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INTRODUCTION TO TOTAL DOMINATOR EDGE CHROMATIC NUMBER

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Abstract. We introduce the total dominator edge chromatic number of a graph G. A total dominator edge coloring (briefly TDE-coloring) of G is a proper edge coloring of G in which each edge of the graph is adjacent to every edge of some color class. The total dominator edge chromatic number (briefly TDEC-number) $\chi_d'^t(G)$ of G is the minimum number of color classes in a TDE-coloring of G. We obtain some properties of $\chi_d'^t(G)$ and compute this parameter for specific graphs. We examine the effects on $\chi_d'^t(G)$ when G is modified by operations on vertices and edges of G. Finally, we consider the k-subdivison of G and study TDEC-number of this kind of graphs.

Keywords: total dominator edge chromatic number; vertex removal; k-subdivision.

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1. Introduction

Let $G = (V, E)$ be a simple graph and $k \in \mathbb{N}$. A mapping $f : V(G) \longrightarrow \{1, 2, ..., k\}$ is called a k-proper coloring of G, if $f(u) \neq f(v)$ whenever the vertices u and v are adjacent in G. A color class of this coloring, is a set consisting of all those vertices assigned the same color. If f is a proper coloring of G with the coloring classes $V_1, V_2, ..., V_k$ such that every vertex in V_i has color i, then sometimes write simply $f = (V_1, V_2, ..., V_k)$. The chromatic number $\chi(G)$ of G is the minimum number of colors needed in a proper coloring of a graph. The total dominating set is a subset D of V such that every vertex of V is adjacent to some vertices of D. The total domination number of G is equal to minimum cardinality of total dominating set in G and it is denoted by $\gamma_t(G)$.

The total dominator coloring, abbreviated TD-coloring, was previously studied in [5, 6]. Let G be a graph with no isolated vertices, the total dominator coloring is a proper coloring of G in which each vertex of the graph is adjacent to every vertex of some (other) color class. The total dominator chromatic number, abbreviated TDC-number, $\chi_d^t(G)$ of G is

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the minimum number of color classes in a TD-coloring of G. Computation of the TDC-number is NP-complete. The TDC-number of some graphs has been computed [2]. Also Henning in [4] established the lower and the upper bounds on the TDC-number of a graph in terms of its total domination number $\gamma_t(G)$. He has shown that, every graph G with no isolated vertices satisfies $\gamma_t(G) \le \chi_d^t(G) \le \gamma_t(G)+\chi(G)$. The properties of TD-colorings in trees have been studied in [4]. Trees T with $\gamma_t(T) = \chi_d^t(T)$ have been characterized in [4]. We have examined the effects on $\chi_d^t(G)$ when G is modified by operations on the vertices and the edges of G, and the TDC-number of some operations on two graphs studied in [3].

Motivated by TDC-number of a graph, we consider the proper edge coloring of G and introduce the total dominator edge chromatic number (TDEC-number) of G, $\chi_d'^t(G)$, obtain some properties of $\chi_d'^t(G)$ and compute this parameter for specific graphs, in the next section. In Section 3, we examine the effects on $\chi_d'^t(G)$ when G is modified by operations on vertics and edges of G. Finally in Section 4, we study the TDEC-number of k-subdivision of graphs.

2. Introduction to total dominator edge chromatic number

In this section, we state the definition of total dominator edge chromatic number and obtain this parameter for some specific graphs.

Definition 2.1. *A* total dominator edge coloring*, briefly TDE-coloring, of a graph G is a proper edge coloring of G in which each edge of the graph is adjacent to every edge of some color class. The total dominator edge chromatic number (TDEC-number) $\chi_d'^t(G)$ of G is the minimum number of color classes in a TDE-coloring of G. A $\chi_d'^t(G)$-coloring of G is any total dominator edge coloring of G with $\chi_d'^t(G)$ colors.*

Remark 2.1. *For every graph G with maximum degree $\Delta(G)$, $\chi_d'^t(G) \ge \Delta(G)$. This inequality is sharp. As an example, for the star graph $K_{1,n}$, $\chi_d'^t(K_{1,n}) = n$.*

The following theorem gives the total dominator edge chromatic number of a path.

Theorem 2.1. *If P_n is the path graph of order $n \ge 9$, then*

$$\chi_d'^t(P_n) = \begin{cases} 2k+2 & \text{if } n = 4k+1, \\ 2k+3 & \text{if } n = 4k+2, \\ 2k+4 & \text{if } n = 4k+3,\ 4k+4. \end{cases}$$

Also $\chi_d'^t(P_3) = \chi_d'^t(P_4) = 2$, $\chi_d'^t(P_5) = 3$, $\chi_d'^t(P_6) = \chi_d'^t(P_7) = 4$ and $\chi_d'^t(P_8) = 5$.

Proof. It is easy to show that $\chi_d'^t(P_3) = \chi_d'^t(P_4) = 2$, $\chi_d'^t(P_5) = 3$, $\chi_d'^t(P_6) = \chi_d'^t(P_7) = 4$ and $\chi_d'^t(P_8) = 5$. Suppose that $n \ge 9$. First we show that in a TDE-coloring, for each four consecutive edges we shall use at least two new colors. We consider two cases. If a used color assign to edge e_{i+1}, then we need to assign a new color to the edge e_{i+2} and e_{i+3} to have a TDE-coloring (see Figure 1). If a new color is assigned to the edge e_{i+1}, then we have to assign a new color to e_{i+2} or e_i to have a TDE-coloring. So we need at least two new colors in every four consecutive vertices.

