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.

CONCORDANCE OF DYNAMIC FRACTIONAL INEQUALITIES INTERCONNECTED ON TIME SCALES

M. J. S. SAHIR[1], §

Abstract. In this work, we present an extension of dynamic reverse Minkowski's inequality by using the time scale Riemann–Liouville type fractional integrals. By using the definitions of delta and nabla time scales Riemann–Liouville type fractional integral operators, we find other general dynamic fractional inequalities. Our findings unify and extend some continuous, discrete and quantum analogues.

Keywords: Time scales, the time scales Riemann–Liouville type fractional integrals, reverse Minkowski's inequality.

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1. Introduction

Bougoffa [11] proved the following result concerning reverse of Minkowski's inequality. Let f and g be positive functions satisfying

$$0 < m \leq \frac{f(y)}{g(y)} \leq M, \ \forall y \in [a, b].$$

Then

$$\left(\int_a^b f^p(y)dy\right)^{\frac{1}{p}} + \left(\int_a^b g^p(y)dy\right)^{\frac{1}{p}} \leq \Omega_1 \left(\int_a^b (f(y) + g(y))^p \, dy\right)^{\frac{1}{p}}, \tag{1}$$

where $p \geq 1$ and $\Omega_1 = \frac{M(m+1)+M+1}{(m+1)(M+1)}$.

The following result is given in [17].

Let $f, g \in L^p(a, b)$ be two positive functions, with $p \geq 1$. If $0 < m \leq \frac{f(y)}{g(y)} \leq M$, $\forall y \in [a, b]$ for $m, M \in (0, \infty)$, then

$$\left(\int_a^b f^p(y)dy\right)^{\frac{2}{p}} + \left(\int_a^b g^p(y)dy\right)^{\frac{2}{p}} \geq \Omega_2 \left(\int_a^b f^p(y)dy\right)^{\frac{1}{p}} \left(\int_a^b g^p(y)dy\right)^{\frac{1}{p}}, \tag{2}$$

where $\Omega_2 = \frac{(m+1)(M+1)}{M} - 2$.

[1] Department of Mathematics, University of Sargodha, Sub-Campus Bhakkar, Pakistan.
e-mail: jibrielshahab@gmail.com; ORCID: https://orcid.org/0000-0002-7854-8266.

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In 2012, Sulaiman [20] proved an integral inequality as follows:
Let $f, g > 0$. If $p \geq 1$ and $1 < m \leq \frac{f(y)}{g(y)} \leq M$ for all $y \in [a, b]$, then

$$\frac{M+1}{M-1}\left(\int_a^b (f(y)-g(y))^p\,dy\right)^{\frac{1}{p}} \leq \left(\int_a^b f^p(y)dy\right)^{\frac{1}{p}} + \left(\int_a^b g^p(y)dy\right)^{\frac{1}{p}}$$
$$\leq \frac{m+1}{m-1}\left(\int_a^b (f(y)-g(y))^p\,dy\right)^{\frac{1}{p}}. \quad (3)$$

We will investigate the unification and extension of the above given results on time scales. The theory of time scales (initiated by Stefan Hilger [13]) is applied to combine results in one comprehensive and hybridized form. The theory of time scales is more general in its nature and is utilized to unify differential calculus, difference calculus, and quantum calculus. The three main partitions of the theory of time scales are delta calculus, nabla calculus, and diamond–α calculus. Generalizations, refinements and extensions of the theory and applications of dynamic inequalities concerning the calculus of time scales have been recently explored.

The usual notation $[a, b]_{\mathbb{T}}$ denotes the intersection of a real interval with the given time scale with $a, b \in \mathbb{T}$ and $a < b$. Moreover, we suppose that all considerable integrals exist and are finite.

2. Preliminaries

We recall basic results related to the delta calculus. The concepts of delta calculus are derived from monographs [8, 9].

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

The mapping $\mu : \mathbb{T} \to \mathbb{R}_0^+ = [0, +\infty)$ such that $\mu(t) := \sigma(t) - t$ is called the forward graininess function. The backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

The mapping $\nu : \mathbb{T} \to \mathbb{R}_0^+ = [0, +\infty)$ such that $\nu(t) := t - \rho(t)$ is called the backward graininess function. If $\sigma(t) > t$, we say that t is right–scattered, while if $\rho(t) < t$, we say that t is left–scattered. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right–dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left–dense. If $\mathbb{T}$ has a left–scattered maximum M, then $\mathbb{T}^k = \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^k = \mathbb{T}$.

