Nunokawa et al. *Journal of Inequalities and Applications* 2012, **2012**:245
http://www.journalofinequalitiesandapplications.com/content/2012/1/245

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Some properties concerning close-to-convexity of certain analytic functions

Mamoru Nunokawa[1], Melike Aydoğan[2*], Kazuo Kuroki[3], Ismet Yildiz[4] and Shigeyoshi Owa[3]

*Correspondence: melike.aydogan@isikun.edu.tr
[2]Department of Mathematics, Işık University, Meşrutiyet Koyu, Şile Kampusu, Istanbul, 34980, Turkey
Full list of author information is available at the end of the article

Abstract
Let $f(z)$ be an analytic function in the open unit disk $\mathbb{D}$ normalized with $f(0) = 0$ and $f'(0) = 1$. With the help of subordinations, for convex functions $f(z)$ in $\mathbb{D}$, the order of close-to-convexity for $f(z)$ is discussed with some example.

MSC: Primary 30C45

Keywords: analytic; starlike; convex; close-to-convex; subordination

1 Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$. A function $f(z) \in \mathcal{A}$ is said to be convex of order α if it satisfies

$$1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} > \alpha \quad \text{in } \mathbb{D}$$

for some real α $(0 \leqq \alpha < 1)$. This family of functions was introduced by Robertson [1] and we denote it by $\mathcal{K}(\alpha)$.

A function $f(z) \in \mathcal{A}$ is called starlike of order α in $\mathbb{D}$ if it satisfies

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha \quad \text{in } \mathbb{D}$$

for some real α $(0 \leqq \alpha < 1)$.

This class was also introduced by Robertson [1] and we denote it by $\mathcal{S}^*(\alpha)$. By the definitions for the classes $\mathcal{K}(\alpha)$ and $\mathcal{S}^*(\alpha)$, we know that $f(z) \in \mathcal{K}(\alpha)$ if and only if $zf'(z) \in \mathcal{S}^*(\alpha)$.

Marx [2] and Strohhäcker [3] showed that $f(z) \in \mathcal{K}(0)$ implies $f(z) \in \mathcal{S}^*(\frac{1}{2})$.

This estimate is sharp for an extremal function

$$f(z) = \frac{z}{1-z}.$$

Jack [4] posed a more general problem: What is the largest number $\beta = \beta(\alpha)$ so that

$$\mathcal{K}(\alpha) \subset \mathcal{S}^*\bigl(\beta(\alpha)\bigr).$$

MacGregor [5] determined the exact value of $\beta(\alpha)$ for each α $(0 \leqq \alpha < 1)$ as the infimum over the disc $\mathbb{D}$ of the real part of a specific analytic function. It has been conjectured that this infimum is attained on the boundary of $\mathbb{D}$ at $z = -1$.

Wilken and Feng [6] asserted MacGregor's conjecture: If $0 \leqq \alpha < 1$ and $f(z) \in \mathcal{K}(\alpha)$, then $f(z) \in \mathcal{S}^*(\beta(\alpha))$, where

$$\beta(\alpha) = \begin{cases} \frac{1-2\alpha}{2^{2(1-\alpha)}-2} & \text{if } \alpha \neq \frac{1}{2}, \\ \frac{1}{2\log 2} & \text{if } \alpha = \frac{1}{2}. \end{cases} \tag{1}$$

Ozaki [7] and Kaplan [8] investigated the following functions: If $f(z) \in \mathcal{A}$ satisfies

$$\operatorname{Re}\frac{f'(z)}{g'(z)} > 0 \quad \text{in } \mathbb{D}$$

for some convex function $g(z)$, then $f(z)$ is univalent in $\mathbb{D}$. In view of Kaplan [8], we say that $f(z)$ satisfying the above inequality is close-to-convex in $\mathbb{D}$.

It is well known that the above definition concerning close-to-convex functions is equivalent to the following condition:

$$\operatorname{Re}\frac{zf'(z)}{g(z)} > 0 \quad \text{in } \mathbb{D}$$

for some starlike function $g(z) \in \mathcal{A}$.

Let us define a function $f(z) \in \mathcal{A}$ which satisfies

$$\operatorname{Re}\frac{zf'(z)}{g(z)} > \alpha \quad \text{in } \mathbb{D}$$

for some real α $(0 \leqq \alpha < 1)$ and for some starlike function $g(z)$ in $\mathbb{D}$.