If $n = 4k+1$, for some $k \in \mathbb{N}$, then we give a TDE-coloring for P_{4k+1} which use only two new colors in every four consecutive edges. Define a function f_0 on $E(P_{4k})$ such that for any edge e_i,

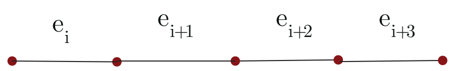

FIGURE 1. Four consecutive edges.

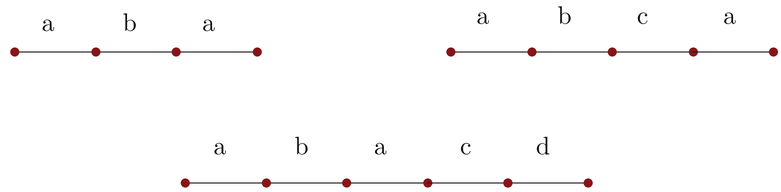

FIGURE 2. TDE-coloring of P_4, P_5 and P_6.

$$f_0(e_i) = \begin{cases} 1 & \text{if } i = 1 + 4s, \\ 2 & \text{if } i = 4s. \\ 2 & \text{if } i = 4s. \end{cases}$$

where s is a natural number and for any e_i, $i \neq 4s$ and $i \neq 4s+1$, $f_0(e_i)$ is a new number. Then this coloring is a TDE-coloring of P_{4k+1} with the minimum number $2k+2$ colors.

If $n = 4k+2$, for some $k \in \mathbb{N}$, then we first color the $4k-4$ edges using f_0. Now for the rest of edges we define f_1 as $f_1(e_{4k-3}) = 1$, $f_1(e_{4k-2}) = 2k+1$, $f_1(e_{4k-1}) = 2k+2$, $f_1(e_{4k}) = 2k+3$ and $f_1(e_{4k+1}) = 2$. Since for every five consecutive edges we have to use at least three new colors, so this edge coloring is a TDE-coloring of P_{4k+1} with the minimum number $2k+3$ colors.

If $n = 4k+3$, for some $k \in \mathbb{N}$, then using f_0 we color the $4k-4$ edges. Now for the rest of edges, define f_2 as $f_2(e_{4k-3}) = 1$, $f_2(e_{4k-2}) = 2k+1$, $f_2(e_{4k-1}) = 2k+2$, $f_2(e_{4k}) = 2k+3$, $f_2(e_{4k+1}) = 2k+4$ and $f_2(e_{4k+2}) = 2$. Since for every six consecutive edges we have to use at least four new colors, so this edge coloring is a TDE-coloring of P_{4k+2} with the minimum number $2k+4$ colors.

If $n = 4k+4$, for some $k \in \mathbb{N}$, then using f_0 we color the $4k-4$ edges and for the rest of edges define f_3 as $f_3(e_{4k-3}) = 1$, $f_3(e_{4k-2}) = 2k+1$, $f_3(e_{4k-1}) = 2k+2$, $f_3(e_{4k}) = 2$, $f_3(e_{4k+1}) = 2k+3$, $f_3(e_{4k+2}) = 2k+4$ and $f_3(e_{4k+2}) = 2$. This coloring is a TDE-coloring of P_{4k+2} with the minimum number $2k+4$ colors. So we have the result. □

Theorem 2.2. *If C_n is the cycle graph of order $n \geq 8$, then*

$$\chi'^t_d(C_n) = \begin{cases} 2k+2, & \text{if } n = 4k, \\ 2k+3, & \text{if } n = 4k+1, \\ 2k+4, & \text{if } n = 4k+2,\ 4k+3. \end{cases}$$

Also $\chi'^t_d(C_3) = 3$, $\chi'^t_d(C_4) = 2$, $\chi'^t_d(C_5) = \chi'^t_d(C_6) = 4$ and $\chi'^t_d(C_7) = 5$.

Proof. It is similar to the Proof of Theorem 2.1. □

The following corollary is an immediate consequence of Theorems 2.1 and 2.2.

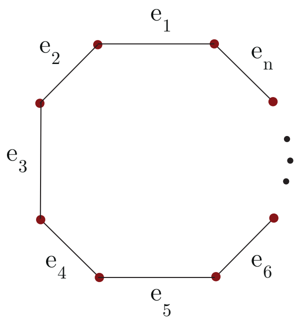

FIGURE 3. Cycle graph of order n, C_n.

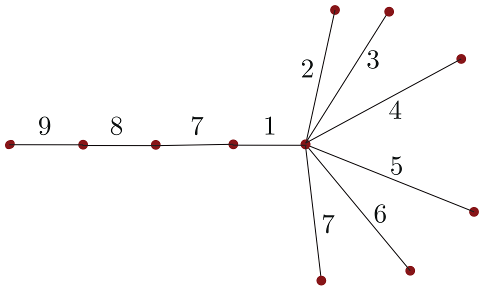

FIGURE 4. Graph G with $\chi_d'^t(G) = \Delta(G) + 2$.

Corollary 2.1. *For every $n \geq 6$, $\chi_d'^t(P_n) = \chi_d'^t(C_{n-1})$.*

The following theorem present a lower bound for TDEC-number of graphs G which have the graph P_6 as induced subgraph.