For a function $f : \mathbb{T} \to \mathbb{R}$, the delta derivative f^Δ is defined as follows:
Let $t \in \mathbb{T}^k$. If there exists $f^\Delta(t) \in \mathbb{R}$ such that for all $\epsilon > 0$, there is a neighborhood U of t, such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|,$$

for all $s \in U$, then f is said to be delta differentiable at t, and $f^\Delta(t)$ is called the delta derivative of f at t.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be right–dense continuous (rd–continuous), if it is continuous at each right–dense point and there exists a finite left–sided limit at every left–dense point. The set of all rd–continuous functions is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

The next definition is given in [8, 9].

Definition 2.1. *A function $F : \mathbb{T} \to \mathbb{R}$ is called a delta antiderivative of $f : \mathbb{T} \to \mathbb{R}$, provided that $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^k$. Then the delta integral of f is defined by*

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

The following results of nabla calculus are taken from [7, 8, 9].

If $\mathbb{T}$ has a right–scattered minimum m, then $\mathbb{T}_k = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}_k = \mathbb{T}$. A function $f : \mathbb{T}_k \to \mathbb{R}$ is called nabla differentiable at $t \in \mathbb{T}_k$, with nabla derivative $f^{\nabla}(t)$, if there exists $f^{\nabla}(t) \in \mathbb{R}$ such that given any $\epsilon > 0$, there is a neighborhood V of t, such that

$$|f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s)| \leq \epsilon|\rho(t) - s|,$$

for all $s \in V$.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be left–dense continuous (ld–continuous), provided it is continuous at all left–dense points in $\mathbb{T}$ and its right–sided limits exist (finite) at all right–dense points in $\mathbb{T}$. The set of all ld–continuous functions is denoted by $C_{ld}(\mathbb{T}, \mathbb{R})$.

The next definition is given in [7, 8, 9].

Definition 2.2. *A function $G : \mathbb{T} \to \mathbb{R}$ is called a nabla antiderivative of $g : \mathbb{T} \to \mathbb{R}$, provided that $G^{\nabla}(t) = g(t)$ holds for all $t \in \mathbb{T}_k$. Then the nabla integral of g is defined by*

$$\int_a^b g(t)\nabla t = G(b) - G(a).$$

The following definition is taken from [4, 6].

Definition 2.3. *Let $f \in C_{rd}$. For $\alpha \geq 1$, the time scale Δ–Riemann–Liouville type fractional integral is defined by*

$$\mathcal{I}_a^{\alpha} f(t) = \int_a^t h_{\alpha-1}(t, \sigma(\tau)) f(\tau)\Delta\tau, \tag{4}$$

which is an integral on $[a, t)_{\mathbb{T}}$, see [10] *and $h_{\alpha} : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, $\alpha \geq 0$ are the coordinate wise rd–continuous functions, such that $h_0(t, s) = 1$,*

$$h_{\alpha+1}(t, s) = \int_s^t h_{\alpha}(\tau, s)\Delta\tau, \ \forall s, t \in \mathbb{T}. \tag{5}$$

Notice that

$$\mathcal{I}_a^1 f(t) = \int_a^t f(\tau)\Delta\tau,$$

which is absolutely continuous in $t \in [a, b]_{\mathbb{T}}$, see [10].

The following definition is taken from [5, 6].

Definition 2.4. *Let $f \in C_{ld}$. For $\alpha \geq 1$, the time scale ∇–Riemann–Liouville type fractional integral is defined by*

$$\mathcal{J}_a^{\alpha} f(t) = \int_a^t \hat{h}_{\alpha-1}(t, \rho(\tau)) f(\tau)\nabla\tau, \tag{6}$$

which is an integral on $(a, t]_{\mathbb{T}}$, see [10] *and $\hat{h}_{\alpha} : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, $\alpha \geq 0$ are the coordinate wise ld–continuous functions, such that $\hat{h}_0(t, s) = 1$,*

$$\hat{h}_{\alpha+1}(t, s) = \int_s^t \hat{h}_{\alpha}(\tau, s)\nabla\tau, \ \forall s, t \in \mathbb{T}. \tag{7}$$

Notice that

$$\mathcal{J}_a^1 f(t) = \int_a^t f(\tau)\nabla\tau,$$

which is absolutely continuous in $t \in [a,b]_{\mathbb{T}}$, *see* [10].

We need the following results.

The following Minkowski's inequality [3] holds:

Let $w, f, g \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$ and $p > 1$. Then

$$\left(\int_a^b |w(x)||f(x)+g(x)|^p \Delta x\right)^{\frac{1}{p}} \leq \left(\int_a^b |w(x)||f(x)|^p \Delta x\right)^{\frac{1}{p}} + \left(\int_a^b |w(x)||g(x)|^p \Delta x\right)^{\frac{1}{p}}. \quad (8)$$

The following Young's inequality [3] holds:

$$a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}, \quad (9)$$

where $a, b \geq 0$ and $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$.

