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APPLICATION OF THE OPERATOR $\phi\left(\begin{matrix} a,b,c \\ d,e \end{matrix};q,fD_q\right)$ FOR THE POLYNOMIALS $Y_n(a,b,c;d,e;x,y|q)$

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Abstract. In this paper, we construct the exponential operator $\phi\left(\begin{matrix} a,b,c \\ d,e \end{matrix};q,fD_q\right)$ that has five parameters a, b, c, d, e and we define a more general polynomials $Y_n(a,b,c;d,e;x,y|q)$, in which case, the bivariate Rogers-Szegö polynomials $h_n(x,y|q)$ become special cases of $Y_n(a,b,c;d,e;x,y|q)$. Furthermore, we involve the operator's technique to give an elegant proof for the generating function with its extension, Mehler's formula with its extension, and Rogers formula for the polynomials $Y_n(a,b,c;d,e;x,y|q)$. As well as, we present some special values for the parameters a, b, c, d, e that will be inserted in the identities of $Y_n(a,b,c;d,e;x,y|q)$ in order to establish the generating function and its extension, Mehler's formula and its extension, and the Rogers formula for $h_n(x,y|q)$.

Keywords: The bivariate Rogers-Szegö polynomials, the generating function, Mehler's formula, Rogers formula.

AMS Subject Classification: 05A30, 33D45.

1. Introduction

In this paper we will use the standard notations for basic hypergeometric series given in [6], we assume that $|q| < 1$.

The q-shifted factorial is defined by

$$(a;q)_n = \begin{cases} 1, & \text{if } n = 0, \\ \prod_{k=0}^{n-1}(1-aq^k), & \text{if } n = 1,2,\cdots. \end{cases}$$

We define

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

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The following notation is used for the multiple q-shifted factorials:

$$(a_1, a_2, \cdots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n, \qquad n = 0, 1, 2, \cdots.$$
$$(a_1, a_2, \cdots, a_m; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty.$$

The generalized basic hypergeometric series is defined by [6]:

$${}_r\phi_s\left(\begin{array}{c} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{array}; q, x\right) = \sum_{n=0}^{\infty} \frac{(a_1, \cdots, a_r; q)_n}{(q, b_1, \cdots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} x^n,$$

The case $r = s + 1$ is the most important class of series

$${}_{s+1}\phi_s\left(\begin{array}{c} a_1, \cdots, a_{s+1} \\ b_1, \cdots, b_s \end{array}; q, x\right) = \sum_{n=0}^{\infty} \frac{(a_1, \cdots, a_{s+1}; q)_n}{(q, b_1, \cdots, b_s; q)_n} x^n, \quad |x| < 1.$$

The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \dfrac{(q; q)_n}{(q; q)_k \, (q; q)_{n-k}}, & \text{if } 0 \leqslant k \leqslant n; \\ 0, & otherwise. \end{cases}$$

One of the most important identities is the Cauchy identity

$$\sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} x^n = \frac{(ax; q)_\infty}{(x; q)_\infty}, \quad |x| < 1. \tag{1}$$

Euler found the following special case of Cauchy identity:

$$\sum_{n=0}^{\infty} \frac{x^n}{(q; q)_n} = \frac{1}{(x; q)_\infty}, \quad |x| < 1. \tag{2}$$

The following identity is the q-Chu-Vandermonde summation formula:

$${}_2\phi_1\left(\begin{array}{c} q^{-n}, b \\ c \end{array}; q, q\right) = \frac{(c/b; q)_n}{(c; q)_n} b^n. \tag{3}$$

The q-differential operator D_q is the one defined by [4]

$$D_q \{f(x)\} = \frac{f(x) - f(xq)}{x}.$$

The Leibniz rule for D_q is [10]

$$D_q^n\{f(x)g(x)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D_q^k\{f(x)\} D_q^{n-k}\{g(xq^k)\}. \tag{4}$$

The following identities are easy to verify:

$$D_q^k \{x^n\} = \frac{(q; q)_n}{(q; q)_{n-k}} x^{n-k}. \tag{5}$$

$$D_q^k \left\{\frac{1}{(xt; q)_\infty}\right\} = \frac{t^k}{(xt; q)_\infty}, \quad |x| < 1. \tag{6}$$