Theorem 2.3. *If G is a connected graph containing P_6 as an induced subgraph, then $\chi_d'^t(G) \geq \Delta(G) + 2$. More generally, if the path graph P_n is an induced subgraph of G, then $\chi_d'^t(G) \geq \Delta(G) + \chi_d'^t(P_{n-2})$.*

Proof. We assign $\Delta(G)$ colors to the edges which are incident to the vertex with maximum degree $\Delta(G)$. Now we consider P_6 as induced subgraph of G. As we have seen in the Proof of Theorem 2.1, we need at least two new colors for each four consecutive edges. So we have $\chi_d'^t(G) \geq \Delta(G)+2$. The proof of inequality $\chi_d'^t(G) \geq \Delta(G)+\chi_d'^t(P_{n-2})$ is similar. □

Remark 2.2. *The graph G in Figure 4 and its coloring shows that the lower bound in Theorem 2.3 is sharp.*

Theorem 2.4. *For every $n \in \mathbb{N}$, $2n-1 \leq \chi_d'^t(K_{2n}) \leq 4n-2$ and $2n \leq \chi_d'^t(K_{2n+1}) \leq 4n-1$.*

Proof. The lower bounds follow from Remark 2.1. To obtain the upper bound, suppose that $V(K_{2n+1}) = \{u_1, \ldots, u_{2n+1}\}$. By removing the vertex u_{2n+1}, we have the complete graph K_{2n}. We know that $\chi'(K_{2n}) = 2n-1$. So we color the edges of K_{2n} with $2n-1$ colors. Now we add the vertex u_{2n+1} and make K_{2n+1} and assign the new colors $2n, 2n+1, \ldots, 4n-1$ to new edges. This is a TDE-coloring for K_{2n+1}. Therefore $\chi_d'^t(K_{2n+1}) \leq 4n-1$. By the similar method we have $\chi_d'^t(K_{2n}) \leq 4n-2$. □

Theorem 2.5. (i) *For every $n \neq m$, $max\{n, m\} \leq \chi_d'^t(K_{n,m}) \leq m + n - 1$.*

(ii) *For every $n \in \mathbb{N}$, $n \leq \chi'^t_d(K_{n,n}) \leq 2n$.*

Proof. (i) The lower bounds follow from Remark 2.1. To obtain the upper bound, suppose that $V(K_{n,m}) = X \cup Y$, where $X = \{u_1, \ldots, u_m\}$ and $Y = \{u_{m+1}, \ldots, u_{m+n}\}$ and $m \geq n$. We have the following cases:
Case 1) $m = n + 1$. By removing the vertex u_1, we have the complete bipartite graph $K_{n,n}$. We know that $\chi'(K_{n,n}) = n$. So we color the edges of $K_{n,n}$ with n colors. Now we add the vertex u_1 and make $K_{n+1,n}$ and assign the new colors $n+1, n+2, \ldots, 2n$ to new edges. This is a TDE-coloring for $K_{n,m}$ and we have $\chi'^t_d(K_{n,m}) \leq 2n = m + n - 1$.
Case 2) $m > n + 1$. By removing the vertex u_1, we have the complete bipartite graph $K_{m-1,n}$. We know that $\chi'(K_{m-1,n}) = m - 1$. So we color the edges of $K_{m-1,n}$ with $m - 1$ colors. Now we add the vertex u_1 and make $K_{m,n}$ and assign the new colors $m, m+1, \ldots, m+n-1$ to new edges. This is a TDE-coloring for $K_{m,n}$ and we have $\chi'^t_d(K_{m,n}) \leq m + n - 1$.

(ii) In this part we have $m = n$. By removing the vertex u_1, we have the complete bipartite graph $K_{n-1,n}$. We know that $\chi'(K_{n-1,n}) = n$. So we color the edges of $K_{n-1,n}$ with n colors. Now we add the vertex u_1 and make $K_{n,n}$ and assign the new colors $n+1, n+2, \ldots, 2n$ to new edges. This is a TDE-coloring for $K_{n,n}$ and we have $\chi'^t_d(K_{n,n}) \leq 2n$.

□

Remark 2.3. *The lower bounds in parts (i) and (ii) of Theorem 2.5 are sharp. It suffices to consider $K_{3,2}$ and $K_{2,2} = C_4$, respectively. Note that $\chi'^t_d(K_{3,2}) = 3$ and $\chi'^t_d(C_4) = 2$. Also the upper bound of part (i) is sharp. It suffices to consider the star graph $K_{1,6}$. Note that $\chi'^t_d(K_{1,6}) = 1 + 6 - 1 = 6$.*

Let n be any positive integer and F_n be the friendship graph with $2n + 1$ vertices and $3n$ edges, formed by the join of K_1 with nK_2. By Remark 2.1, and TDE-coloring which has shown in Figure 5, we have the following result for the wheel of order n, W_n and the friendship graph F_n.

Theorem 2.6. (i) *For any $n \geq 3$, $\chi'^t_d(W_n) = n - 1$.*
(ii) *For $n \geq 2$, $\chi'^t_d(F_n) = 2n$.*

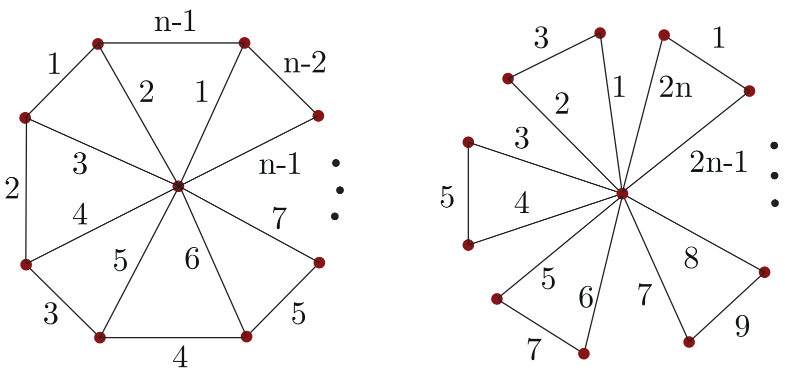

FIGURE 5. TDE-coloring of wheel and friendship graph.