Then we call $f(z)$ close-to-convex of order α in $\mathbb{D}$ with respect to $g(z)$.

It is the purpose of the present paper to investigate the order of close-to-convexity of the functions which satisfy $f(z) \in \mathcal{K}(\alpha)$ and $0 \leqq \alpha < 1$.

2 Preliminary

To discuss our problems, we have to give here the following lemmas.

Lemma 1 *Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be analytic in $\mathbb{D}$ and suppose that*

$$p(z) \prec \frac{1-\alpha z}{1+\beta z} \quad \textit{in } \mathbb{D},$$

where $\prec$ means the subordination, $0 < \alpha < 1$ and $0 < \beta < 1$.

Then we have

$$\frac{1-\alpha}{1+\beta} < \operatorname{Re} p(z) < \frac{1+\alpha}{1-\beta}.$$

This shows that

$$\operatorname{Re} p(z) > 0 \quad \text{in } \mathbb{D}.$$

A proof is very easily obtained.

Lemma 2 *Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be analytic in $\mathbb{D}$, and suppose that there exists a point $z_0 \in \mathbb{D}$ such that*

$$\operatorname{Re} p(z) > c \quad \text{for } |z| < |z_0|$$

and

$$\operatorname{Re} p(z_0) = c, \qquad p(z_0) \neq c$$

for some real c $(0 < c < 1)$. Then we have

$$\operatorname{Re} \frac{z_0 p'(z_0)}{p(z_0)} \leqq \begin{cases} -\frac{1-c}{2c} & \text{when } \frac{1}{2} \leqq c < 1, \\ -\frac{c}{2(1-c)} & \text{when } 0 < c < \frac{1}{2}. \end{cases}$$

Proof Let us put

$$q(z) = \frac{p(z) - c}{1 - c}, \qquad q(0) = 1.$$

Then $q(z)$ is analytic in $\mathbb{D}$ and

$$\operatorname{Re} q(z) > 0 \quad \text{for } |z| < |z_0|$$

and

$$\operatorname{Re} q(z_0) = 0, \qquad q(z_0) \neq 0.$$

Then, from [9, Theorem 1], we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = i\ell,$$

where

$$\ell \geqq \frac{1}{2}\left(a + \frac{1}{a}\right) \geqq 1 \quad \text{when } \arg q(z_0) = \frac{\pi}{2}$$

and

$$\ell \leqq -\frac{1}{2}\left(a + \frac{1}{a}\right) \leqq -1 \quad \text{when } \arg q(z_0) = -\frac{\pi}{2},$$

where $q(z_0) = \pm ia$ and $a > 0$.

For the case $\arg q(z_0) = \frac{\pi}{2}$, we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = \frac{z_0 p'(z_0)}{p(z_0) - c} = i\ell$$

and so

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{p(z_0) - c}{p(z_0)} i\ell,$$

$$\begin{aligned}\operatorname{Re}\frac{z_0 p'(z_0)}{p(z_0)} &= \operatorname{Re}\left(\frac{i(1-c)a}{c + i(1-c)a} i\ell\right) = \operatorname{Re}\left\{\left(\frac{c - i(1-c)a}{c^2 + (1-c)^2 a^2}\right)\left(-a(1-c)\ell\right)\right\} \\ &= \frac{-(1-c)ca\ell}{c^2 + (1-c)^2 a^2} \leqq -\frac{c(1-c)}{2}\left(\frac{1+a^2}{c^2 + (1-c)^2 a^2}\right).\end{aligned}$$

If we put

$$h(x) = \frac{1 + x^2}{c^2 + (1-c)^2 x^2} \quad (x > 0),$$

then it is easy to see that

$$\frac{1}{c^2} < h(x) < \frac{1}{(1-c)^2} \quad \text{when } \frac{1}{2} \leqq c < 1$$

and

$$\frac{1}{(1-c)^2} < h(x) < \frac{1}{c^2} \quad \text{when } 0 < c < \frac{1}{2}.$$