Based on the Euler identity (2), Chen and Liu [4] defined the q-exponential operator $T(bD_q)$ as follows:

$$T(bD_q) = \sum_{n=0}^{\infty} \frac{(bD_q)^n}{(q; q)_n}.$$

They used the q-exponential operator $T(bD_q)$ to derive the generating function, Mehler's formula and Rogers formula for classical Rogers-Szegö polynomials $h_n(x|q)$ which is defined by

$$h_n(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k.$$

The Cauchy polynomials is defined by [7, 8]

$$P_k(x,y) = \begin{cases} (x-y)(x-qy)\cdots(x-yq^{k-1}), & \text{if } k>0, \\ 1, & \text{if } k=0. \end{cases}$$

In 2003, Chen *et. al.* [3] constructed the homogenous q-difference operator D_{xy} as follows:

$$D_{xy}\{f(x,y)\} = \frac{f(x,q^{-1}y) - f(qx,y)}{x - q^{-1}y}.$$

Based on the operator D_{xy}, they construct the following homogeneous q-shift operator

$$\mathbb{E}(D_{xy}) = \sum_{n=0}^{\infty} \frac{D_{xy}^n}{(q;q)_n}.$$

Also, they defined the bivariate Rogers-Szegö polynomials as follows:

$$h_n(x,y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} P_k(x,y).$$

By using the homogeneous q-shift operator $E(D_{xy})$, they derived the generating function for $h_n(x,y|q)$

$$\sum_{n=0}^{\infty} h_n(x,y|q) \frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(t,xt;q)_\infty}, \tag{7}$$

provided that $\max\{|t|,|xt|\} < 1$.

In 2007, Chen *et. al.* [5] used the q-exponential operator $T(bD_q)$ and the homogeneous q-shift operator $E(D_{xy})$ to derive Mehler's formula and Rogers formula for the polynomials $h_n(x,y|q)$.

In 2009, Saad and Sukhi [11] observed that $h_n(x,y|q)$ can be rewritten in the form

$$h_n(x,y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (y;q)_k x^{n-k}. \tag{8}$$

In 2013, Saad and Sukhi [12] defined the q-exponential operator $R(bD_q)$ as follows:

$$R(bD_q) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k} (bD_q)^k.$$

By using this operator, they derived Mehler's formula and Rogers formula for the polynomials $h_n(x,y|q)$.

Based on the q-Chu-Vandermonde summation formula (3), Zhang and Yang [14] considered the finite q-exponential operator with two parameters

$${}_2\mathscr{T}_1\left[\begin{matrix} q^{-N}, v \\ w \end{matrix} ; q, tD_q\right] = \sum_{n=0}^{N} \frac{(q^{-N},v;q)_n}{(q,w;q)_n} (tD_q)^n.$$

Inspired by the basic hypergeometric series ${}_2\phi_1$, Li and Tan [9] introduced the generalized q-exponential operator with three parameters

$$\mathbb{T}\left[\begin{matrix} u, v \\ w \end{matrix} \Big| q; tD_q\right] = \sum_{n=0}^{\infty} \frac{(u, v; q)_n}{(q, w; q)_n}(tD_q)^n.$$

Our work embraces four major parts that can be evidently organized as follows: In the first part, we set up the exponential operator $\phi\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, fD_q\right)$ and then we define the polynomials $Y_n(a, b, c; d, e; x, y|q)$ which generalizes the the bivariate Rogers-Szegö polynomials $h_n(x, y|q)$. Then, we proceed further to demonstrate three factors in the polynomials which are the generating function with its extension, Mehler's formula with its extension, and Rogers formula for the polynomials $Y_n(a, b, c; d, e; x, y|q)$ by using an appropriate operator. In the final step, we employ some special values for the parameters a, b, c, d, e of the operator ϕ that would be utilized in the identities of $Y_n(a, b, c; d, e; x, y|q)$ to obtain the generating function and its extension, Mehler's formula and its extension, and the Rogers formula for $h_n(x, y|q)$.

2. The q-exponential Operator ϕ and its Identities

We define the q-exponential operator with five parameters as follows:

$$\phi\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, fD_q\right) = \sum_{n=0}^{\infty} \frac{(a, b, c; q)_n}{(q, d, e; q)_n}(fD_q)^n. \tag{9}$$

The finite q-exponential operator with two parameters ${}_2\mathscr{T}_1\left[\begin{matrix} q^{-N}, v \\ w \end{matrix}; q, tD_q\right]$ defined by Zhang and Yang [14] can be considered as special case of our operator for $a = q^{-N}$, $b = v$, $d = w$ and $c = e = 0$. Also the generalized q-exponential operator with three parameters $\mathbb{T}\left[\begin{matrix} u, v \\ w \end{matrix} \Big| q; tD_q\right]$ defined by Li and Tan [9] can be considered as special case of our operator for $a = u$, $b = v$, $c = 0$, $d = w$, $e = 0$ and $f = t$. In this section, we give some operator identities that will be used later to give a proof operator for some identities.