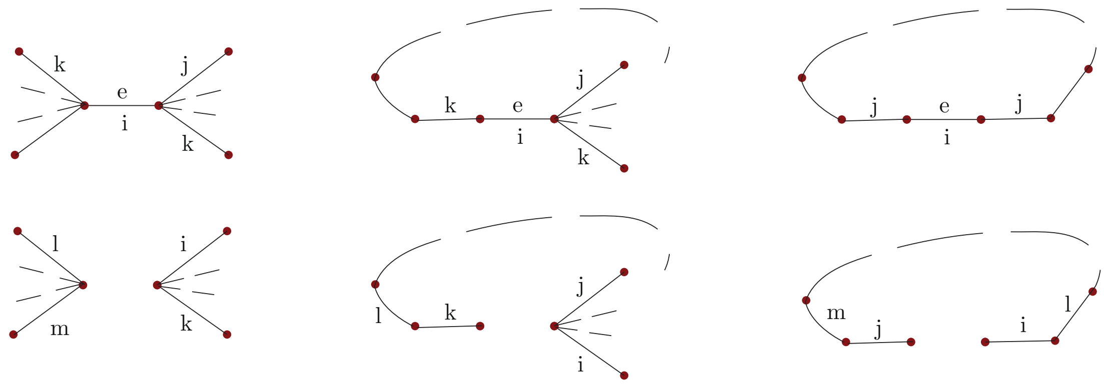

FIGURE 6. Cases which has been considered in the proof of Theorem 3.1.

3. TDEC-number of some operations on a graph

The graph $G-v$ is a graph that is made by deleting the vertex v and all edges incident to v from the graph G and the graph $G-e$ is a graph that obtained from G by simply removing the edge e. In this section we present bounds for TDEC-number of $G-v$ and $G-e$. We begin with $G-e$.

Theorem 3.1. *If G is a connected graph, and $e\in E(G)$ is not a bridge of G, then*

$$\chi_d'^t(G)-2\leq \chi_d'^t(G-e)\leq \chi_d'^t(G)+2.$$

Proof. First we prove the right inequality. Suppose that the edge e in a TDE-coloring of G has color i. If no edges of G use the color class i, then TDE-coloring of G is a TDE-coloring of $G-e$, too. So $\chi_d'^t(G-e)\leq \chi_d'^t(G)$. If some edges of G use the color class i in TDE-coloring, then we have at most two edges with color i. If two edges of G have color i, then removing e does not effect on TDE-coloring and any edge uses the old color class in TDE-coloring of G. So $\chi_d'^t(G-e)\leq \chi_d'^t(G)$. If only one edge e has the color i, then we change the color of some edges in $G-e$ to have a TDE-coloring for $G-e$. In this case the edge e uses some color class, say k, and is adjacent to all color class k. We can not have more than two k in this case. Suppose that we have two k. Then we have only three cases for the graph G as we see in Figure 6. In Figure 6, the colors l and m are new colors and we only change the color of some edges in $G-e$ and assign the other edges their old color in G. This coloring is a TDE-coloring for $G-e$. In any case we do not use more than two new colors. Therefore we have $\chi_d'^t(G-e)\leq \chi_d'^t(G)+2$. Now suppose that we have only one color k. Then we have only six cases for the graph G as we see in Figure 7. In Figure 7, the colors l and m are new colors and we only change the color of some edges in $G-e$ and assig the other edges their old color in G. This kind of coloring is a TDE-coloring for $G-e$. In any case we do not use more than two new colors. So we have $\chi_d'^t(G-e)\leq \chi_d'^t(G)+2$.

Now we prove that $\chi_d'^t(G)-2\leq \chi_d'^t(G-e)$. To do this, first we color $G-e$ and then we add edge e. We assign new color i to e and new color j to one edge which is adjacent to e. So we have a TDE-coloring for G and $\chi_d'^t(G)\leq \chi_d'^t(G-e)+2$. Therefore we have the result. □

Theorem 3.2. *If G is a connected graph, and $v\in V(G)$ is not a cut vertex of G, then*