This shows that

$$\operatorname{Re}\frac{z_0 p'(z_0)}{p(z_0)} \leqq \begin{cases} -\frac{1-c}{2c} & \text{when } \frac{1}{2} \leqq c < 1, \\ -\frac{c}{2(1-c)} & \text{when } 0 < c < \frac{1}{2}. \end{cases}$$

For the case $\arg q(z_0) = -\frac{\pi}{2}$, applying the same method as above, we have the same conclusion

$$\operatorname{Re}\frac{z_0 p'(z_0)}{p(z_0)} \leqq \begin{cases} -\frac{1-c}{2c} & \text{when } \frac{1}{2} \leqq c < 1, \\ -\frac{c}{2(1-c)} & \text{when } 0 < c < \frac{1}{2}. \end{cases}$$

This completes the proof of the lemma. □

Our next lemma is

Lemma 3 *Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be analytic in $\mathbb{D}$ and suppose that there exists a point $z_0 \in \mathbb{D}$ such that*

$$\operatorname{Re} p(z) > c \quad \textit{for } |z| < |z_0|$$

and

$$\operatorname{Re} p(z_0) = c, \qquad p(z_0) \neq c$$

for some real c $(c < 0)$.

Then we have

$$\operatorname{Re} \frac{z_0 p'(z_0)}{p(z_0)} > -\frac{c}{2(1-c)} > 0. \tag{2}$$

Proof Let us put

$$q(z) = \frac{p(z) - c}{1 - c}, \qquad q(0) = 1.$$

Then $q(z)$ is analytic in $\mathbb{D}$. If $p(z)$ satisfies the hypothesis of the lemma, then there exists a point $z_0 \in \mathbb{D}$ such that

$$\operatorname{Re} q(z) > 0 \quad \text{for } |z| < |z_0|$$

and

$$\operatorname{Re} q(z_0) = 0 \quad \text{and} \quad q(z_0) \neq 0,$$

then $p(z)$ satisfies the conditions of the lemma.

For the case $\arg q(z_0) = \frac{\pi}{2}$, applying the same method as in the proof of Lemma 2, we have

$$\operatorname{Re} \frac{z_0 p'(z_0)}{p(z_0)} = -\frac{(1-c)ca\ell}{c^2 + (1-c)^2 a^2} \geqq -\frac{c(1-c)}{2}\left(\frac{1+a^2}{c^2 + (1-c)^2 a^2}\right).$$

Putting

$$h(x) = \frac{1 + x^2}{c^2 + (1-c)^2 x^2} \quad (x > 0),$$

it follows that

$$h'(x) = \frac{(2c-1)x}{(c^2 + (1-c)^2 x^2)^2} < 0 \quad (x > 0). \tag{3}$$

Therefore, from (3) we obtain (2) .

For the case $\arg q(z_0) = -\frac{\pi}{2}$, applying the same method as above, we have the same conclusion as in the case $\arg q(z_0) = \frac{\pi}{2}$. □

3 The order of close-to-convexity

Now, we discuss the close-to-convexity of $f(z)$ with the help of lemmas.

Theorem 1 *Let* $f(z) \in \mathcal{A}$, *and suppose that there exists a starlike function* $g(z)$ *such that*

(i) *for the case* $\frac{1}{2} \leqq c < 1$,

$$1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} > \operatorname{Re} \frac{zg'(z)}{g(z)} - \frac{1-c}{2c} \quad \textit{in } \mathbb{D},$$

$$\frac{zf'(z)}{g(z)} \neq c \quad \textit{in } \mathbb{D}$$

and

(ii) *for the case* $0 < c < \frac{1}{2}$,

$$1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} > \operatorname{Re} \frac{zg'(z)}{g(z)} - \frac{c}{2(1-c)} \quad \textit{in } \mathbb{D},$$

$$\frac{zf'(z)}{g(z)} \neq c \quad \textit{in } \mathbb{D}.$$

Then we have

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > c \quad \textit{in } \mathbb{D}.$$

This means that $f(z)$ *is close-to-convex of order* c *in* $\mathbb{D}$.