Lemma 2.1. *We have*

$$\phi\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, fD_q\right)\left\{\frac{1}{(xt, xs; q)_\infty}\right\} = \frac{1}{(xt, xs; q)_\infty} \times \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{(a, b, c; q)_{k+j}}{(d, e; q)_{k+j}(q; q)_k} \frac{(xt; q)_j}{(q; q)_j}(fs)^j(ft)^k, \tag{10}$$

provided that $\max\{|xs|, |xt|\} < 1$.

Proof. By the definition of the operator $\phi\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, fD_q\right)$, we have

$$\phi\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, fD_q\right)\left\{\frac{1}{(xt, xs; q)_\infty}\right\} = \sum_{k=0}^{\infty} \frac{(a, b, c; q)_k}{(c, d; q)_k} f^k D_q^k\left\{\frac{1}{(xt, xs; q)_\infty}\right\}$$

By using Leibniz rule (4), we have

$$\begin{aligned}
&\phi\left(\begin{matrix}a,b,c\\ d,e\end{matrix};q,fD_q\right)\left\{\frac{1}{(xt,xs;q)_\infty}\right\}\\
&=\sum_{k=0}^{\infty}\frac{(a,b,c;q)_k}{(q,d,e;q)_k}f^k\sum_{j=0}^{k}\begin{bmatrix}k\\ j\end{bmatrix}q^{j(j-k)}D_q^j\left\{\frac{1}{(xs;q)_\infty}\right\}D_q^{k-j}\left\{\frac{1}{(xtq^j;q)_\infty}\right\}\\
&=\sum_{k=0}^{\infty}\frac{(a,b,c;q)_k}{(q,d,e;q)_k}f^k\sum_{j=0}^{k}\begin{bmatrix}k\\ j\end{bmatrix}q^{j(j-k)}\frac{s^j}{(xs;q)_\infty}\frac{(q^jt)^{k-j}}{(xtq^j;q)_\infty}\quad\text{(by using (6))}\\
&=\sum_{k=0}^{\infty}\frac{(a,b,c;q)_k}{(d,e;q)_k}f^k\sum_{j=0}^{k}\frac{s^jt^{k-j}(xt,q)_j}{(q;q)_{k-j}(q;q)_j(xs,xt;q)_\infty}\\
&=\frac{1}{(xt,xs;q)_\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(a,b,c;q)_{k+j}}{(d,e;q)_{k+j}(q;q)_k}\frac{(xt;q)_j}{(q;q)_j}(fs)^j(ft)^k.
\end{aligned}$$

□

Setting $s=0$ in (10), we get the following corollary:

Corollary 2.1.

$$\phi\left(\begin{matrix}a,b,c\\ d,e\end{matrix};q,fD_q\right)\left\{\frac{1}{(xt;q)_\infty}\right\}=\frac{1}{(xt;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix};q,ft\right),\tag{11}$$

provided that $\max\{|xt|,|ft|\}<1$.

Lemma 2.2. *For a nonnegative integer* n*, we have*

$$\begin{aligned}\phi\left(\begin{matrix}a,b,c\\ d,e\end{matrix};q,fD_q\right)\left\{\frac{x^n}{(xt;q)_\infty}\right\}&=\frac{x^n}{(xt;q)_\infty}\\
&\sum_{k=0}^{\infty}\sum_{j=0}^{n}\begin{bmatrix}n\\ j\end{bmatrix}\frac{(a,b,c;q)_{k+j}(xt;q)_j(ft)^k(f/x)^j}{(d,e;q)_{k+j}(q;q)_k},\end{aligned}\tag{12}$$

provided that $|xt|<1$.