$$\chi_d'^t(G)-deg(v)\leq \chi_d'^t(G-v)\leq \chi_d'^t(G)+deg(v).$$

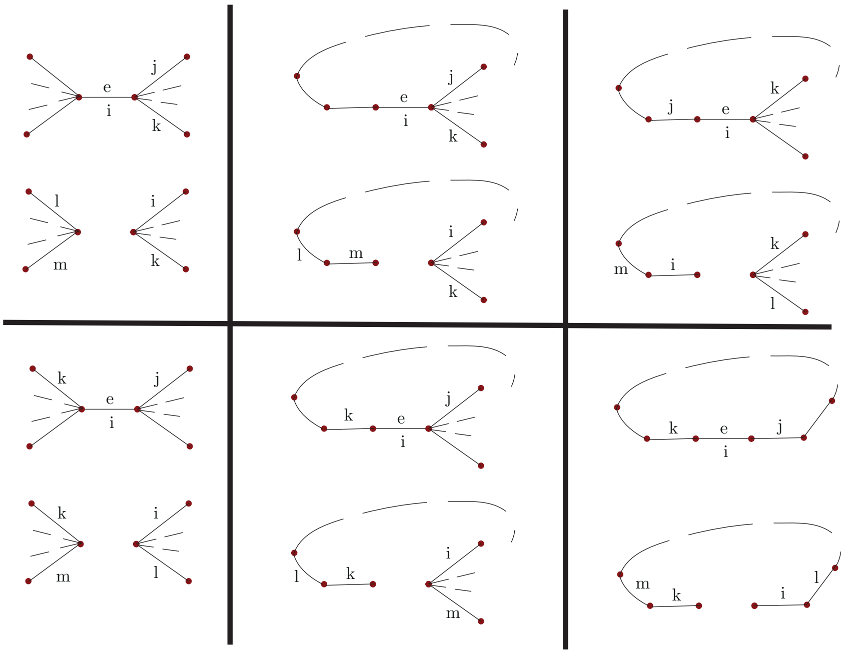

FIGURE 7. Another cases which has been considered in the proof of Theorem 3.1.

Proof. First we prove the left inequality. We give a TDE-coloring to $G-v$, add v and all the corresponding edges. Then we assign $deg(v)$ new colors to these edges and do not change the color of other edges. So this is a TDE-coloring of G and $\chi_d^{\prime t}(G) \leq \chi_d^{\prime t}(G-v) + deg(v)$.

For the right inequality, first we give a TDE-coloring to G. In this case, since v is not a cut vertex, each edge which is adjacent to an edge with endpoint v has an other adjacent edge too. we change the color of this edge to a new color and do this $deg(v)$ times and do not change the color of the other edges. So this is a TDE-coloring of $G-v$ and $\chi_d^{\prime t}(G-v) \leq \chi_d^{\prime t}(G) + deg(v)$. Therefore we have the result. □

The following theorem is an immediate consequence of Theorems 2.2 and 2.6.

Theorem 3.3. *There is a connected graph G, and a vertex $v \in V(G)$ which is not a cut vertex of G such that $|\chi_d^{\prime t}(G) - \chi_d^{\prime t}(G-v)|$ can be arbitrarily large.*

In a graph G, contraction of an edge e with endpoints u, v is the replacement of u and v with a single vertex such that edges incident to the new vertex are the edges other than e that were incident with u or v. The resulting graph G/e has one less edge than G. We denote this graph by G/e. We end this section with the following theorem which gives bounds for $\chi_d^{\prime t}(G/e)$.

Theorem 3.4. *If G is a connected graph and $e = uv \in E(G)$, then*

$$\chi_d^{\prime t}(G) - 2 \leq \chi_d^{\prime t}(G/e) \leq \chi_d^{\prime t}(G) + min\{deg(u), deg(v)\} - 1.$$

Proof. First we prove the left inequality. We give a TDE-coloring to G/e, add e and assign it a new color, say i and change the color of one of its adjacent edges to new color j and do not change other colors. This is a TDE-coloring of G. So we have $\chi_d^{\prime t}(G) \leq \chi_d^{\prime t}(G/e) + 2$. For the right inequality, we give a TDE-coloring to G. Suppose that $min\{deg(u), deg(v)\} = deg(u)$. Now we make G/e and change the color of adjacent edges of e with the endpoint u to new colors. So we have the result. □

Remark 3.1. *The lower bound in Theorem 3.4 is sharp. It suffices to consider the cycle graph C_5 as G. Note that $\chi_d'^t(C_5) = 4$ and $\chi_d'^t(C_4) = 2$.*

4. TDEC-number of k-subdivision of a graph

The *k-subdivision* of G, denoted by $G^{\frac{1}{k}}$, is constructed by replacing each edge v_iv_j of G with a path of length k., say $P_{\{v_i,v_j\}}$. These k-paths are called *superedges*, any new vertex is an internal vertex, and is denoted by $x_l^{\{v_i,v_j\}}$ if it belongs to the superedge $P_{\{v_i,v_j\}}$, $i < j$ with distance l from the vertex v_i, where $l \in \{1, 2, \dots, k-1\}$. Note that for $k = 1$, we have $G^{1/1} = G^1 = G$, and if the graph G has v vertices and e edges, then the graph $G^{\frac{1}{k}}$ has $v + (k-1)e$ vertices and ke edges. The total dominator chromatic number of a graph has been studied in [1]. In this section we study TDEC-number of k-subdivision of a graph. In particular, we obtain some bounds for $\chi_d'^t(G^{\frac{1}{k}})$ and prove that for any $k \geq 2$, $\chi_d'^t(G^{\frac{1}{k}}) \leq \chi_d'^t(G^{\frac{1}{k+1}})$.

