TWMS J. App. and Eng. Math. V.12, N.3, 2022, pp. 864-875

‘DEL’ RELATION AND PARALLELISM IN FUZZY LATTICES

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Abstract. The notions of ‘del’ relation, a neutral element and parallelism from lattice theory are introduced in a fuzzy lattice and their properties are obtained.

Keywords: Fuzzy lattice, fuzzy modular element, parallelism, atom-free fuzzy lattice.

AMS Subject Classification: 03B52, 03E72, 06D72.

1. Introduction

Ore [10] introduced the concept of a distributive element in a lattice. A generalization of this concept, namely, a ‘neutral element’ in a lattice was introduced by Birkhoff [3]. Several researchers have developed many equivalent conditions for an element of a lattice to be neutral. Maeda and Maeda [4] have studied modular pairs in lattices.

Fuzzy sets and fuzzy relations were introduced by Zadeh [11]. The concept of a fuzzy binary relation and a fuzzy partial order relation are due to Zadeh [12]. Fuzzy lattices were defined by Ajmal and Thomas [1] and Chon [2]. Mezzomo *et. al.* [6] defined a new notion of a fuzzy ideal and a fuzzy filter in a fuzzy lattice. Recently, Wasadikar and Khubchandani [8] have defined a fuzzy modular pair in a fuzzy lattice. As a continuation of the study of fuzzy modular pairs in [8], in this paper, we consider fuzzy distributive triples in a fuzzy lattice. In section 3, we define a neutral element in a fuzzy lattice $\mathcal{L} = (X,\ A)$. We prove that the set of all neutral elements of a fuzzy lattice is a distributive sublattice of $\mathcal{L} = (X,\ A)$. In section 4, we define the concept of parallelism in a fuzzy lattice, while in section 5, we define the notion of atom free parallelism and prove some properties and relations among them in a fuzzy lattice.

2. Preliminaries

Throughout this paper, (X, A) denotes a fuzzy lattice, where A is a fuzzy partial order relation on a nonempty set X.

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§ Manuscript received: May 10, 2020; accepted: June 09, 2021.
TWMS Journal of Applied and Engineering Mathematics, Vol.12, No.3

For the definitions of a fuzzy partial order relation, fuzzy equivalence relation, fuzzy supremum, fuzzy infimum, fuzzy lattice etc. we refer to Chon [2].

We use the notations $a \vee_F b$ and $a \wedge_F b$ to denote the fuzzy supremum and the fuzzy infimum of $a, b \in X$ to distinguish the supremum and infimum of a, b in the lattice sense, if these exist in X.

Definition 2.1. [7, Definition 3.4] *A fuzzy lattice (X, A) is said to be bounded if there exist elements $\perp$ and $\top$ in X, such that $A(\perp, a) > 0$ and $A(a, \top) > 0$, for every $a \in X$. In this case, $\perp$ and $\top$ are respectively called bottom and top elements of X.*

We illustrate these concepts in the following example.

Example 2.1. *Let $X = \{\perp, a, b, c, d, e, \top\}$ and let $A : X \times X \longrightarrow [0, 1]$ be a fuzzy relation defined as follows:*
$A(\perp, \perp) = A(a, a) = A(b, b) = A(c, c) = A(d, d) = A(e, e) = A(\top, \top) = 1,$
$A(\perp, a) = 0.07,\ A(\perp, b) = 0.16,\ A(\perp, c) = 0.34,\ A(\perp, d) = 0.51,\ A(\perp, e) = 0.62,$
$A(\perp, \top) = 0.82,$
$A(a, \perp) = 0,\ A(a, b) = 0,\ A(a, c) = 0.19,\ A(a, d) = 0.36,\ A(a, e) = 0,\ A(a, \top) = 0.67,$
$A(b, \perp) = 0,\ A(b, a) = 0,\ A(b, c) = 0.09,\ A(b, d) = 0,\ A(b, e) = 0.38,\ A(b, \top) = 0.55,$
$A(c, \perp) = 0,\ A(c, a) = 0,\ A(c, b) = 0,\ A(c, d) = 0,\ A(c, e) = 0,\ A(c, \top) = 0.39,$
$A(d, \perp) = 0,\ A(d, a) = 0,\ A(d, b) = 0,\ A(d, c) = 0,\ A(d, e) = 0,\ A(d, \top) = 0.20,$
$A(e, \perp) = 0,\ A(e, a) = 0,\ A(e, b) = 0,\ A(e, c) = 0,\ A(e, d) = 0,\ A(e, \top) = 0.10,$
$A(\top, \perp) = 0,\ A(\top, a) = 0,\ A(\top, b) = 0,\ A(\top, c) = 0,\ A(\top, d) = 0,\ A(\top, e) = 0.$
Then A is a fuzzy partial order relation.

The fuzzy join and fuzzy meet tables are as follows:

$\vee_F$	$\perp$	a	b	c	d	e	$\top$
$\perp$	$\perp$	a	b	c	d	e	$\top$
a	a	a	c	c	d	$\top$	$\top$
b	b	c	b	c	$\top$	e	$\top$
c	c	c	c	c	$\top$	$\top$	$\top$
d	d	d	$\top$	$\top$	d	$\top$	$\top$
e	e	$\top$	e	$\top$	$\top$	e	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

$\wedge_F$	$\perp$	a	b	c	d	e	$\top$
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
a	$\perp$	a	$\perp$	a	a	$\perp$	a
b	$\perp$	$\perp$	b	b	$\perp$	b	b
c	$\perp$	a	b	c	a	b	$\perp$
d	$\perp$	a	$\perp$	a	d	$\perp$	d
e	$\perp$	$\perp$	b	b	$\perp$	e	e
$\top$	$\top$	a	b	c	d	e	$\top$

We note that (X, A) is a fuzzy lattice.
We recall some known results which we shall use in this paper.