3. Main Results

To present our main results, first we give a simple proof for an extension of dynamic reverse Minkowski's inequality on time scales by using the Δ–Riemann–Liouville type fractional integral.

Theorem 3.1. *Let* $w, f, g \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$ *be* Δ*–integrable functions,* $h_{\alpha-1}(.,.) > 0$ *and* $p \geq 1$. *Then for* $\alpha \geq 1$, *we have*

$$(\mathcal{I}_a^\alpha (|w(x)||f(x)|^p))^{\frac{1}{p}} + (\mathcal{I}_a^\alpha (|w(x)||g(x)|^p))^{\frac{1}{p}} \leq \Omega_3 \left(\mathcal{I}_a^\alpha (|w(x)| (|f(x)| + |g(x)|)^p)\right)^{\frac{1}{p}}, \quad (10)$$

with some positive constants m_1, m_2, M_1 *and* M_2 *such that* $\frac{m_1}{M_2} \leq \left|\frac{f(y)}{g(y)}\right| \leq \frac{M_1}{m_2}$ *on the set* $[a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$, *where* $\Omega_3 = \frac{M_1(m_1+M_2)+M_2(M_1+m_2)}{(M_1+m_2)(m_1+M_2)}$.

Proof. From the given conditions, we obtain

$$|f(y)|^p \leq \left(\frac{M_1}{m_2+M_1}\right)^p (|f(y)| + |g(y)|)^p \quad (11)$$

and

$$|g(y)|^p \leq \left(\frac{M_2}{m_1+M_2}\right)^p (|f(y)| + |g(y)|)^p, \ \forall y \in [a,x]_{\mathbb{T}}. \quad (12)$$

Multiplying by $h_{\alpha-1}(x,\sigma(y))|w(y)|$, where $h_{\alpha-1}(x,\sigma(y)) > 0$, $\forall x \in [a,b]_{\mathbb{T}}$ on both sides of inequality (11) and integrating with respect to the variable y, we get

$$\int_a^x h_{\alpha-1}(x,\sigma(y))|w(y)||f(y)|^p \Delta y \leq \left(\frac{M_1}{m_2+M_1}\right)^p \int_a^x h_{\alpha-1}(x,\sigma(y))|w(y)| (|f(y)| + |g(y)|)^p \Delta y. \quad (13)$$

Thus, it follows from (13)

$$\left(\mathcal{I}_a^{\alpha}\left(|w(x)||f(x)|^p\right)\right)^{\frac{1}{p}} \leq \frac{M_1}{M_1+m_2}\left(\mathcal{I}_a^{\alpha}\left(|w(x)|\left(|f(x)|+|g(x)|\right)^p\right)\right)^{\frac{1}{p}}. \tag{14}$$

Similarly, we get from (12)

$$\left(\mathcal{I}_a^{\alpha}\left(|w(x)||g(x)|^p\right)\right)^{\frac{1}{p}} \leq \frac{M_2}{m_1+M_2}\left(\mathcal{I}_a^{\alpha}\left(|w(x)|\left(|f(x)|+|g(x)|\right)^p\right)\right)^{\frac{1}{p}}. \tag{15}$$

Adding (14) and (15), we get the desired claim. □

Next, we give an extension of dynamic reverse Minkowski's inequality on time scales by using the ∇–Riemann–Liouville type fractional integral.

Theorem 3.2. *Let $w, f, g \in C_{ld}\left([a,b]_{\mathbb{T}}, \mathbb{R}\right)$ be ∇–integrable functions, $\hat{h}_{\alpha-1}(.,.) > 0$ and $p \geq 1$. Then for $\alpha \geq 1$, we have*

$$\left(\mathcal{J}_a^{\alpha}\left(|w(x)||f(x)|^p\right)\right)^{\frac{1}{p}} + \left(\mathcal{J}_a^{\alpha}\left(|w(x)||g(x)|^p\right)\right)^{\frac{1}{p}} \leq \Omega_3\left(\mathcal{J}_a^{\alpha}\left(|w(x)|\left(|f(x)|+|g(x)|\right)^p\right)\right)^{\frac{1}{p}}, \tag{16}$$

with some positive constants m_1, m_2, M_1 and M_2 such that $\frac{m_1}{M_2} \leq \left|\frac{f(y)}{g(y)}\right| \leq \frac{M_1}{m_2}$ on the set $[a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$, where $\Omega_3 = \frac{M_1(m_1+M_2)+M_2(M_1+m_2)}{(M_1+m_2)(m_1+M_2)}$.