Theorem 4.1. *If G is a graph with m edges, then $\chi_d'^t(G^{\frac{1}{k}}) \geq m$, for $k \geq 3$.*

Proof. For $k = 3$, in any superedge $P^{\{v,w\}}$ such as $\{v, x_1^{\{v,w\}}, x_2^{\{v,w\}}, w\}$,The edge $x_1^{\{v,w\}}x_2^{\{v,w\}}$ need to use a new color in at least one of its adjacent edges, and we cannot use this color in any other superedges. So we have the result. □

Theorem 4.2. *If G is a connected graph with m edges and $k \geq 2$, then*

$$\chi_d'^t(P_{k+1}) \leq \chi_d'^t(G^{\frac{1}{k}}) \leq m\chi_d'^t(P_{k+1}).$$

Proof. First we prove the the right inequality. Suppose that $e = uu_1$ be an arbitrary edge of G. This edge is replaced with the super edge $P^{\{u,u_1\}}$ in $G^{\frac{1}{k}}$, with vertices $\{u, x_1^{\{u,u_1\}}, \dots, x_{k-1}^{\{u,u_1\}}, u_1\}$. We color this superedge with $\chi_d'^t(P_{k+1})$ colors as a total dominator edge coloring of P_{k+1}. We do this for all superedges. Thus we need at most $m\chi_d'^t(P_{k+1})$ new colors for a total dominator edge coloring of $G^{\frac{1}{k}}$.

For the left inequality, if G is a path then the result is true. So we suppose that G is a connected graph which is not a path. Let $P^{\{v,w\}}$ be an arbitrary superedge of $G^{\frac{1}{k}}$ with vertex set $\{v, x_1^{\{v,w\}}, \dots, x_{k-1}^{\{v,w\}}, w\}$. Since G is not a path, so at least one of v and w is adjacent to some vertices of $G^{\frac{1}{k}}$ except $x_1^{\{v,w\}}$ and $x_{k-1}^{\{v,w\}}$, respectively. Let c' be a total dominator edge coloring of $G^{\frac{1}{k}}$. The two following cases can be occured: either the restriction of c' to edges of $P^{\{v,w\}}$ is a total dominator edge coloring and so we have the result, or not. If the restriction of c' to edges of $P^{\{v,w\}}$ is not a total dominator coloring then since c' is a total dominator edge coloring of $G^{\frac{1}{k}}$ we conclude that at least one of edges $vx_1^{\{v,w\}}$ and $wx_{k-1}^{\{v,w\}}$, as the edges of the induced subgraph $P^{\{v,w\}}$, are not adjacent to every edge of some color class. Without loss of generality we assume that the edge $vx_1^{\{v,w\}}$, as the edge of the induced subgraph $P^{\{v,w\}}$, is not adjacent to every vertex of some color class. But c' is a total dominator coloring of $G^{\frac{1}{k}}$ so the edge $vx_1^{\{v,w\}}$ is adjacent to every edge of some color class, as the edge of $G^{\frac{1}{k}}$. Hence there is a new color for an adjacent edge of $vx_1^{\{v,w\}}$, except the edge $x_1^{\{v,w\}}x_2^{\{v,w\}}$. Thus if we use this new color for the edge $x_1^{\{v,w\}}x_2^{\{v,w\}}$ and consider the restriction of c' for the remaining edges of superedge $P^{\{v,w\}}$, then $P^{\{v,w\}}$ has a total dominator edge coloring. Therefore the total edge coloring c' has at least $\chi_d'^t(P_{k+1})$ colors. □

The lower bound of Theorem 4.2 is sharp for P_2 and by the following Proposition we show that the upper bound of this Theorem is sharp for $G = K_{1,n}$ and $k = 3$.

Proposition 4.1. *For every $n \geq 3$, $\chi_d''^t(K_{1,n}^{\frac{1}{3}}) = 2n$.*

Proof. Let $e_1, \ldots, e_n$ be the pendant edges of $K_{1,n}^{\frac{1}{3}}$. The adjacent edges to e_i is denoted by f_i, and the adjacent edge to f_i is denoted by g_i for any $1 \leq i \leq n$. Since edge f_i is the only edge adjacent to e_i, so the color of f_i should not be used for any other edges of graph, where $1 \leq i \leq n$. Thus we color the edges $f_1, \ldots, f_n$ with colors $1, \ldots, n$, respectively, and do not use these colors any more. For every $1 \leq i \leq n$, the edge f_i is adjacent to e_i and g_i, thus we need a new color for at least one of e_i and g_i. So we need at least $2n$ color to have a TDE-coloring of $K_{1,n}^{\frac{1}{3}}$. Now for every e_i and g_i we use the new color $i + n$. Obviously this is a TDE-coloring of $K_{1,n}^{\frac{1}{3}}$ and we have the result. □

Here we improve the lower bound of Theorem 4.2 for $k \geq 10$.

Theorem 4.3. *If G is a connected graph with m edges and maximum degree $\Delta(G)$ and $k \geq 10$, then*

$$m(\chi_d''^t(P_{k-1}) - 2) + 2 \leq \chi_d''^t(G^{\frac{1}{k}}).$$

Proof. Let $e = vw$ be an edge of G. We consider the superedge $P^{\{v,w\}}$ with vertex set $\{v, x_1^{\{v,w\}}, \ldots, x_{k-1}^{\{v,w\}}, w\}$. It is clear that $P^{\{v,w\}} \setminus \{v, w\}$ is the path graph P_{k-1}. Since we use repetitious colors for the edges $x_1^{\{v,w\}}x_2^{\{v,w\}}$ and $x_{k-2}^{\{v,w\}}x_{k-1}^{\{v,w\}}$ in the TDE-coloring of paths, so we need at least $\chi_d''^t(P_{k-1}) - 2$ colors for each superedges and we cannot use these colors anymore. Also we need two colors for edges$x_1^{\{v,w\}}x_2^{\{v,w\}}$ and $x_{k-2}^{\{v,w\}}x_{k-1}^{\{v,w\}}$ and some other edges hence the result follows. □