Proof. From definition of the operator $\phi\left(\begin{matrix}a,b,c\\ d,e\end{matrix};q,fD_q\right)$, we have

$$\phi\left(\begin{matrix}a,b,c\\ d,e\end{matrix};q,fD_q\right)\left\{\frac{x^n}{(xt;q)_\infty}\right\}=\sum_{k=0}^{\infty}\frac{(a,b,c;q)_k}{(q,d,e;q)_k}f^kD^k\left\{\frac{x^n}{(xt;q)_\infty}\right\}.$$

By using Leibniz rule (4), we have

$$
\begin{aligned}
&\sum_{k=0}^{\infty}\frac{(a,b,c;q)_k}{(q,d,e;q)_k}f^kD^k\left\{\frac{x^n}{(xt;q)_\infty}\right\}\\
&=\sum_{k=0}^{\infty}\frac{(a,b,c;q)_k}{(q,d,e;q)_k}f^k\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}q^{j(j-k)}D^j\left\{x^n\right\}D^{k-j}\left\{\frac{1}{(q^jxt;q)_\infty}\right\}\\
&=\sum_{k=0}^{\infty}\frac{(a,b,c;q)_k}{(q,d,e;q)_k}f^k\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}q^{j(j-k)}\frac{(q;q)_n}{(q;q)_{n-j}}x^{n-j}\frac{(q^jt)^{k-j}}{(q^jxt;q)_\infty}\quad\text{(by using (5) and (6))}\\
&=\sum_{k=0}^{\infty}\frac{(a,b,c;q)_k}{(d,e;q)_k}f^k\sum_{j=0}^{k}\begin{bmatrix}n\\j\end{bmatrix}x^{n-j}t^{k-j}\frac{1}{(q^jxt;q)_\infty(q;q)_{k-j}}\\
&=\sum_{k=0}^{\infty}\sum_{j=0}^{n}\frac{(a,b,c;q)_{k+j}}{(d,e;q)_{k+j}}f^{k+j}\begin{bmatrix}n\\j\end{bmatrix}x^{n-j}t^k\frac{(xt,q)_j}{(xt;q)_\infty(q;q)_k}\\
&=\frac{x^n}{(xt;q)_\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}\frac{(a,b,c;q)_{k+j}(xt;q)_j(ft)^k(f/x)^j}{(d,e;q)_{k+j}(q;q)_k}.
\end{aligned}
$$

□

3. The Generating Function for $Y_n(a,b,c,d,e,f;q;x)$

We define the following polynomials:

$$Y_n(a,b,c;d,e;x,y|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(a,b,c;q)_k}{(d,e;q)_k}y^kx^{n-k}.$$

The bivariate Rogers-Szegö polynomials $h_n(x,y|q)$ can be regarded as special case of the polynomials $Y_n(a,b,c;d,e;x,y|q)$ for $b=c=d=e=0$, $y=1$ and then $a=y$. Setting $b=c=d=e=0$ and exchange x and y, we get the generalized Hahn polynomials [1]. In this section, we provide a working guide for the generating function and its extension for $Y_n(a,b,c;d,e;x,y|q)$. Then we give some parameter values to get the generating function and its extension for $h_n(x,y|q)$.

The following result is easy to verify:

$$\phi\left(\begin{matrix}a,b,c\\d,e\end{matrix};q,yD_q\right)\{x^n\}=Y_n(a,b,c;d,e;x,y|q). \tag{13}$$

Theorem 3.1. (The generating function for $Y_n(a,b,c;d,e;x,y|q)$). *We have*

$$\sum_{n=0}^{\infty}Y_n(a,b,c;d,e;x,y|q)\frac{t^n}{(q;q)_n}=\frac{1}{(xt;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}a,b,c\\d,e\end{matrix};q,yt\right), \tag{14}$$

provided that $\max\{|xt|,|ft|\}<1$.

Proof.

$$\begin{aligned}
&\sum_{n=0}^{\infty} Y_n(a,b,c;d,e;x,y|q)\frac{t^n}{(q;q)_n} \\
&= \sum_{n=0}^{\infty} \phi\left(\begin{matrix} a,b,c \\ d,e \end{matrix};q,yD_q\right)\{x^n\}\frac{t^n}{(q;q)_n} \quad \text{(by using (13))} \\
&= \phi\left(\begin{matrix} a,b,c \\ d,e \end{matrix};q,yD_q\right)\left\{\frac{1}{(xt;q)_\infty}\right\} \quad \text{(by using (2))} \\
&= \frac{1}{(xt;q)_\infty}\,{}_3\phi_2\left(\begin{matrix} a,b,c \\ d,e \end{matrix};q,yt\right). \quad \text{(by using (11))}
\end{aligned}$$

□

Setting $b=c=d=e=0$, $y=1$ and then $a=y$ in (14) and by using (8) and (1), we recover the generating function for the polynomials $h_n(x,y|q)$ (7).