Proposition 2.1. [2, Proposition 3.3] *and* [6, Proposition 2.4] *Let (X, A) be a fuzzy lattice. For $a, b, c \in X$, the following statements hold:*
(i) $A(a, b) > 0$ iff $a \wedge_F b = a$ iff $a \vee_F b = b$.
(ii) If $A(b, c) > 0$, then $A(a \wedge_F b, a \wedge_F c) > 0$ and $A(a \vee_F b, a \vee_F c) > 0$.

For the definitions of a fuzzy distributive and fuzzy modular lattice, we refere to [2]. We note that from the distributive inequalities and fuzzy antisymmetry property, (X, A) is distributive iff $A(a \wedge_F (b \vee_F c), (a \wedge_F b) \vee_F (a \wedge_F c)) > 0$ and
$A((a \vee_F b) \wedge_F (a \vee_F c), a \vee_F (b \wedge_F c)) > 0$.

Definition 2.2. [8, Definition 3.1] *Let X be a nonempty set and $\mathcal{L} = (X, A)$ be a fuzzy lattice with $\perp$. Let $a, b \in X$. We say that (a, b) is a fuzzy meet-modular pair and we write $(a, b)_F M_m$, if whenever $A(c, b) > 0$, then $(c \vee_F a) \wedge_F b = c \vee_F (a \wedge_F b)$.*

We say that (a, b) is a fuzzy join-modular pair and we write $(a, b)_F M_j$, if whenever $A(b, c) > 0$, then $(c \wedge_F a) \vee_F b = c \wedge_F (a \vee_F b)$.

Remark 2.1. [8, Remark 3.1] *If* $a \in X$, *then* $(\bot, a)_F M_m$, $(a, \bot)_F M_m$, $(a, a)_F M_m$, $(\top, a)_F M_m$, $(a, \top)_F M_m$ *and* $(\bot, \top)_F M_m$ *hold if* $\bot$ *and* $\top$ *exist.*

Definition 2.3. [8, Definition 4.1] *A fuzzy lattice* $\mathcal{L} = (X,\ A)$ *with* $\bot$ *is called fuzzy weakly modular when in* $\mathcal{L} = (X,\ A)$, $a \wedge_F b \neq \bot$ *implies* $(a, b)_F M_m$.

Definition 2.4. [8, Definition 4.4] *Let* $\mathcal{L} = (X,\ A)$ *be a fuzzy lattice. Let* $a, b \in X$, *then* $b \prec_F a$ *(a "fuzzy covers" b) if* $0 < A(b, a) < 1$ *and* $A(b, x) > 0$ *and* $A(x, a) > 0$ *imply* $x = a$ *or* $x = b$.

Definition 2.5. [8, Definition 3.3] *Let* P *denote the set of all* $a \in X$ *such that* $\bot \prec_F a$. *The elements of* P *are called fuzzy atoms.*

Lemma 2.1. [9, Lemma 5.3] *If* $b \prec_F a \vee_F b$, *then* $(a, b)_F M_j$.

Definition 2.6. [9, Definition 5.3] *Let* $\mathcal{L} = (X,\ A)$ *be a fuzzy lattice with* $\bot$. *If* p *is a fuzzy atom and* $a \wedge_F p = \bot$, *then* $a \prec_F a \vee_F p$ *is called the fuzzy covering property.*

Definition 2.7. [5, Definition 3.1] *Let* $\mathcal{L} = (X,\ A)$ *be a fuzzy lattice and* $Y \subseteq X$. Y *is an ideal of* $\mathcal{L}$ *if it satisfies the following conditions:*
(i) If $x \in X$, $y \in Y$ *and* $A(x, y) > 0$, *then* $x \in Y$.
(ii) If $x, y \in Y$, *then* $x \vee_F y \in Y$.

Definition 2.8. [9, Definition 5.1] *A fuzzy poset* $\mathcal{L} = (X,\ A)$ *with a least element* $\bot$ *is called fuzzy atomic if for every nonzero* $b \in X$ *there is some fuzzy atom* a *such that* $A(a, b) > 0$.

3. 'Del' relation in fuzzy lattices

Definition 3.1. *Let* $\mathcal{L} = (X, A)$ *be a fuzzy lattice. Let* $a, b, c \in X$. *We write* $(a, b, c)_F D$ *if* $(a \vee_F b) \wedge_F c = (a \wedge_F c) \vee_F (b \wedge_F c)$ *(I)*
and we write $(a, b, c)_F D^*$ *if* $(a \wedge_F b) \vee_F c = (a \vee_F c) \wedge_F (b \vee_F c)$. *(II)*
If (I) and (II) hold for all permutations of a, b *and* c, *then we say that* $\{a, b, c\}$ *is a fuzzy distributive triplet and we write* $(a, b, c)_F T$.

$\mathcal{L} = (X, A)$ *is called fuzzy distributive when* $(a, b, c)_F D$ *and* $(a, b, c)_F D^*$ *hold for all elements* $a, b, c \in X$.

Example 3.1. *In Example 2.1, both* $(a, e, b)_F D$ *and* $(a, e, b)_F D^*$ *hold.*

Definition 3.2. *A fuzzy lattice* $\mathcal{L} = (X, A)$ *is called a* FM_j*-symmetric fuzzy lattice if* $(a, b)_F M_j$ *implies* $(b, a)_F M_j$.

Example 3.2. *Consider the fuzzy lattice in Example 2.1.*
For $A(b, e) > 0$, $(b, c)_F M_j$ *holds as* $(e \wedge_F c) \vee_F b = b \vee_F b = b$ *and* $e \wedge_F (c \vee_F b) = e \wedge_F c = b$.
For $A(c, \top) > 0$, $(c, b)_F M_j$ *holds as* $(\top \wedge_F b) \vee_F c = b \vee_F c = c$ *and* $\top \wedge_F (b \vee_F c) = \top \wedge_F c = c$.