Theorem 4.4. *If G is a connected graph with m edges and maximum degree $\Delta(G)$ and $k \geq 10$, then*

$$\chi_d''^t(G^{\frac{1}{k}}) \geq \begin{cases} m(\frac{k}{2}) + 2 & k \equiv 0 \ (mod\ 4) \\ m(\frac{k-1}{2}) + 2 & k \equiv 1 \ (mod\ 4) \\ m(\frac{k-2}{2}) + 2 & k \equiv 2 \ (mod\ 4) \\ m(\frac{k-1}{2}) + 2 & k \equiv 3 \ (mod\ 4). \end{cases}$$

Proof. It follows by Theorems 2.1 and 4.3. □

Theorem 4.5. *If G is a connected graph with m edges with maximum degree $\Delta(G)$ and $k \geq 10$, then*

$$\chi_d''^t(G^{\frac{1}{k}}) \leq m(\chi_d''^t(P_{k+1}) - 2) + \Delta(G).$$

Proof. As we see in the TDE-coloring of paths, we can use the same color for the pendant edges. So we assign the colors $1, 2, \ldots, \Delta(G)$ to all the edges incident to the vertices belong to G and we color other edges of any superedges with $\chi_d''^t(P_{k+1}) - 2$ colors. This is a TDE-coloring for $G^{\frac{1}{k}}$ and hence the result follows. □

Theorem 4.6. *If G is a connected graph with m edges and $k \geq 10$, then*

$$\chi_d''^t(G^{\frac{1}{k}}) \leq \begin{cases} \frac{mk}{2} + \Delta(G), & k \equiv 0 \ (mod\ 4) \\ m(\frac{k+1}{2}) + \Delta(G), & k \equiv 1 \ (mod\ 4) \\ m(\frac{k+2}{2}) + \Delta(G), & k \equiv 2 \ (mod\ 4) \\ m(\frac{k+1}{2}) + \Delta(G), & k \equiv 3 \ (mod\ 4). \end{cases}$$

Proof. It follows by Theorems 2.1 and 4.5. □

Theorem 4.7. *For any $k \geq 4$, $\chi_d'^t(G^{\frac{1}{k}}) \leq \chi_d'^t(G^{\frac{1}{k+1}})$.*

Proof. First we give a TDE-coloring to the edges of $G^{\frac{1}{k+1}}$. Let $P^{\{v,w\}}$ be an arbitrary superedge of $G^{\frac{1}{k+1}}$ with vertex set $\{v, x_1^{\{v,w\}}, \ldots, x_k^{\{v,w\}}, w\}$. We have the following cases:

Case 1) There exists an edge $u \in \{x_1^{\{v,w\}}x_2^{\{v,w\}}, \ldots, x_{k-1}^{\{v,w\}}x_k^{\{v,w\}}\}$ such that other edges of graph are not adjacent to all edges with color class of the edge u. Consider the graph in Figure 8. Suppose that the edge u has the color i and the edge n has the color α. The edge m is adjacent to all edges with color class j and $j \neq i$ and the edge n is adjacent to all edges with color class k and $k \neq i$. Since $k \geq 4$, without loss of generality, suppose that $m \neq vx_1^{\{v,w\}}$. We have two subcases:

Subcase i) The color of the edge m is not α. In this case, we make G/u and do not change the color of any edges. So without adding a new color we have a TDE-coloring for this new graph.

Subcase ii) The color of the edge m is α. Since the edge u is adjacent to color class α, so any other edges does not have color α. In this case, by making G/m and keeping the color of any edges as before, we have a TDE-coloring for this new graph. Because the edge t is adjacent to color class which is not α, the color of the edge t is not i (because if the color of the edge t is i it has contradiction with our assumptions), the edge n is adjacent to all edges with color class k and the edge u is adjacent to all edges with color class α.

Case 2) For every edge $u \in \{x_1^{\{v,w\}}x_2^{\{v,w\}}, \ldots, x_{k-1}^{\{v,w\}}x_k^{\{v,w\}}\}$, there exists an edge such that is adjacent to all edges with color of edge u. Consider the graph in Figure 8. Suppose that the edge u has the color i and the edge p has the color j and the edge p is adjacent to all edges with color i. We have two subcases:

Subcase i) The color of the edge q is not i. We make G/r and do not change the color of any edges. So without adding a new color we have a TDEC for this new graph since there is no other edges with color i.

Subcase ii) The color of the edge q is i. In this case the edge r is adjacent to color class of edge s and the color of the edge s does not use for other edges. Now we make G/u and do not change the color of any edges. Now we consider the color of edge r. If the color of r is j, then we change it to i and since obviously the edge s was adjacent to a color class except j, so we have a TDE-coloring. If the color of the edge r is not j we do not change the color of that and we have a TDE-coloring again.

Now we do the same algorithm for all superedges. So we have $\chi_d'^t(G^{\frac{1}{k}}) \leq \chi_d'^t(G^{\frac{1}{k+1}})$. □

FIGURE 8. A part of a superedge in the proof of Theorem 4.7.