Proof. Similar to the proof of Theorem 3.1. □

Remark 3.1. *Let $\alpha = 1$, $\mathbb{T} = \mathbb{R}$, $x = b$, $w \equiv 1$, $f, g \in (0, +\infty)$, $m = \frac{m_1}{M_2}$ and $M = \frac{M_1}{m_2}$. Then (10) reduces to (1).*

Theorem 3.3. *Let $w, f, g \in C_{rd}\left([a,b]_{\mathbb{T}}, \mathbb{R}\right)$ be Δ–integrable functions, $h_{\alpha-1}(.,.) > 0$ and $p \geq 1$. Then for $\alpha \geq 1$, we have*

$$\left(\mathcal{I}_a^{\alpha}\left(|w(x)||f(x)|^p\right)\right)^{\frac{2}{p}} + \left(\mathcal{I}_a^{\alpha}\left(|w(x)||g(x)|^p\right)\right)^{\frac{2}{p}} \geq \Omega_4\left(\mathcal{I}_a^{\alpha}\left(|w(x)||f(x)|^p\right)\right)^{\frac{1}{p}}\left(\mathcal{I}_a^{\alpha}\left(|w(x)||g(x)|^p\right)\right)^{\frac{1}{p}}, \tag{17}$$

with some positive constants m_1, m_2, M_1 and M_2 such that $\frac{m_1}{M_2} \leq \left|\frac{f(y)}{g(y)}\right| \leq \frac{M_1}{m_2}$ on the set $[a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$, where $\Omega_4 = \frac{(M_1+m_2)(m_1+M_2)}{M_1M_2} - 2$.

Proof. Multiplying (14) and (15), we have that

$$\left(\mathcal{I}_a^{\alpha}\left(|w(x)||f(x)|^p\right)\right)^{\frac{1}{p}}\left(\mathcal{I}_a^{\alpha}\left(|w(x)||g(x)|^p\right)\right)^{\frac{1}{p}} \leq \frac{M_1M_2}{(M_1+m_2)(m_1+M_2)}\left(\mathcal{I}_a^{\alpha}\left(|w(x)|\left(|f(x)|+|g(x)|\right)^p\right)\right)^{\frac{2}{p}}. \tag{18}$$

By applying dynamic Minkowski's inequality similar to (8) on right–hand side of (18), we obtain

$$\left(\mathcal{I}_a^{\alpha}\left(|w(x)||f(x)|^p\right)\right)^{\frac{1}{p}}\left(\mathcal{I}_a^{\alpha}\left(|w(x)||g(x)|^p\right)\right)^{\frac{1}{p}} \leq \frac{M_1M_2}{(M_1+m_2)(m_1+M_2)}\left\{\left(\mathcal{I}_a^{\alpha}\left(|w(x)||f(x)|^p\right)\right)^{\frac{1}{p}} + \left(\mathcal{I}_a^{\alpha}\left(|w(x)||g(x)|^p\right)\right)^{\frac{1}{p}}\right\}^2. \tag{19}$$

Inequality (19) directly yields (17). □

Theorem 3.4. *Let $w, f, g \in C_{ld}([a,b]_{\mathbb{T}}, \mathbb{R})$ be ∇–integrable functions, $\hat{h}_{\alpha-1}(.,.) > 0$ and $p \geq 1$. Then for $\alpha \geq 1$, we have*

$$\begin{aligned}(\mathcal{J}_a^\alpha (|w(x)||f(x)|^p))^{\frac{2}{p}} &+ (\mathcal{J}_a^\alpha (|w(x)||g(x)|^p))^{\frac{2}{p}} \\ &\geq \Omega_4 (\mathcal{J}_a^\alpha (|w(x)||f(x)|^p))^{\frac{1}{p}} (\mathcal{J}_a^\alpha (|w(x)||g(x)|^p))^{\frac{1}{p}}, \quad (20)\end{aligned}$$

with some positive constants m_1, m_2, M_1 and M_2 such that $\frac{m_1}{M_2} \leq \left|\frac{f(y)}{g(y)}\right| \leq \frac{M_1}{m_2}$ on the set $[a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$, where $\Omega_4 = \frac{(M_1+m_2)(m_1+M_2)}{M_1 M_2} - 2$.