Definition 3.3. *Let* $\mathcal{L} = (X, A)$ *be a fuzzy lattice with* $\bot$. *Let* $a, b \in X$. *We write* $a \nabla_F b$ *if* $(x \vee_F a) \wedge_F b = x \wedge_F b$ *for every* $x \in X$. $(*)$

Let $Y \subseteq X$. We write $Y^{\nabla_F} = \{a \in X |\ a \nabla_F b, \text{ for all } b \in Y\}$.

Remark 3.1. *If* $a \nabla_F b$, *then* $a \wedge_F b = \bot$ *and* $(a, b)_F M_m$ *hold.*
If we put $x = \bot$ *in* $(*)$, *then we get* $a \wedge_F b = \bot$.
To show that $(a, b)_F M_m$ *holds. Let* $c \in X$ *be such that* $A(c, b) > 0$. *Since* $a \nabla_F b$ *holds we have* $(c \vee_F a) \wedge_F b = c \wedge_F b = c = c \vee_F \bot = c \vee_F (a \wedge_F b)$.

Remark 3.2. $a \nabla_F b$ *is equivalent to* $a \wedge_F b = \bot$ *and* $(x, a, b)_F D$ *for every* $x \in X$.

Proof. Suppose that $a\nabla_F b$ holds. Then by Remark 3.1 we have $a \wedge_F b = \bot$ and $(a, b)_F M_m$. Hence for every $x \in X$ satisfying $A(x, b) > 0$ we have

$$(x \vee_F a) \wedge_F b = x \vee_F (a \wedge_F b) = (x \wedge_F b) \vee_F (a \wedge_F b) \quad \text{as } A(x, b) > 0.$$

So, $(x, a, b)_F D$ holds.

Conversely, suppose that $a \wedge_F b = \bot$ and $(x, a, b)_F D$ hold.
Then we have,

$$\begin{aligned}(x \vee_F a) \wedge_F b &= (x \wedge_F b) \vee_F (a \wedge_F b), \text{ as } (x, a, b)_F D\\ &= (x \wedge_F b) \vee_F \bot,\\ &= x \wedge_F b.\end{aligned}$$

Thus $a\nabla_F b$ holds. □

Lemma 3.1. *Let $\mathcal{L} = (X, A)$ be a fuzzy lattice with $\bot$. Then the following statements hold for $a, b, a_1, b_1 \in X$.*
(i) If $a\nabla_F b$ holds and a_1, b_1 are such that $A(a_1, a) > 0$ and $A(b_1, b) > 0$, then $a_1\nabla_F b_1$ holds.
(ii) If $a_1\nabla_F b$ and $a_2\nabla_F b$ hold, then $(a_1 \vee_F a_2)\nabla_F b$ holds.
(iii) Y^{∇_F} is an ideal of $\mathcal{L}$ for every subset Y of X.

Proof. Let $\mathcal{L} = (X, A)$ be a fuzzy lattice with $\bot$.
(i): Suppose that $a\nabla_F b$ hold and a_1, b_1 are such that $A(a_1, a) > 0$ and $A(b_1, b) > 0$.
To show that $a_1\nabla_F b_1$ holds, i.e., to show for any $x \in X$, $A((x \vee_F a_1) \wedge_F b_1, x \wedge_F b_1) > 0$.
We note that $A((x \vee_F a_1) \wedge_F b_1, (x \vee_F a_1) \wedge_F b_1) = 1 > 0$.
Hence $A((x \vee_F a_1) \wedge_F b_1, (x \vee_F a_1) \wedge_F b_1 \wedge_F b) > 0$, as $A(b_1, b) > 0$ implies $b_1 \wedge_F b = b_1$.
We have $A((x \vee_F a_1) \wedge_F b_1, (a_1 \vee_F x) \wedge_F (a \vee_F x) \wedge_F b \wedge_F b_1) > 0$, since $A(a_1, a) > 0$.
Hence $A((x \vee_F a_1) \wedge_F b_1, (a_1 \vee_F x) \wedge_F x \wedge_F b \wedge_F b_1) > 0$, since $a\nabla_F b$.
Thus, $A((x \vee_F a_1) \wedge_F b_1, x \wedge_F b \wedge_F b_1) > 0$, by absorption identity.
Hence $A((x \vee_F a_1) \wedge_F b_1, x \wedge_F b_1) > 0$. Since $A(x \wedge_F b_1, (x \vee_F a_1) \wedge_F b_1) > 0$ always holds, by fuzzy antisymmetry of A we get $(x \vee_F a_1) \wedge_F b_1 = x \wedge_F b_1$.
Hence $a_1\nabla_F b_1$ holds.

(ii): If $a_1\nabla_F b$ and $a_2\nabla_F b$, then for any $x \in X$ we have
$A((x \vee_F a_1 \vee_F a_2) \wedge_F b, (x \vee_F a_1 \vee_F a_2) \wedge_F b) = 1 > 0$,
i.e., $A((x \vee_F a_1 \vee_F a_2) \wedge_F b, ((x \vee_F a_1) \vee_F a_2) \wedge_F b) > 0$,
i.e., $A((x \vee_F a_1 \vee_F a_2) \wedge_F b, (x \vee_F a_1) \wedge_F b) > 0$, using $a_2\nabla_F b$ and Remark 3.2.
i.e., $A((x \vee_F a_1 \vee_F a_2) \wedge_F b, (x \wedge_F b) > 0$, by $a_1\nabla_F b$ and Remark 3.2.
As $A((x \wedge_F b, (x \vee_F a_1 \vee_F a_2) \wedge_F b) > 0$ always holds, by fuzzy antisymmetry of A we have $(x \vee_F a_1 \vee_F a_2) \wedge_F b = (x \wedge_F b)$.
Hence $(a_1 \vee_F a_2)\nabla_F b$ holds.