Theorem 4.8. *For any graph G, $\chi_d'^t(G^{\frac{1}{3}}) \leq \chi_d'^t(G^{\frac{1}{4}})$.*

Proof. First we give a TDE-coloring to the edges of $G^{\frac{1}{4}}$. Let $P^{\{i,j\}}$ be an arbitrary superedge of $G^{\frac{1}{4}}$ with edge set $\{s, v, u, w\}$ (see Figure 9) and suppose that the edge v has the color α. We have the following cases:

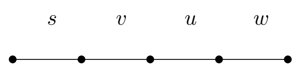

FIGURE 9. A superedge in $G^{\frac{1}{4}}$.

Case 1) The edges u and s are adjacent with an edge with a color class which is not α. we have two subcases:

Subcase i) The color of edges u and s are different. In this case, we make G/v and don't change the color of any edges. So we have a TDE-coloring for this new graph. Because two edges u and s are adjacent with an edge with color class which is not α.

Subcase ii) The color of edges u and s are the same. Suppose that u and s have color β. In this case β does not use for any other edges. So w is adjacent with an edge with color class except β. Now we make G/u. So we have a TDE-coloring for this new graph.

Case 2) The edge u is adjacent to all edges with color class α. we have two subcases:

Subcase i) The color of the edge w is not α. Suppose that the edge u has color γ. If the edge v is adjacent with all edges with color γ, and if the color of s is γ, we make G/u. But if the color of edge s is not γ, then we make G/u and assign the color γ to the edge w. So we have a TDE-coloring for this new graph. If the edge v is adjacent to all edges with color except γ (edge s), then we make G/u. So we have a TDE-coloring for this new graph.

Subcase ii) The color of the edge w is α. We have two new cases. First, the edge v is adjacent to an edge with color class γ. Any adjacent edge with w is not adjacent to edge with color class α (except u). So we make G/u and assign the color γ to w. This is a TDE-coloring for this new graph. Second, v is not adjacent with color class γ. So the color of the edge s does not use any more. Also the edge s is not adjacent to edge with color class α. So we make G/v. This is a TDE-coloring for this new graph.

Case 3) The edge s is adjacent to all edges with color class α. We have two subcases:

Subcase i) If v is the only edge which has color α, then we make G/u when v is adjacent with color class of edge s and make G/s when v is adjacent with color class of edge u. So this is a TDE-coloring for this new graph.

Subcase ii) If there exist some edges with color α, then the edge u is adjacent with color class except α. So we make G/v. This is a TDE-coloring for this new graph.

We apply this TDE-coloring for all superedges. So we obtain a TDE-coloring for $G^{\frac{1}{3}}$. Therefore we have $\chi_d^{\prime t}(G^{\frac{1}{3}}) \leq \chi_d^{\prime t}(G^{\frac{1}{4}})$. □

Theorem 4.9. *For any graph G, $\chi_d^{\prime t}(G^{\frac{1}{2}}) \leq \chi_d^{\prime t}(G^{\frac{1}{3}})$.*

Proof. First we give a TDE-coloring to the edges of $G^{\frac{1}{3}}$. Let $P^{\{i,j\}}$ be an arbitrary superedge of $G^{\frac{1}{3}}$ with edge set $\{s, v, u\}$ (see Figure 10) and suppose that the edge v has the color α. We have the following cases:

Case 1) The edges u and s are adjacent with an edge with a color class which is not α. we have two subcases:

Subcase i) The color of edges u and s are different. In this case, we make G/v and don't change the color of any edges. So we have a TDE-coloring for this new graph. Because two edges u and s are adjacent with an edge with color class which is not α.

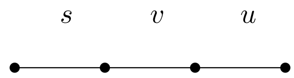

FIGURE 10. A superedge in $G^{\frac{1}{3}}$.

Subcase ii) The color of edges u and s are the same. Suppose that u and s have color β. In this case any other edges is not adjacent with color class β, because i and j are not adjacent vertices (Because of the definition of $G^{\frac{1}{3}}$). Now we make G/u. So we have a TDE-coloring for this new graph.

Case 2) The edge s is adjacent to all edges with color class α. We have two subcases:

Subcase i) If v is the only edge which has color α, then we make G/u when v is adjacent with color class of edge s and make G/s when v is adjacent with color class of edge u. So this is a TDE-coloring for this new graph.

Subcase ii) If there exist some edges with color α, then the edge u is adjacent with color class except α. If the edges u and s have the same color then we make G/u and if u and s have different colors, then we make G/v. This is a TDE-coloring for this new graph.

Case 3) The edges u and s are adjacent to all edges with color class α. So there is no other edge with color α. We have two subcases:

Subcase i) The edges u and s have the same color then we make G/u.

Subcase ii) The edges u and s have different colors, then we make G/u when v is adjacent with color class of edge s and make G/s when v is adjacent with color class of edge u.

We apply this TDE-coloring for all superedges. So we obtain a TDE-coloring for $G^{\frac{1}{2}}$. Therefore we have $\chi_d'^t(G^{\frac{1}{2}}) \leq \chi_d'^t(G^{\frac{1}{3}})$. □

5. CONCLUSIONS

A total dominator edge coloring (briefly TDE-coloring) of a graph G is a proper edge coloring of G in which each edge of the graph is adjacent to every edge of some color class. The total dominator edge chromatic number (briefly TDEC-number) $\chi_d'^t(G)$ of G is the minimum number of color classes in a TDE-coloring of G. We obtained some properties of $\chi_d'^t(G)$ and computed this parameter for specific graphs. We also examined the effects on $\chi_d'^t(G)$ when G is modified by operations on vertices and edges of G. Finally, we investigated TDEC-number of the k-subdivison of G.

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