Proof. Similar to the proof of Theorem 3.3. □

Remark 3.2. *Let $\alpha = 1$, $\mathbb{T} = \mathbb{R}$, $x = b$, $w \equiv 1$, $f, g \in (0, +\infty)$, $m = \frac{m_1}{M_2}$ and $M = \frac{M_1}{m_2}$. Then (17) reduces to (2).*

Theorem 3.5. *Let $w, f, g \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$ be Δ–integrable functions, $h_{\alpha-1}(.,.) > 0$ and $p \geq 1$. Then for $\alpha \geq 1$, we have*

$$\begin{aligned}\frac{M+1}{M-c} &(\mathcal{I}_a^\alpha (|w(x)| (|f(x)| - c|g(x)|)^p))^{\frac{1}{p}} \\ &\leq (\mathcal{I}_a^\alpha (|w(x)||f(x)|^p))^{\frac{1}{p}} + (\mathcal{I}_a^\alpha (|w(x)||g(x)|^p))^{\frac{1}{p}} \\ &\leq \frac{m+1}{m-c} (\mathcal{I}_a^\alpha (|w(x)| (|f(x)| - c|g(x)|)^p))^{\frac{1}{p}}, \quad (21)\end{aligned}$$

with some positive constants c, m and M such that $0 < c < m \leq \left|\frac{f(y)}{g(y)}\right| \leq M$ on the set $[a,x]_{\mathbb{T}}$, where $\forall x \in [a,b]_{\mathbb{T}}$.

Proof. We note that

$$m - c \leq \left|\frac{f(y)}{g(y)}\right| - c \leq M - c, \ \forall y \in [a,x]_{\mathbb{T}}.$$

Therefore

$$\left(\frac{|f(y)| - c|g(y)|}{M-c}\right)^p \leq |g(y)|^p \leq \left(\frac{|f(y)| - c|g(y)|}{m-c}\right)^p, \ \forall y \in [a,x]_{\mathbb{T}}. \quad (22)$$

Multiplying by $h_{\alpha-1}(x, \sigma(y))|w(y)|$, where $h_{\alpha-1}(x, \sigma(y)) > 0$, $\forall x \in [a,b]_{\mathbb{T}}$ on both sides of inequality (22) and integrating with respect to the variable y, we get

$$\begin{aligned}\frac{1}{M-c} (\mathcal{I}_a^\alpha (|w(x)| (|f(x)| - c|g(x)|)^p))^{\frac{1}{p}} &\leq (\mathcal{I}_a^\alpha (|w(x)||g(x)|^p))^{\frac{1}{p}} \\ &\leq \frac{1}{m-c} (\mathcal{I}_a^\alpha (|w(x)| (|f(x)| - c|g(x)|)^p))^{\frac{1}{p}}. \quad (23)\end{aligned}$$

To obtain similar analogue, we have that

$$\frac{1}{c} - \frac{1}{m} \leq \frac{1}{c} - \left|\frac{g(y)}{f(y)}\right| \leq \frac{1}{c} - \frac{1}{M}, \ \forall y \in [a,x]_{\mathbb{T}}.$$

$$\left(\frac{M}{M-c}\right)^p (|f(y)| - c|g(y)|)^p \leq |f(y)|^p \leq \left(\frac{m}{m-c}\right)^p (|f(y)| - c|g(y)|)^p, \quad (24)$$

$\forall y \in [a,x]_{\mathbb{T}}$. Multiplying by $h_{\alpha-1}(x,\sigma(y))|w(y)|$, where $h_{\alpha-1}(x,\sigma(y)) > 0$, $\forall x \in [a,b]_{\mathbb{T}}$ on both sides of inequality (24) and integrating with respect to the variable y, we get

$$\frac{M}{M-c}\left(\mathcal{I}_a^{\alpha}\left(|w(x)|\left(|f(x)|-c|g(x)|\right)^p\right)\right)^{\frac{1}{p}} \leq \left(\mathcal{I}_a^{\alpha}\left(|w(x)||f(x)|^p\right)\right)^{\frac{1}{p}} \leq \frac{m}{m-c}\left(\mathcal{I}_a^{\alpha}\left(|w(x)|\left(|f(x)|-c|g(x)|\right)^p\right)\right)^{\frac{1}{p}}. \quad (25)$$