(iii): To show that Y^{∇_F} is an ideal of $\mathcal{L}$ for every subset Y of X.
(a): Let $a \in Y^{\nabla_F}$ and $A(c, a) > 0$, so we get $c\nabla_F b$ for all $b \in Y$. Hence $c \in Y^{\nabla_F}$ holds.
(b): It follows from (ii) that if $a_1, a_2 \in Y^{\nabla_F}$, then $a_1 \vee_F a_2 \in Y^{\nabla_F}$. □

Definition 3.4. *Let $\mathcal{L} = (X, A)$ be a fuzzy lattice with $\bot$. Let $Y_1, Y_2, \cdots, Y_n$ be subsets of X. Suppose that each Y_i contains $\bot$. We say that X is the direct sum of $Y_1, Y_2, \cdots, Y_n$ and we write $X = Y_1 \oplus Y_2 \oplus \cdots, \oplus Y_n$, if every element $a \in X$ can be written as $a = a_1 \vee_F a_2 \vee_F \cdots \vee_F a_n$, $a_i \in Y_i$ (for $i = 1, 2, \cdots, n$) and $Y_i \subset Y_j^{\nabla_F}$ for $i \neq j$.*

Lemma 3.2. *Let $\mathcal{L} = (X, A)$ be a fuzzy lattice with $\bot$ is a direct sum of $Y_1, Y_2, \cdots, Y_n$, then every element $a \in X$ can be expressed in the form $a = a_1 \vee_F \cdots \vee_F a_n$, $a_i \in Y_i$ (for $i = 1, 2, \cdots, n$) uniquely and the sets $Y_1, \cdots, Y_n$ are ideals of $\mathcal{L}$.*

Proof. Let $\mathcal{L} = (X, A)$ be a fuzzy lattice with $\perp$.
(i): Let $a = a_1 \vee_F a_2 \vee_F \cdots \vee_F a_n = b_1 \vee_F b_2 \vee_F \cdots \vee_F b_n$ where $a_i \in Y_i$ and $b_i \in Y_i$ (for $i = 1, 2, \cdots, n$). Then by (iii) of Lemma 3.1 and by Definition 3.4, we have $b_2 \vee_F \cdots \vee_F b_n \in Y_1^{\nabla_F}$. Hence $(b_2 \vee_F \cdots \vee_F b_n)\nabla a_1$. As $a = a_1 \vee_F a_2 \vee_F \cdots \vee_F a_n$ we have $A(a_1, a) > 0$. Hence

$$\begin{aligned} a_1 &= a \wedge_F a_1, \\ &= (b_1 \vee_F (b_2 \vee_F \cdots \vee_F b_n)) \wedge_F a_1, \\ &= b_1 \wedge_F a_1 \text{ as } (b_2 \vee_F \cdots \vee_F b_n)\nabla a_1 \text{ holds} \end{aligned}$$

So, we get $A(a_1, b_1) > 0$. (I)
Since $a_2 \vee_F \cdots \vee_F a_n \in Y_1^{\nabla_F}$, $(a_2 \vee_F \cdots \vee_F a_n)\nabla_F b_1$ holds. This implies that $(a_2 \vee_F \cdots \vee_F a_n) \wedge_F b_1 = \perp$. We have

$$\begin{aligned} b_1 &= a \wedge_F b_1, \\ &= (a_1 \vee_F (a_2 \vee_F \cdots \vee_F a_n)) \wedge_F b_1, \text{ as } a = a_1 \vee_F a_2 \vee_F \cdots \vee_F a_n \\ &= a_1 \wedge_F b_1 \text{ as } (a_2 \vee_F \cdots \vee_F a_n)\nabla_F b_1. \end{aligned}$$

So, we get $A(b_1, a_1) > 0$. (II)
Therefore, from (I) and (II) by fuzzy antisymmetry of A we get $a_1 = b_1$.
More generally, we have $a_i = b_i$ for every i.

(ii): To show that $Y_1, \cdots, Y_n$ are ideals of $\mathcal{L}$. We shall show that Y_1 is an ideal of X.
(a): Let $a \in Y_1$ and $A(b, a) > 0$. Suppose that $b = b_1 \vee_F \cdots \vee_F b_n$, $b_i \in Y_i$.
Let $i \neq 1$. Since $A(b_i, b) > 0$, $A(b, a) > 0$, by fuzzy transitivity of A we get $A(b_i, a) > 0$; i.e., $b_i = a \wedge_F b_i$.
If $a \in Y_i^{\nabla_F}$, then $a \wedge_F b_i = \perp$ for all b_i. This implies $b_i = \perp$. Hence $b = b_1 \in Y_1$.
(b): If $a, b \in Y_1$, $a \vee_F b$ is expressed in the form $a \vee_F b = c_1 \vee_F \cdots \vee_F c_n$, $c_i \in Y_i$.
If $i \neq 1$, then since $a, b \in Y_1 \subset Y_i^{\nabla_F}$, we have $c_i = (a \vee_F b) \wedge_F c_i = a \wedge_F c_i = \perp$.
Hence $a \vee_F b = c_1 \in Y_1$.
Therefore, Y_1 is an ideal. In general Y_i is an ideal for every i. □

Definition 3.5. *An element z of a fuzzy lattice $\mathcal{L} = (X, A)$ is called a fuzzy neutral element when $(z, a, b)_F T$ for all $a, b \in X$.*

Lemma 3.3. *The set of all fuzzy neutral elements of a fuzzy lattice $\mathcal{L} = (X, A)$ is a fuzzy distributive sublattice of $\mathcal{L}$.*