Theorem 3.2. (Extension of generating function of $Y_n(a,b,c;d,e;x,y|q)$)

$$\begin{aligned}
&\sum_{n=0}^{\infty} Y_{n+m}(a,b,c;d,e;x,y|q)\frac{t^n}{(q;q)_n} \\
&= \frac{x^m}{(xt;q)_\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{m}\begin{bmatrix} m \\ j \end{bmatrix}\frac{(a,b,c;q)_{k+j}(xt;q)_j(yt)^k(y/x)^j}{(d,e;q)_{k+j}(q;q)_k}.
\end{aligned} \tag{15}$$

Proof.

$$\begin{aligned}
&\sum_{n=0}^{\infty} Y_{n+m}(a,b,c;d,e;x,y|q)\frac{t^n}{(q;q)_n} \\
&= \sum_{n=0}^{\infty} \phi\left(\begin{matrix} a,b,c \\ d,e \end{matrix};q,yD_q\right)\{x^{m+n}\}\frac{t^n}{(q;q)_n} \quad \text{(by using (13))} \\
&= \phi\left(\begin{matrix} a,b,c \\ d,e \end{matrix};q,yD_q\right)\left\{\frac{x^m}{(xt;q)_\infty}\right\} \quad \text{(by using 2)} \\
&= \frac{x^m}{(xt;q)_\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{m}\begin{bmatrix} m \\ j \end{bmatrix}\frac{(a,b,c;q)_{k+j}(xt;q)_j(yt)^k(y/x)^j}{(d,e;q)_{k+j}(q;q)_k}. \quad \text{(by using 12)}
\end{aligned}$$

□

Setting $b=c=d=e=0$, $y=1$ and then $a=y$ in the extension of generating function for the polynomials $Y_n(a,b,c;d,e;x,y|q)$ (15), we get the following extension of the generating function for the polynomials $h_n(x,y|q)$:

$$\sum_{n=0}^{\infty} h_{n+m}(x,y|q)\frac{t^n}{(q;q)_n} = \frac{x^m(yt;q)_\infty}{(xt,t;q)_\infty}\,{}_3\phi_1\left(\begin{matrix} q^{-m},xt,y \\ yt \end{matrix};q,q^m/x\right).$$

4. Mehler's Formula for $Y_n(a,b,c;d,e;x,y|q)$ and $h_n(x,y|q)$

In this section we use the operator $\phi\left(\begin{matrix} a,b,c \\ d,e \end{matrix};q,yD_q\right)$ to derive Mehler's formula and its extension for the polynomials $Y_n(a,b,c;d,e;x,y|q)$. Then we give some special values to the parameters to obtain Mehler's formula and its extension for $h_n(x,y|q)$.

Theorem 4.1. (Mehler's formula for $Y_n(a,b,c;d,e;x,y|q)$). *We have*

$$\sum_{n=0}^{\infty} Y_n(a,b,c;d,e;x,y|q)Y_n(a_1,b_1,c_1;d_1,e_1;u,v|q)\frac{t^n}{(q;q)_n}$$
$$= \frac{1}{(xut;q)_\infty}\sum_{k=0}^{\infty}\frac{(a,b,c;q)_k}{(q,d,e;q)_k}(yut)^k\sum_{i=0}^{\infty}\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\frac{(a_1,b_1,c_1;q)_{i+j}(xut;q)_j(vxt)^i(v/u)^j}{(d_1,e_1;q)_{i+j}(q;q)_i}. \quad (16)$$

Proof.