Proof Let us put

$$p(z) = \frac{zf'(z)}{g(z)}, \qquad p(0) = 1.$$

Then it follows that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zg'(z)}{g(z)} + \frac{zp'(z)}{p(z)}.$$

(i) For the case $\frac{1}{2} \leqq c < 1$, if there exists a point $z_0 \in \mathbb{D}$ such that

$$\operatorname{Re} p(z) > c \quad \text{for } |z| < |z_0|$$

and

$$\operatorname{Re} p(z_0) = c,$$

then, applying Lemma 2 and the hypothesis of Theorem 1, we have

$$p(z_0) \neq c$$

and

$$\operatorname{Re} \frac{z_0 p'(z_0)}{p(z_0)} \leqq -\frac{1-c}{2c}.$$

Thus, it follows that

$$\begin{aligned}1+\operatorname{Re}\frac{z_0 f''(z_0)}{f'(z_0)} &= \operatorname{Re}\frac{z_0 g'(z_0)}{g(z_0)}+\operatorname{Re}\frac{z_0 p'(z_0)}{p(z_0)}\\ &\leqq \operatorname{Re}\frac{z_0 g'(z_0)}{g(z_0)}-\frac{1-c}{2c},\end{aligned}$$

which contradicts the hypothesis of Theorem 1. (ii) For the case $0<c<\frac{1}{2}$, applying the same method as above, we also have that

$$\operatorname{Re}\frac{zf'(z)}{g(z)}>c \quad \text{in } \mathbb{D}.$$

This completes the proof of the theorem. □

Applying Theorem 1, we have the following corollary.

Corollary 1 *Let $f(z)\in\mathcal{A}$ be convex of order α $(0<\alpha<1)$, and suppose that there exists a starlike function $g(z)$ such that*

(i) *for the case $\frac{1}{2}\leqq\alpha<c$,*

$$1+\operatorname{Re}\frac{zf''(z)}{f'(z)}>\operatorname{Re}\frac{zg'(z)}{g(z)}-\frac{1-\beta(c)}{2\beta(c)} \quad \text{in } \mathbb{D},$$

$$\frac{zf'(z)}{g(z)}\neq\beta(c) \quad \text{in } \mathbb{D}$$

and

(ii) *for the case $0<\alpha<c\leqq\frac{1}{2}$,*

$$1+\operatorname{Re}\frac{zf''(z)}{f'(z)}>\operatorname{Re}\frac{zg'(z)}{g(z)}-\frac{\beta(c)}{2(1-\beta(c))} \quad \text{in } \mathbb{D},$$

$$\frac{zf'(z)}{g(z)}\neq\beta(c) \quad \text{in } \mathbb{D}.$$

Then we have

$$\operatorname{Re}\frac{zf'(z)}{g(z)}>\beta(c)>\beta(\alpha)>\alpha \quad \text{in } \mathbb{D}.$$

Remark 1 For the case $0<\alpha<c<1$, it is trivial that

$$\alpha<\beta(\alpha)<\beta(c)<1.$$

Example 1 Let $f(z)\in\mathcal{A}$ satisfy

$$1+\operatorname{Re}\frac{zf''(z)}{f'(z)}>\operatorname{Re}\frac{1-Az}{1+Az}-\frac{1-\beta(\frac{1}{2})}{2\beta(\frac{1}{2})}>-\frac{1}{10} \quad \text{in } \mathbb{D}, \tag{4}$$

where

$$A=\frac{32\beta(\frac{1}{2})-10}{8\beta(\frac{1}{2})+10}\fallingdotseq 0.29605$$

and $\beta(\frac{1}{2}) = \frac{1}{2\log 2}$. If we consider the starlike function $g(z)$ given by

$$g(z) = \frac{z}{(1+Az)^2},$$

then we have

$$\operatorname{Re}\frac{zf'(z)}{g(z)} > \beta\left(\frac{1}{2}\right) \fallingdotseq 0.7213,$$

which means that $f(z)$ is close-to-convex of order $\beta(\frac{1}{2})$ in $\mathbb{D}$.