Proof. Let z_1 and z_2 be fuzzy neutral elements. We shall show that $z_1 \vee_F z_2$ is fuzzy neutral element. Let $a, b \in X$. We note that $A((z_1 \vee_F z_2 \vee_F a) \wedge_F b, (z_1 \vee_F z_2 \vee_F a) \wedge_F b) = 1 > 0$,
i.e., $A((z_1 \vee_F z_2 \vee_F a) \wedge_F b, (z_1 \vee_F (z_2 \vee_F a)) \wedge_F b) > 0$,
i.e., $A((z_1 \vee_F z_2 \vee_F a) \wedge_F b, (z_1 \wedge_F b) \vee_F ((z_2 \vee_F a) \wedge_F b)) > 0$,
i.e., $A((z_1 \vee_F z_2 \vee_F a) \wedge_F b, (z_1 \wedge_F b) \vee_F ((z_2 \wedge_F b) \vee_F (a \wedge_F b))) > 0$,
i.e., $A((z_1 \vee_F z_2 \vee_F a) \wedge_F b, ((z_1 \vee_F z_2) \wedge_F b) \vee_F (a \wedge_F b)) > 0$.
Therefore, $A((z_1 \vee_F z_2 \vee_F a) \wedge_F b, ((z_1 \vee_F z_2) \wedge_F b) \vee_F (a \wedge_F b)) > 0$.
As $A(((z_1 \vee_F z_2) \wedge_F b) \vee_F (a \wedge_F b), (z_1 \vee_F z_2 \vee_F a) \wedge_F b) > 0$ always holds.
By fuzzy antisymmetry of A we get $(z_1 \vee_F z_2 \vee_F a) \wedge_F b = ((z_1 \vee_F z_2) \wedge_F b) \vee_F (a \wedge_F b)$.
Hence $(z_1 \vee_F z_2, a, b)_F D$ holds.
We note that $A((a \vee_F b) \wedge_F (z_1 \vee_F z_2), (a \vee_F b) \wedge_F (z_1 \vee_F z_2)) = 1 > 0$,
i.e., $A((a \vee_F b) \wedge_F (z_1 \vee_F z_2), ((a \vee_F b) \wedge_F z_1) \vee_F ((a \vee_F b) \wedge_F z_2)) > 0$,
i.e., $A((a \vee_F b) \wedge_F (z_1 \vee_F z_2), (a \wedge_F z_1) \vee_F (b \wedge_F z_1) \vee_F (a \wedge_F z_2) \vee_F (b \wedge_F z_2)) > 0$,
i.e., $A((a \vee_F b) \wedge_F (z_1 \vee_F z_2), (a \wedge_F (z_1 \vee_F z_2)) \vee_F (b \wedge_F (z_1 \vee_F z_2))) > 0$.
As $A((a \wedge_F (z_1 \vee_F z_2)) \vee_F (b \wedge_F (z_1 \vee_F z_2)), (a \vee_F b) \wedge_F (z_1 \vee_F z_2)) > 0$ always holds.

By fuzzy antisymmetry of A we get
$(a \vee_F b) \wedge_F (z_1 \vee_F z_2) = (a \wedge_F (z_1 \vee_F z_2)) \vee_F (b \wedge_F (z_1 \vee_F z_2))$.
Hence $(a, b, z_1 \vee_F z_2)_F D$ holds.
We note that $A(((z_1 \vee_F z_2) \wedge_F a) \vee_F b, ((z_1 \vee_F z_2) \wedge_F a) \vee_F b) = 1 > 0$,
i.e., $A(((z_1 \vee_F z_2) \wedge_F a) \vee_F b, (z_1 \wedge_F a) \vee_F (z_2 \wedge_F a) \vee_F b) > 0$,
i.e., $A(((z_1 \vee_F z_2) \wedge_F a) \vee_F b, (z_1 \wedge_F a) \vee_F (z_1 \wedge_F b) \vee_F (z_2 \wedge_F a) \vee_F b) > 0$,
by putting $b = (z_1 \wedge_F b) \vee_F b$
i.e., $A(((z_1 \vee_F z_2) \wedge_F a) \vee_F b, (z_1 \wedge_F (a \vee_F b)) \vee_F ((z_2 \wedge_F a) \vee_F b)) > 0$,
i.e., $A(((z_1 \vee_F z_2) \wedge_F a) \vee_F b, (z_1 \wedge_F (a \vee_F b)) \vee_F ((z_2 \vee_F b) \wedge_F (a \vee_F b))) > 0$,
i.e., $A(((z_1 \vee_F z_2) \wedge_F a) \vee_F b, (z_1 \vee_F z_2 \vee_F b) \wedge_F (a \vee_F b)) > 0$.
As $A((z_1 \vee_F z_2 \vee_F b) \wedge_F (a \vee_F b), ((z_1 \vee_F z_2) \wedge_F a) \vee_F b) > 0$ always holds.
By fuzzy antisymmetry of A we get $((z_1 \vee_F z_2) \wedge_F a) \vee_F b = (z_1 \vee_F z_2 \vee_F b) \wedge_F (a \vee_F b)$.
Hence $(z_1 \vee_F z_2, a, b)_F D^*$ holds.
We note that $A((a \wedge_F b) \vee_F (z_1 \vee_F z_2), (a \wedge_F b) \vee_F (z_1 \vee_F z_2)) = 1 > 0$,
i.e., $A((a \wedge_F b) \vee_F (z_1 \vee_F z_2), ((a \wedge_F b) \vee_F z_1) \vee_F z_2)) > 0$,
i.e., $A((a \wedge_F b) \vee_F (z_1 \vee_F z_2), ((a \vee_F z_1) \wedge_F (b \vee_F z_1)) \vee_F z_2) > 0$,
i.e., $A((a \wedge_F b) \vee_F (z_1 \vee_F z_2), (a \vee_F z_1 \vee_F z_2) \wedge_F (b \vee_F z_1 \vee_F z_2)) > 0$.
As $A((a \vee_F z_1 \vee_F z_2) \wedge_F (b \vee_F z_1 \vee_F z_2), (a \wedge_F b) \vee_F (z_1 \vee_F z_2)) > 0$ always holds.
By fuzzy antisymmetry of A we get $(a \wedge_F b) \vee_F (z_1 \vee_F z_2) = (a \vee_F z_1 \vee_F z_2) \wedge_F (b \vee_F z_1 \vee_F z_2)$.
Hence $(a, b, z_1 \vee_F z_2)_F D^*$ holds. Thus $z_1 \vee_F z_2$ is fuzzy neutral element.
Similarly, we can show that $z_1 \wedge_F z_2$ is fuzzy neutral element.
Therefore, the set of fuzzy neutral elements forms a sublattice which is obviously fuzzy distributive. □

4. Fuzzy parallelism in a fuzzy lattice

The notion of fuzzy parallelism is well-known in lattices, see [4]. In this section, we introduce this notion in a fuzzy lattice and prove some properties.