Combining (23) and (25), we get the desired claim. □

Theorem 3.6. *Let $w,f,g \in C_{ld}([a,b]_{\mathbb{T}},\mathbb{R})$ be ∇–integrable functions, $\hat{h}_{\alpha-1}(.,.) > 0$ and $p \geq 1$. Then for $\alpha \geq 1$, we have*

$$\frac{M+1}{M-c}\left(\mathcal{J}_a^{\alpha}\left(|w(x)|\left(|f(x)|-c|g(x)|\right)^p\right)\right)^{\frac{1}{p}} \leq \left(\mathcal{J}_a^{\alpha}\left(|w(x)||f(x)|^p\right)\right)^{\frac{1}{p}} + \left(\mathcal{J}_a^{\alpha}\left(|w(x)||g(x)|^p\right)\right)^{\frac{1}{p}} \leq \frac{m+1}{m-c}\left(\mathcal{J}_a^{\alpha}\left(|w(x)|\left(|f(x)|-c|g(x)|\right)^p\right)\right)^{\frac{1}{p}}, \quad (26)$$

with some positive constants c, m and M such that $0 < c < m \leq \left|\frac{f(y)}{g(y)}\right| \leq M$ on the set $[a,x]_{\mathbb{T}}$, where $\forall x \in [a,b]_{\mathbb{T}}$.

Proof. Similar to the proof of Theorem 3.5. □

Remark 3.3. *Let $\alpha = 1$, $\mathbb{T} = \mathbb{R}$, $c = 1$, $x = b$, $w \equiv 1$ and $f,g \in (0,+\infty)$. Then* (21) *reduces to* (3).

Theorem 3.7. *Let $w,f,g \in C_{rd}([a,b]_{\mathbb{T}},\mathbb{R})$ be Δ–integrable functions, $h_{\alpha-1}(.,.) > 0$, $\alpha \geq 1$ and $p \geq 1$. If $0 < m \leq \left|\frac{f(y)}{g(y)}\right| \leq M$, for $M \geq 1$ on the set $[a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$, then*

$$\left(\mathcal{I}_a^{\alpha}\left(|w(x)||f(x)|^p\right)\right)^{\frac{1}{p}} + \left(\mathcal{I}_a^{\alpha}\left(|w(x)||g(x)|^p\right)\right)^{\frac{1}{p}} \leq 2\left(\mathcal{I}_a^{\alpha}\left(|w(x)|\Lambda^p\left(|f(x)|,|g(x)|\right)\right)\right)^{\frac{1}{p}}, \quad (27)$$

with

$$\Lambda\left(|f(y)|,|g(y)|\right) = \max\left\{M\left(\left(\frac{M}{m}+1\right)|f(y)| - M|g(y)|\right), \frac{(m+M)|g(y)|-|f(y)|}{m}\right\}.$$

Proof. It follows from given hypothesis that

$$0 < m \leq m + M - \left|\frac{f(y)}{g(y)}\right| \leq M, \ \forall y \in [a,x]_{\mathbb{T}}.$$

Therefore

$$|g(y)| \leq \frac{(m+M)|g(y)|-|f(y)|}{m} \leq \Lambda\left(|f(y)|,|g(y)|\right), \ \forall y \in [a,x]_{\mathbb{T}}. \quad (28)$$

Multiplying by $h_{\alpha-1}(x,\sigma(y))|w(y)|$, where $h_{\alpha-1}(x,\sigma(y)) > 0$, $\forall x \in [a,b]_{\mathbb{T}}$ on both sides of inequality (28) and integrating with respect to the variable y, we get

$$\left(\mathcal{I}_a^{\alpha}\left(|w(x)||g(x)|^p\right)\right)^{\frac{1}{p}} \leq \left(\mathcal{I}_a^{\alpha}\left(|w(x)|\Lambda^p\left(|f(x)|,|g(x)|\right)\right)\right)^{\frac{1}{p}}. \quad (29)$$

It also follows from given hypothesis that

$$\frac{1}{M} \leq \frac{1}{m} + \frac{1}{M} - \left|\frac{g(y)}{f(y)}\right| \leq \frac{1}{m}, \ \forall y \in [a,x]_{\mathbb{T}}.$$

Thus,

$$|f(y)| \leq \left(\frac{M}{m}+1\right)|f(y)| - M|g(y)|, \forall y \in [a,x]_{\mathbb{T}},$$

and

$$|f(y)| \leq M\left(\left(\frac{M}{m}+1\right)|f(y)|-M|g(y)|\right) \leq \Lambda\left(|f(y)|,|g(y)|\right), \ \forall y \in [a,x]_{\mathbb{T}}. \tag{30}$$

Multiplying by $h_{\alpha-1}(x,\sigma(y))|w(y)|$, where $h_{\alpha-1}(x,\sigma(y))>0$, $\forall x \in [a,b]_{\mathbb{T}}$ on both sides of inequality (30) and integrating with respect to the variable y, we get

$$\left(\mathcal{I}_a^{\alpha}\left(|w(x)||f(x)|^p\right)\right)^{\frac{1}{p}} \leq \left(\mathcal{I}_a^{\alpha}\left(|w(x)|\Lambda^p\left(|f(x)|,|g(x)|\right)\right)\right)^{\frac{1}{p}}. \tag{31}$$