$$\sum_{n=0}^{\infty} Y_n(a,b,c;d,e;x,y|q)Y_n(a_1,b_1,c_1;d_1,e_1;u,v|q)\frac{t^n}{(q;q)_n}$$
$$= \sum_{n=0}^{\infty} Y_n(a,b,c;d,e;x,y|q)\phi\left(\begin{matrix}a_1,b_1,c_1\\d_1,e_1\end{matrix};q,vD_q\right)\{u^n\}\frac{t^n}{(q;q)_n}$$
$$= \phi\left(\begin{matrix}a_1,b_1,c_1\\d_1,e_1\end{matrix};q,vD_q\right)\left\{\sum_{n=0}^{\infty} Y_n(a,b,c;d,e;x,y|q)\frac{(ut)^n}{(q;q)_n}\right\}$$
$$= \phi\left(\begin{matrix}a_1,b_1,c_1\\d_1,e_1\end{matrix};q,vD_q\right)\left\{\frac{1}{(xut;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}a,b,c\\d,e\end{matrix};q,yut\right)\right\} \quad \text{(by using (14))}$$
$$= \sum_{k=0}^{\infty}\frac{(a,b,c;q)_k}{(q,d,e;q)_k}(yt)^k\phi\left(\begin{matrix}a_1,b_1,c_1\\d_1,e_1\end{matrix};q,vD_q\right)\left\{\frac{u^k}{(xut;q)_\infty}\right\}.$$

Applying the operator $\phi\left(\begin{matrix}a_1,b_1,c_1\\d_1,e_1\end{matrix};q,vD_q\right)$ with respect to the parameter u and by using (12), we get the required result. □

By using special values of the parameters in the Mehler's formula for the polynomials $Y_n(a,b,c;d,e;x,y|q)$ (16), we recover Mehler's formula for $h_n(x,y|q)$.

Corollary 4.1. (Mehler's formula for $h_n(x,y|q)$). *We have*

$$\sum_{n=0}^{\infty} h_n(x,y|q)h_n(u,v|q)\frac{t^n}{(q;q)_n} = \frac{(yt,xvt;q)_\infty}{(yut,t;q)}\,{}_3\phi_2\left(\begin{matrix}y,xt,v/u\\xvt,yt\end{matrix};q,ut\right).$$

Proof. Setting $b = c = d = e = 0$, $y = 1$ and then $a = y$, and $h = u = v = w = 0$, $r = 1$ and then $g = v$ in Mehler's formula for $Y_n(a, b, c, d, e, f; x; q)$ (16), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} h_n(x, y|q)h_n(u, v|q)\frac{t^n}{(q;q)_n} \\
&= \frac{1}{(xut;q)_\infty}\sum_{k=0}^{\infty}\frac{(y;q)_k}{(q;q)_k}(ut)^k\sum_{i=0}^{\infty}\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\frac{(v;q)_{i+j}(xut;q)_j}{(q;q)_i}(xt)^i(1/u)^j \\
&= \frac{1}{(xut;q)_\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(y;q)_{k+j}}{(q;q)_{k+j}}(ut)^{k+j}\frac{(q;q)_{k+j}}{(q;q)_k(q;q)_j}\frac{(v;q)_{i+j}(xut;q)_j}{(q;q)_i}(xt)^i(1/u)^j \\
&= \frac{1}{(xut;q)_\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(y;q)_{k+j}(ut)^{k+j}}{(q;q)_k}\frac{(v, xut;q)_j(1/u)^j}{(q;q)_j}\sum_{i=0}^{\infty}\frac{(vq^j;q)_i(xt)^i}{(q;q)_i} \\
&= \frac{1}{(xut;q)_\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(y;q)_{k+j}(ut)^{k+j}}{(q;q)_k}\frac{(v, xut;q)_j(1/u)^j}{(q;q)_j}\frac{(vxtq^j;q)_\infty}{(xt;q)_\infty} \quad \text{(by using (1))} \\
&= \frac{(xvt;q)_\infty}{(xut, xt;q)_\infty}\sum_{j=0}^{\infty}\frac{(v, xut, y;q)_j t^j}{(q, xvt;q)_j}\sum_{k=0}^{\infty}\frac{(yq^j;q)_k(ut)^k}{(q;q)_k} \\
&= \frac{(xvt;q)_\infty}{(xut, xt;q)_\infty}\sum_{j=0}^{\infty}\frac{(v, xut, y;q)_j t^j}{(q, xvt;q)_j}\frac{(yutq^j;q)_\infty}{(ut;q)_\infty} \quad \text{(by using (1))} \\
&= \frac{(xvt, uty;q)_\infty}{(xut, xt, ut;q)_\infty}\sum_{j=0}^{\infty}\frac{(v, xut, y;q)_j t^j}{(q, xvt, yut;q)_j} \\
&= \frac{(xvt, uty;q)_\infty}{(xut, xt, ut;q)_\infty}\, {}_3\phi_2\left(\begin{matrix} v, y, xut \\ yut, xvt\end{matrix}; q, t\right). \qquad (17)
\end{aligned}$$