Definition 4.1. *Let $\mathcal{L} = (X, A)$ be a fuzzy lattice with $\perp$. Let $a, b \in X - \{\perp\}$. We write $a <|_F b$ when $a \wedge_F b = \perp$ and $b \prec_F a \vee_F b$. If $a <|_F b$ and $b <|_F a$ hold, then we say that a and b are fuzzy parallel and we write $a \,||_F\, b$.*

Example 4.1. *In Example 2.1 $a \wedge_F b = \perp$ and $b \prec_F a \vee_F b = c$ also $a \prec_F a \vee_F b = c$. As $a <|_F b$ and $b <|_F a$ hold we say a and b are parallel.*

Remark 4.1. *Let $\mathcal{L} = (X, A)$ be a fuzzy lattice with the fuzzy covering property. If p is a fuzzy atom and if $p \wedge_F a = \perp$, then by the fuzzy covering property we have $a \prec_F a \vee_F p$. Hence $p <|_F a$. In particular, if p and q are different fuzzy atoms of X, then putting $a = q$, we get $p <|_F q$. Interchanging the roles of p and q we get $q <|_F p$. Hence $p \,||_F\, q$.*

Definition 4.2. *$\mathcal{L} = (X, A)$ is called a fuzzy atomistic lattice, in short FAC if $\mathcal{L} = (X, A)$ is an fuzzy atomistic lattice with the fuzzy covering property.*

Lemma 4.1. *In a fuzzy lattice $\mathcal{L} = (X, A)$ with $\perp$. If $a <|_F b$, then $a_1 \vee_F b = a \vee_F b$ for any $a_1 \in X$ satisfying $a_1 \neq \perp$ and $A(a_1, a) > 0$.*

Proof. Suppose that $a <|_F b$ holds. Then $b \prec_F a \vee_F b$ holds and $a \wedge_F b = \perp$.
As $A(a_1, a) > 0$, by (ii) of Proposition 2.1, we have $A(a_1 \wedge_F b, a \wedge_F b) > 0$,
i.e., $A(a_1 \wedge_F b, \perp) > 0$. Since $A(\perp, a_1 \wedge_F b) > 0$ always holds, by fuzzy antisymmetry of A we have $a_1 \wedge_F b = \perp$.
As $A(a_1, a) > 0$ so by (ii) of Proposition 2.1, we have $A(a_1 \vee_F b, a \vee_F b) > 0$ and $A(b, a_1 \vee_F b) > 0$.

As $b \prec_F a \vee_F b$ holds we have either $b = a_1 \vee_F b$ or $a_1 \vee_F b = a \vee_F b$.
If $b = a_1 \vee_F b$, then $A(a_1, b) > 0$, a contradiction to $a_1 \wedge_F b = \perp$.
Therefore, we get $a_1 \vee_F b = a \vee_F b$. □

Lemma 4.2. *Let $\mathcal{L} = (X, A)$ be a fuzzy lattice with $\perp$. Suppose that $a <|_F b$ holds.*
(i) If $a_1 \neq \perp$ is not a fuzzy atom such that $A(a_1, a) > 0$, then $(b, a_1)_F M_m$ does not hold;
(ii) If $A(a_1, a) > 0$, then $(a_1, b)_F M_j$ holds;
(iii) If $a_1 \neq \perp$ and $A(a_1, a) > 0$, then $(b, a_1)_F M_j$ does not hold.

Proof. Suppose that $a <|_F b$ holds. Then $b \prec_F a \vee_F b$ holds and $a \wedge_F b = \perp$.
(i): Suppose that $a_1 \neq \perp$ and a_1 is not a fuzzy atom and $A(a_1, a) > 0$. Then there exists $c \in X$ such that $c \neq \perp$ and $A(c, a_1) > 0$. Clearly, $A(c, a) > 0$
Also, given $A(a_1, a) > 0$, then by Lemma 4.1, we have $c \vee_F b = a \vee_F b$ and $a_1 \vee_F b = a \vee_F b$.
From $A(a_1, a) > 0$ and (ii) of Proposition 2.1, we have $A(a_1 \wedge_F b, a \wedge_F b) > 0$. Since $a \wedge_F b = \perp$, we get $A(a_1 \wedge_F b, \perp) > 0$.
As $A(\perp, a_1 \wedge_F b) > 0$ always holds, by fuzzy antisymmetry of A we get $a_1 \wedge_F b = \perp$.
Hence $(c \vee_F b) \wedge_F a_1 = (a_1 \vee_F b) \wedge_F a_1 = a_1$ and $c \vee_F (b \wedge_F a_1) = c \vee_F \perp = c \neq a_1$.
Thus, $(b, a_1)_F M_m$ cannot hold.
(ii): If $a_1 = \perp$, then by Remark 2.1, we have $(a_1, b)_F M_j$. Suppose that $a_1 \neq \perp$ and $A(a_1, a) > 0$ hold. As $a <|_F b$ holds we have $b \prec_F a \vee_F b$. By Lemma 4.1 we have $a \vee_F b = a_1 \vee_F b$. Hence by Lemma 2.1, we have $(a_1, b)_F M_j$.
(iii): If $a_1 \neq \perp$ and $A(a_1, a) > 0$, then by Lemma 4.1, we have $a_1 \vee_F b = a \vee_F b$.
We have $a \wedge_F (b \vee_F a_1) = a \wedge_F (b \vee_F a) = a$ and $(a \wedge_F b) \vee_F a_1 = a_1 \neq a$.
Hence $(b, a_1)_F M_j$ does not hold. □

