anti-Gallai middle graph. Further, we have presented some simple properties of Gallai and anti-Gallai middle graphs. Next, we have established the results related to Eulerian and Hamiltonian properties of Gallai and anti-Gallai middle graphs.

Acknowledgement. The authors would like to extend their gratitude to the anonymous referee(s) for careful reading and constructive suggestion for the improvement of the paper.

References

[1] Beineke, L. W., (1968), Derived graphs and digraphs, Beiträge zur Graphentheorie (Eds. H. Sachs, H. Voss and H. Walter), Teubner, Leipzig, pp. 17-33.
[2] Beineke, L. W., (1970), Characterizations of derived graphs, J. Combinatorial Theory, 9, pp. 129-135.
[3] Chvatal, V. and Sbihi, N., (1988), Recognizing claw-free perfect graphs, J. Combin. Theory Ser., B., 44, pp. 154-176.
[4] Gallai, T., (1967), Transitive orientierbare graphen, Acta Math. Acad. Sci. Hung., 18, pp. 25-66.
[5] Garg, P., Sinha, D. and Goyal, S., (2015), Eulerian and Hamiltonian properties of Gallai and anti-Gallai total graphs, J. Indones. Math. Soc., 21, (2), pp. 105-116.
[6] Godsil, C. D. and Royle, G., (2001), Algebraic Graph Theory, Springer, New York.
[7] Hamada, T. and Yoshimura, I., (1967), Traversibility and connectivity of the middle graph of a graph, Discrete Math., 14, pp. 247-256.
[8] Lakshmanan, A. S., Rao, S. B. and Vijayakumar, A., (2007), Gallai and anti-Gallai graphs of a graph, Math. Bohem., 132, (1), pp. 43-54.
[9] Le, V. B., (1996), Gallai and anti-Gallai graphs, Discrete Math., 159, pp. 179-189.
[10] Ray-Chaudhuri, D., (1967), Characterization of line graphs, J. Combinatorial Theory, 3, pp. 201-214.
[11] Sampathkumar, E. and Chikkodimath, S. B., (1973), Semitotal graphs of a graph-I, Journal of Karnatak University Science, 18, pp. 274-280.
[12] Sampathkumar, E. and Chikkodimath, S. B., (1973), Semitotal graphs of a graph-II, Journal of Karnatak University Science, 18, pp. 281-284.
[13] Sampathkumar, E. and Chikkodimath, S. B., (1973), Semitotal graphs of a graph-III, Journal of Karnatak University Science, 18, pp. 285-296.
[14] Sun, L., (1977), Two classes of perfect graphs, J. Combin. Theory Ser., B, 23, pp. 143-149.
[15] Whitney, H., (1932), Congruent graphs and the connectivity of graphs, Amer. J. of Math., 54, (1), pp. 150-168.

Dr. Shanu Goyal graduated from Banasthali University, Newai, Tonk, Rajasthan and she is an assistant professor in the same institute. Her area of interest includes Graph Theory.

Dr. Dilip Jain is a post graduated of IIT Delhi, New Delhi and he holds a PhD from S.V. Sardar Vallabhbhai National Institute of Technlogy, Surat Gujarat, India. He is a lecutrer in Mathematics in Technical Education Department, Rajasthan, India. His area of interest includes Graph Theory and Fixed Point Theory.