Combining (29) and (31), we get the desired claim. $\square$

Theorem 3.8. *Let $w,f,g \in C_{ld}\left([a,b]_{\mathbb{T}},\mathbb{R}\right)$ be ∇–integrable functions, $\hat{h}_{\alpha-1}(.,.)>0$, $\alpha \geq 1$ and $p \geq 1$. If $0<m \leq \left|\frac{f(y)}{g(y)}\right| \leq M$, for $M \geq 1$ on the set $[a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$, then*

$$\left(\mathcal{J}_a^{\alpha}\left(|w(x)||f(x)|^p\right)\right)^{\frac{1}{p}}+\left(\mathcal{J}_a^{\alpha}\left(|w(x)||g(x)|^p\right)\right)^{\frac{1}{p}} \leq 2\left(\mathcal{J}_a^{\alpha}\left(|w(x)|\Lambda^p\left(|f(x)|,|g(x)|\right)\right)\right)^{\frac{1}{p}}, \tag{32}$$

with

$$\Lambda\left(|f(y)|,|g(y)|\right)=\max\left\{M\left(\left(\frac{M}{m}+1\right)|f(y)|-M|g(y)|\right), \frac{(m+M)|g(y)|-|f(y)|}{m}\right\}.$$

Proof. Similar to the proof of Theorem 3.7. $\square$

Remark 3.4. *Let $\alpha=1$, $\mathbb{T}=\mathbb{R}$, $x=b$, $w \equiv 1$ and $f,g \in (0,+\infty)$. Then (27) reduces to*

$$\left(\int_a^b f^p(y)dy\right)^{\frac{1}{p}}+\left(\int_a^b g^p(y)dy\right)^{\frac{1}{p}} \leq 2\left(\int_a^b \Lambda^p\left(f(y),g(y)\right)dy\right)^{\frac{1}{p}}, \tag{33}$$

where

$$\Lambda\left(f(y),g(y)\right)=\max\left\{M\left(\left(\frac{M}{m}+1\right)f(y)-Mg(y)\right), \frac{(m+M)g(y)-f(y)}{m}\right\}.$$

The inequality (33) may be found in [20].

Theorem 3.9. *Let $w,f,g \in C_{rd}\left([a,b]_{\mathbb{T}},\mathbb{R}\right)$ be Δ–integrable, $h_{\alpha-1}(.,.)>0$ and $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$. Then for $\alpha \geq 1$, we have*

$$\begin{aligned}\mathcal{I}_a^{\alpha}\left(|w(x)||f(x)g(x)|\right) \leq\ & \frac{\Omega_5}{2}\left\{\mathcal{I}_a^{\alpha}\left(|w(x)||f(x)|^p\right)+\mathcal{I}_a^{\alpha}\left(|w(x)||g(x)|^p\right)\right\} \\ & +\frac{\Omega_6}{2}\left\{\mathcal{I}_a^{\alpha}\left(|w(x)||f(x)|^q\right)+\mathcal{I}_a^{\alpha}\left(|w(x)||g(x)|^q\right)\right\},\end{aligned} \tag{34}$$

with some positive constants m and M such that $0<m \leq \left|\frac{f(y)}{g(y)}\right| \leq M$ on the set $[a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$, where $\Omega_5=\frac{2^p}{p}\left(\frac{M}{M+1}\right)^p$ and $\Omega_6=\frac{2^q}{q}\left(\frac{1}{m+1}\right)^q$.

Proof. From $0<m \leq \left|\frac{f(y)}{g(y)}\right| \leq M$, $\forall y \in [a,x]_{\mathbb{T}}$, we have that