Next we show

Theorem 2 *Let $f(z) \in \mathcal{A}$ and $g(z) \in \mathcal{A}$ be given by*

$$g(z) = \begin{cases} \frac{z}{(1+\beta z)^{\frac{\alpha+\beta}{\beta}}} & (\beta \neq 0), \\ ze^{-\alpha z} & (\beta = 0) \end{cases}$$

for some α $(0 \leqq \alpha < 1)$ and some β $(0 \leqq \beta < 1)$. Further suppose that for arbitrary r $(0 < r < 1)$,

$$\min_{|z|=r}\left(\operatorname{Re}\frac{zf'(z)}{g(z)}\right) = \left(\operatorname{Re}\frac{z_0 f'(z_0)}{g(z_0)}\right)_{|z_0|=r} \neq \frac{z_0 f'(z_0)}{g(z_0)}$$

and

$$1 + \operatorname{Re}\frac{zf''(z)}{f'(z)} \leqq -\frac{c}{2(1-c)} + \frac{1-\alpha}{1+\beta}$$

for $c < 0$. Then we have

$$\operatorname{Re}\frac{zf'(z)}{g(z)} > c \quad \text{in } \mathbb{D}.$$

Proof Let us define the function $p(z)$ by

$$p(z) = \frac{zf'(z)}{g(z)}, \qquad p(0) = 1$$

for $c < 0$. If there exists a point $z_0 \in \mathbb{D}$ such that

$$\operatorname{Re} p(z) > c \quad \text{for } |z| < |z_0|$$

and

$$\operatorname{Re} p(z_0) = c$$

for $c < 0$, then from the hypothesis of Theorem 2, we have

$$\operatorname{Re} p(z_0) \neq p(z_0).$$

Therefore, applying Lemma 1 and Lemma 3, we have

$$\begin{aligned} 1 + \operatorname{Re} \frac{z_0 f''(z_0)}{f'(z_0)} &= \operatorname{Re} \frac{z_0 p'(z_0)}{p(z_0)} + \operatorname{Re} \frac{z_0 g'(z_0)}{g(z_0)} \\ &> -\frac{c}{2(1-c)} + \frac{1-\alpha}{1+\beta}. \end{aligned}$$

This is a contradiction, and therefore we have

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > c \quad \text{in } \mathbb{D}.$$

□

Remark 2 In view of the definition for close-to-convex functions, if $f(z)$ satisfies

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > 0 \quad \text{in } \mathbb{D},$$

then we can say that $f(z)$ is close-to-convex in $\mathbb{D}$. But c should be a negative real number in Theorem 2. Therefore, we cannot say that $f(z)$ is close-to-convex in $\mathbb{D}$ in Theorem 2.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

Author details
[1]University of Gunma, Hoshikuki 798-8, Chuou-Ward, Chiba, 260-0808, Japan. [2]Department of Mathematics, Işık University, Meşrutiyet Koyu, Şile Kampusu, Istanbul, 34980, Turkey. [3]Department of Mathematics, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan. [4]Department of Mathematics, Duzce University, Konuralp Yerleskesi, Duzce, 81620, Turkey.

Acknowledgements
The authors thank the referees for their helpful comments and suggestions to improve our manuscript.

Received: 1 August 2012 Accepted: 8 October 2012 Published: 24 October 2012

References
1. Robertson, MS: On the theory of univalent functions. Ann. Math. **37**, 374-408 (1936)
2. Marx, A: Untersuchungen über schlichte Abbildungen. Math. Ann. **107**, 40-67 (1932/33)
3. Strohhäcker, E: Beiträge zur Theorie der schlichten Funktionen. Math. Z. **37**, 356-380 (1933)
4. Jack, IS: Functions starlike and convex of order α. J. Lond. Math. Soc. **3**, 469-474 (1971)
5. MacGregor, TH: A subordination for convex functions of order α. J. Lond. Math. Soc. **9**, 530-536 (1975)
6. Wilken, DR, Feng, J: A remark on convex and starlike functions. J. Lond. Math. Soc. **21**, 287-290 (1980)
7. Ozaki, S: On the theory of multivalent functions. Sci. Rep. Tokyo Bunrika Daigaku, Sect. A. **2**, 167-188 (1935)
8. Kaplan, W: Close-to-convex schlicht functions. Mich. Math. J. **1**, 169-185 (1952)
9. Nunokawa, M: On properties of non-Carathéodory functions. Proc. Jpn. Acad., Ser. A, Math. Sci. **68**(6), 152-153 (1992)

doi:10.1186/1029-242X-2012-245
Cite this article as: Nunokawa et al.: Some properties concerning close-to-convexity of certain analytic functions. *Journal of Inequalities and Applications* 2012 **2012**:245.