Lemma 4.3. *Let $\mathcal{L} = (X, A)$ is a fuzzy lattice with $\perp$. Suppose that $\mathcal{L}$ is a FM_j-symmetric fuzzy lattice. If $a <|_F b$, then a is a fuzzy atom in X.*

Proof. If $a <|_F b$ holds, then $b \prec_F a \vee_F b$ holds and $a \wedge_F b = \perp$. Hence by Lemma 2.1, we have $(a, b)_F M_j$. By (ii) of Lemma 4.2, $(a_1, b)_F M_j$ holds for every $A(a_1, a) > 0$.
As $\mathcal{L} = (X, A)$ is a FM_j-symmetric fuzzy lattice, we have $(b, a_1)_F M_j$.
By (iii) of Lemma 4.2, a is a fuzzy atom in X. □

Lemma 4.4. *Let $a, b \in X - \{\perp\}$ be elements of a fuzzy atomic lattice $\mathcal{L} = (X, A)$ with the fuzzy covering property.*
(i) $a <|_F b$ if and only if $a \wedge_F b = \perp$ and there exists a fuzzy atom p such that $A(p, a) > 0$ and $p \vee_F b = a \vee_F b$;
(ii) $a \,||_F\, b$ if and only if $a \wedge_F b = \perp$ and there exist fuzzy atoms $p, q \in X$ such that $A(p, a) > 0$, $A(q, b) > 0$ and $a \vee_F q = b \vee_F p$.

Proof. Let $a, b \in X - \{\perp\}$.
(i): Let $a <|_F b$ hold. Since $\mathcal{L} = (X, A)$ is fuzzy atomic there exists a fuzzy atom p with $A(p, a) > 0$. By Lemma 4.1, we have $p \vee_F b = a \vee_F b$.

Conversely, suppose that $a \wedge_F b = \perp$ and p is a fuzzy atom satisfying $p \vee_F b = a \vee_F b$ and $A(p, a) > 0$. By (ii) of Proposition 2.1, we have $A(p \wedge_F b, a \wedge_F b) > 0$.
Since $a \wedge_F b = \perp$, we get $A(p \wedge_F b, \perp) > 0$. But $A(\perp, p \wedge_F b) > 0$ always holds. Hence by fuzzy antisymmetry of A we get $p \wedge_F b = \perp$. As p is a fuzzy atom and $p \wedge_F b = \perp$ by the fuzzy covering property we have $b \prec_F p \vee_F b$, that is, $b \prec_F a \vee_F b$. Hence $a <|_F b$ holds.
(ii): $a \,||_F\, b$ means $a <|_F b$ and $b <|_F a$ hold. Then by (i) there exist fuzzy atoms $p, q \in X$ such that $A(p, a) > 0$, $A(q, b) > 0$ and $p \vee_F b = q \vee_F a = a \vee_F b$.

Conversely, suppose that $a \wedge_F b = \perp$, $A(p, a) > 0$, $A(q, b) > 0$ and $a \vee_F q = b \vee_F p$ hold
We have $A(b, a \vee_F q) > 0$ and $A(p, a \vee_F q) > 0$ by (ii) of Proposition 2.1, we have

$A(a \vee_F b, a \vee_F q) > 0$. As $A(p, a) > 0$ by (ii) of Proposition 2.1, we have
$A(p \vee_F b, a \vee_F b) > 0$. (I)
As $a \vee_F q = b \vee_F p$ so, we have $A(a, b \vee_F p) > 0$ and $A(q, b \vee_F p) > 0$.
by (ii) of Proposition 2.1, we get $A(a \vee_F b, b \vee_F p) > 0$. (II)
From (I) and (II) by fuzzy antisymmetry of A we get $a \vee_F b = p \vee_F b$.
Similarly, $a \vee_F q = a \vee_F b$.
Therefore, we get $a <|_F b$ and $b <|_F a$ by (i). Thus, $a \,||_F\, b$ holds. □

Lemma 4.5. *Let $\mathcal{L} = (X, A)$ be a fuzzy weakly modular lattice with the fuzzy covering property. If $a <|_F b$ and if q is a fuzzy atom with $A(q, b) > 0$, then $a \,||_F\, (a \vee_F q) \wedge_F b$.*