$$|f(y)| \leq \frac{M}{M+1}\left(|f(y)|+|g(y)|\right) \tag{35}$$

and

$$|g(y)| \leq \frac{1}{m+1}\left(|f(y)|+|g(y)|\right). \tag{36}$$

By applying Young's inequality (9), we get

$$\begin{aligned}
&\int_a^x |w(y)||f(y)g(y)|\Delta y \\
\leq\ & \frac{1}{p}\int_a^x |w(y)||f(y)|^p\Delta y + \frac{1}{q}\int_a^x |w(y)||g(y)|^q\Delta y \\
\leq\ & \frac{1}{p}\left(\frac{M}{M+1}\right)^p \int_a^x |w(y)|\left(|f(y)|+|g(y)|\right)^p \Delta y \\
+\ & \frac{1}{q}\left(\frac{1}{m+1}\right)^q \int_a^x |w(y)|\left(|f(y)|+|g(y)|\right)^q \Delta y \\
\leq\ & \frac{1}{p}\left(\frac{M}{M+1}\right)^p 2^{p-1}\int_a^x |w(y)|\left(|f(y)|^p+|g(y)|^p\right) \Delta y \\
+\ & \frac{1}{q}\left(\frac{1}{m+1}\right)^q 2^{q-1}\int_a^x |w(y)|\left(|f(y)|^q+|g(y)|^q\right) \Delta y,
\end{aligned}$$

we have used the elementary inequality, such that

$$(\beta+\gamma)^p \leq 2^{p-1}(\beta^p+\gamma^p),\ p>1,\ \beta,\gamma\in[0,+\infty).$$

Replacing $|w(y)|$ by $h_{\alpha-1}(x,\sigma(y))|w(y)|$, where $h_{\alpha-1}(x,\sigma(y))>0$, $\forall x\in[a,b]_{\mathbb{T}}$, we conclude the desired result. □

Theorem 3.10. *Let $w,f,g\in C_{ld}\left([a,b]_{\mathbb{T}},\mathbb{R}\right)$ be ∇–integrable, $\hat{h}_{\alpha-1}(.,.)>0$ and $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$. Then for $\alpha\geq 1$, we have*

$$\begin{aligned}
\mathcal{J}_a^\alpha\left(|w(x)||f(x)g(x)|\right) \leq \frac{\Omega_5}{2}&\left\{\mathcal{J}_a^\alpha\left(|w(x)||f(x)|^p\right)+\mathcal{J}_a^\alpha\left(|w(x)||g(x)|^p\right)\right\} \\
&+\frac{\Omega_6}{2}\left\{\mathcal{J}_a^\alpha\left(|w(x)||f(x)|^q\right)+\mathcal{J}_a^\alpha\left(|w(x)||g(x)|^q\right)\right\}, \quad (37)
\end{aligned}$$

with some positive constants m and M such that $0<m\leq\left|\frac{f(y)}{g(y)}\right|\leq M$ on the set $[a,x]_{\mathbb{T}}$, $\forall x\in[a,b]_{\mathbb{T}}$, where $\Omega_5=\frac{2^p}{p}\left(\frac{M}{M+1}\right)^p$ and $\Omega_6=\frac{2^q}{q}\left(\frac{1}{m+1}\right)^q$.

Proof. Similar to the proof of Theorem 3.9. □

Remark 3.5. *Let $\alpha=1$, $\mathbb{T}=\mathbb{R}$, $x=b$, $w\equiv 1$ and $f,g\in(0,+\infty)$. Then (34) reduces to*

$$\begin{aligned}
&\int_a^b f(y)g(y)dy \\
&\leq \frac{\Omega_5}{2}\left\{\int_a^b f^p(y)dy+\int_a^b g^p(y)dy\right\}+\frac{\Omega_6}{2}\left\{\int_a^b f^q(y)dy+\int_a^b g^q(y)dy\right\}. \quad (38)
\end{aligned}$$

The inequality (38) may be found in [17].

4. Conclusion

By using the definition of a fractional integral, recently proposed by Katugampola, many generalized Minkowski type fractional inequalities are proved, see [18]. By using the definition of the Riemann–Liouville fractional integral, some new results of integral inequalities related to the Minkowski inequality are also established, see [12]. The integral inequality concerning reverse of Minkowski's inequality is proved in generalized form in [19]. Some inequalities involving Hadamard–type k–fractional integral operators are proved in [1]. An extension by means of ω–weighted classes of the generalized Rieman–Liouville k–fractional integral inequalities is explored in [2].

By considering an axiomatic definition of fractional calculus on time scales, many results have been developed concerning the time scale Riemann–Liouville type fractional integrals, see [4, 5, 6, 14, 15, 16].

In the future research, we may generalize several classical inequalities and their applications on time scales. If a result is established on time scales, then we get its discrete version by setting $\mathbb{T} = \mathbb{N}$ and continuous version by setting $\mathbb{T} = \mathbb{R}$. Further, we get quantum form of a result by general setting of $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$, where $q > 1$.

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Muhammad Jibril Shahab Sahir is currently working as a research student in the University of Sargodha, Pakistan and principal at GHSS, Gohar Wala, Bhakkar, Pakistan. His area of interest includes Dynamic inequalities, Fractional calculus and Time Scales Calculus.