By using transformation of ${}_3\phi_2$ series [6, Appendix , Eq. (III.9)], we get

$${}_3\phi_2\left(\begin{matrix} y, v, xut \\ xvt, yut\end{matrix}; q, t\right) = \frac{(ut, yt;q)_\infty}{(yut, t;q)}\, {}_3\phi_2\left(\begin{matrix} y, xt, v/u \\ xvt, yt\end{matrix}; q, ut\right).$$

Substituting the above equation into (17), we get the required result. □

Theorem 4.2. (Extension of Mehler's formula for $Y_n(a, b, c, d, e, f; x; q)$)

$$\begin{aligned}
&\sum_{n=0}^{\infty} Y_{n+m}(a, b, c; d, e; x, y|q)Y_n(a_1, b_1, c_1; d_1, e_1; u, v|q)\frac{t^n}{(q;q)_n} \\
&= \frac{x^m}{(vtx;q)_\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{m}\begin{bmatrix}m\\j\end{bmatrix}\frac{(a, b, c;q)_{k+j}(y/x)^j(ytv)^k(vtx;q)_j}{(d, e;q)_{k+j}(q;q)_k} \\
&\quad\times\sum_{i=0}^{\infty}\sum_{l=0}^{k}\begin{bmatrix}k\\l\end{bmatrix}\frac{(a_1, b_1, c_1;q)_{i+l}(vtxq^j;q)_l(xvtq^j)^i(v/u)^l}{(d_1, e_1;q)_{l+i}(q;q)_i}. \qquad (18)
\end{aligned}$$

Proof.

$$\sum_{n=0}^{\infty} Y_{n+m}(a,b,c;d,e;x,y;q) Y_n(a_1,b_1,c_1;d_1,e_1;u,v;q)\frac{t^n}{(q;q)_n}$$
$$= \sum_{n=0}^{\infty} Y_{n+m}(a,b,c;d,e;x,y;q)\phi\left(\begin{matrix} a_1,b_1,c_1 \\ d_1,e_1 \end{matrix};q,vD_q\right)\{u^n\}\frac{t^n}{(q;q)_n} \quad \text{(by using (13))}$$
$$= \phi\left(\begin{matrix} a_1,b_1,c_1 \\ d_1,e_1 \end{matrix};q,vD_q\right)\left\{\sum_{n=0}^{\infty} Y_{n+m}(a,b,c;d,e;x,y;q)\frac{(ut)^n}{(q;q)_n}\right\}$$
$$= \phi\left(\begin{matrix} a_1,b_1,c_1 \\ d_1,e_1 \end{matrix};q,vD_q\right)\left\{\frac{x^m}{(xut;q)_\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{m}\begin{bmatrix} m \\ j \end{bmatrix}\frac{(a,b,c;q)_{k+j}(xut;q)_j(yut)^k(y/x)^j}{(d,e;q)_{k+j}(q;q)_k}\right\}$$
$$\text{(by using (15))}$$
$$= z^m\sum_{k=0}^{\infty}\sum_{j=0}^{m}\frac{(a,b,c;q)_{k+j}(yt)^k(y/x)^j}{(d,e;q)_{k+j}(q;q)_k}\phi\left(\begin{matrix} a_1,b_1,c_1 \\ d_1,e_1 \end{matrix};q,vD_q\right)\left\{\frac{u^k}{(uxtq^j;q)_\infty}\right\}$$
$$= x^m\sum_{k=0}^{\infty}\sum_{j=0}^{m}\begin{bmatrix} m \\ j \end{bmatrix}\frac{(a,b,c;q)_{k+j}(yt)^k(y/x)^j}{(d,e;q)_{k+j}(q;q)_k}\frac{u^k}{(uxtq^j;q)_\infty}\sum_{i=0}^{\infty}\sum_{l=0}^{k}\begin{bmatrix} k \\ l \end{bmatrix}\frac{(a_1,b_1,c_1;q)_{i+l}}{(d_1,e_1;q)_{l+i}(q;q)_i}$$
$$\times\frac{(uxtq^j;q)_l(vxtq^j)^i(v/u)^l}{(q;q)_i} \quad \text{(by using (12))}$$
$$= \frac{x^m}{(utx;q)_\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{m}\begin{bmatrix} m \\ j \end{bmatrix}\frac{(a,b,c;q)_{k+j}(y/x)^j(ytu)^k(utx;q)_j}{(d,e;q)_{k+j}(q;q)_k}$$
$$\times\sum_{i=0}^{\infty}\sum_{l=0}^{k}\begin{bmatrix} k \\ l \end{bmatrix}\frac{(a_1,b_1,c_1;q)_{i+l}(utxq^j;q)_l(xvtq^j)^i(v/u)^l}{(d_1,e_1;q)_{l+i}(q;q)_i}.$$

□

Setting $a=y$, $b=c=d=e=0$, $f=1$, $z=x$, $g=v$, $h=u=v=w=0$, $r=1$ and $x=u$ in the extension of Mehler's formula for $Y_n(a,b,c,d,e,f;x;q)$ (18) and by using (8), we get the following extension of Mehler's formula for the polynomials $h_n(x,y|q)$:

$$\sum_{n=0}^{\infty} h_{n+m}(x,y|q)h_n(u,v|q)\frac{t^n}{(q;q)_n}$$
$$= \frac{x^m(vxt,yut;q)_\infty}{(uxt,xt,ut;q)_\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{m}\frac{(y;q)_{j+l}(uxt,xt;q)_j(v,xuq^j;q)_l(1/x)^j t^l}{(xvt,yut;q)_{j+l}(q;q)_l}.$$

5. The Rogers Formula for $Y_n(a,b,c;d,e;x,y|q)$ and $h_n(x,y|q)$

In this section we use the operator $\phi\left(\begin{matrix} a,b,c \\ d,e \end{matrix};q,yD_q\right)$ to derive Rogers formula for the polynomials $Y_n(a,b,c;d,e;x,y|q)$. Then we give some special values for the parameters in Rogers formula for polynomial $Y_n(a,b,c;d,e;x,y|q)$ to recover Rogers formula for $h_n(x,y|q)$.

Theorem 5.1. (The Rogers formula for $Y_n(a,b,c;d,e;x,y|q)$). We have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} Y_{n+m}(a,b,c,d,e,f;x;q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$
$$= \frac{1}{(xt,xs;q)_\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(a,b,c;q)_{k+j}(xt;q)_j(ft)^k(fs)^j}{(d,e;q)_{k+j}(q;q)_k(q;q)_j}, \tag{19}$$

provided that $\max\{|xt|,|xs|\}<1$.

Proof.

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} Y_{n+m}(a,b,c,d,e,f;x;q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\phi\left(\begin{matrix} a,b,c \\ d,e\end{matrix};q,yD_q\right)\left\{x^{n+m}\right\}\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$
$$= \phi\left(\begin{matrix} a,b,c \\ d,e\end{matrix};q,yD_q\right)\left\{\frac{1}{(xt,xs;q)_\infty}\right\}. \quad \text{(by using (2))}$$

By using (10), we get the required result. □

Setting $a=y$, $b=c=d=e=0$ and $f=1$ in the Rogers formula of $Y_n(a,b,c,d,e,f;q;x)$ (5.1), we recover the following Rogers formula for the bivariate Rogers-Szegö polynomials $h_n(x,y|q)$ [5]:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(x,y|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m} = \frac{(ys;q)_\infty}{(xs,s,xt;q)_\infty}\,{}_2\phi_1\left(\begin{matrix} y,xs \\ ys\end{matrix};q,t\right),$$

provided that $\max\{|s|,|xt|,|xs|\}<1$.

6. Conclusions

(1) The finite q-exponential operator ${}_2\mathscr{T}_1\left[\begin{matrix} q^{-N},v \\ w\end{matrix};q,tD_q\right]$ and the generalized q-exponential operator $\mathbb{T}\left[\begin{matrix} u,v \\ w\end{matrix}\Big|q;tD_q\right]$ are special cases of the operator $\phi\left(\begin{matrix} a,b,c \\ d,e\end{matrix};q,yD_q\right)$.

(2) We may give special values in polynomial identities for $Y_n(a,b,c;d,e;x,y|q)$ to obtain polynomial identities for $h_n(x,y|q)$ versus $a=y$, $b=0$, $c=0$, $d=0$, $e=0$ and $f=1$ as seen in the generating function, Rogers formula and Mehler's formula.

(3) Generalized Han polynomials [1] are a special case of our polynomial $Y_n(a,b,c;d,e;x,y|q)$ versus $b=c=d=e=0$, $x=y$ and then $f=x$.

(4) The operator proof is simpler than the classical proof.

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