Proof. Suppose that $a <|_F b$ holds. Then $b \prec_F a \vee_F b$ holds and $a \wedge_F b = \bot$.
Put $b_1 = (a \vee_F q) \wedge_F b$. (I)
As $A(b_1, b) > 0$ then by (ii) of Proposition 2.1, we have $A(a \wedge_F b_1, a \wedge_F b) > 0$.
Since $a \wedge_F b = \bot$ we have $A(a \wedge_F b_1, \bot) > 0$. As $A(\bot, a \wedge_F b_1) > 0$ always holds, by fuzzy antisymmetry of A we have $a \wedge_F b_1 = \bot$. Also, by (I) we have $A(b_1, a \vee_F q) > 0$ by (ii) of Proposition 2.1, we have $A(a \vee_F b_1, a \vee_F q) > 0$. (II)
As $A(q, a \vee_F q) > 0$ always holds, by (ii) of Proposition 2.1, we get
$A(q \wedge_F b, (a \vee_F q) \wedge_F b) > 0$. As $A(q, b) > 0$ we have $A(q, (a \vee_F q) \wedge_F b) > 0$ which gives $A(q, b_1) > 0$ by (I).
By (ii) of Proposition 2.1, we have $A(a \vee_F q, a \vee_F b_1) > 0$. (III)
From (II) and (III) by fuzzy antisymmetry of A we have $a \vee_F q = a \vee_F b_1$.
To prove that $a \,||_F\, b_1$ holds, it is sufficient to show that $b_1 \prec_F a \vee_F q$ and $a \prec_F a \vee_F q$.
Since, $A(q, b) > 0$ by (ii) of Proposition 2.1, we get $A(a \wedge_F q, a \wedge_F b) > 0$. This implies $A(a \wedge_F q, \bot) > 0$ as $a \wedge_F b = \bot$. Since $A(\bot, a \wedge_F q) > 0$ always holds by fuzzy antisymmetry of A we have $a \wedge_F q = \bot$. Hence by fuzzy covering property we get $a \prec_F a \vee_F q$.
Let $c \in X$ be such that $A(b_1, c) > 0$, $A(c, a \vee_F q) > 0$.
Case (1): Let $A(c, b) > 0$. We have $A(c, a \vee_F q) > 0$. Then by (ii) of Proposition 2.1, we have $A(c \wedge_F b, b \wedge_F (a \vee_F q)) > 0$. This implies that $A(c, b_1) > 0$ by (I). By fuzzy antisymmetry of A we have $b_1 = c$.
Case (2): Let $A(c, b) = 0$, $0 < A(b, b \vee_F c) < 1$ and $A(c, a \vee_F q) > 0$ by (ii) of Proposition 2.1, we have $A(b \vee_F c, b \vee_F (a \vee_F q)) > 0$. Thus $A(b \vee_F c, a \vee_F b) > 0$ as $A(q, b) > 0$. So, we have $0 < A(b, b \vee_F c) < 1$, $A(b \vee_F c, a \vee_F b) > 0$.
Then $b = b \vee_F c$ and $b \vee_F c = a \vee_F b$. (IV)
As $b \prec_F a \vee_F b$ holds and $A(q, b) > 0$ by (ii) of Proposition 2.1 we have $A(a \vee_F q, a \vee_F b) > 0$.
By (IV) we have $A(a \vee_F q, b \vee_F c) > 0$.
Since $\mathcal{L} = (X, A)$ is fuzzy weakly modular fuzzy lattice, using $0 < A(\bot, q) < 1$ and $A(q, (a \vee_F q) \wedge_F b) > 0$ we have $(b, a \vee_F q)_F M_m$.
Hence

$$\begin{aligned} c &= c \vee_F b_1, \\ &= c \vee_F \{b \wedge_F (a \vee_F q)\}, \quad \text{as } b_1 = (a \vee_F q) \wedge_F b \\ &= (c \vee_F b) \wedge_F (a \vee_F q), \quad \text{because } (b, a \vee_F q)_F M_m \end{aligned}$$

Hence $c = a \vee_F q$. Therefore, $b_1 \prec_F a \vee_F q$ holds. Hence $a \,||_F\, (a \vee_F q) \wedge_F b$ holds. □

5. Fuzzy atom-free parallelism in a fuzzy lattice

In this section, we define a fuzzy modular element in a fuzzy lattice $\mathcal{L} = (X, A)$ and introduce fuzzy atom-free parallelism in a fuzzy lattice.

Definition 5.1. *An element a in a fuzzy lattice $\mathcal{L} = (X, A)$ is called a fuzzy modular element if $(x, a)_F M_m$ for every $x \in X$.*

The elements $\perp$, $\top$ and every fuzzy atom, if they exist, are fuzzy modular elements.

Definition 5.2. *Let $\mathcal{L} = (X, A)$ be a fuzzy lattice with $\perp$ and let $a, b \in X - \{\perp\}$. If $a \wedge_F b = \perp$ and there exists a fuzzy modular element $m \in X$ such that $m \vee_F b = a \vee_F b$ and $A(m, a) > 0$, then we write $a <|_{(m)} b$.*

We have $0 < A(\perp, m) < 1$ and $0 < A(m, \top) < 1$.
(i) If $m = \perp$, then $b = a \vee_F b$. Hence $a \wedge_F b = a = \perp$;
(ii) If $m = \top$, then $\top \vee_F b = a \vee_F b$, which gives $\top = a \vee_F b$.
Hence we have $A(a, \top) > 0$; (I)
Since $A(m, a) > 0$ we get $A(\top, a) > 0$. (II)
From (I) and (II) we have $a = \top$. This gives $a \wedge_F b = \top \wedge_F b = b = \perp$.

If $a <|_{(m)} b$ and $b <|_{(n)} a$ hold, then we say that *a and b are parallel with axes m and n* and we write $a \,||_{(m,n)}\, b$. Since fuzzy modular elements m and n are not necessarily fuzzy atoms, we may say that this parallelism is fuzzy atom-free.

Example 5.1. *Consider the fuzzy lattice in Example 2.1. Here $d \wedge_F e = \perp$. As $a, b \in X$ are modular elements such that $a \vee_F e = d \vee_F e$, $b \vee_F d = e \vee_F d$, $A(a, d) > 0$ and $A(b, e) > 0$ hold. Hence $d <|_{(a)} e$ and $e <|_{(b)} d$ hold.*
Thus, $d||_{(a,b)} e$ holds.

Lemma 5.1. *Let $a <|_{(m)} b$ hold in a fuzzy lattice with $\perp$.*
(i) If $0 < A(m, a_1) < 1$ and $A(a_1, a) > 0$, then $(b, a_1)_F M_m$ does not hold;
(ii) If $A(m, a_1) > 0$, $0 < A(a_1, a) < 1$, then $(b, a_1)_F M_j$ does not hold;
(iii) If $\mathcal{L} = (X, A)$ is a FM-symmetric fuzzy lattice and if $A(m, a_1) > 0$, $A(a_1, a) > 0$, then $(a_1, b)_F M_j$ holds.

Proof. Let $a <|_{(m)} b$ hold in a fuzzy lattice $\mathcal{L} = (X, A)$ with $\perp$.
Since $a <|_{(m)} b$ we have $a \wedge_F b = \perp$ and $m \vee_F b = a \vee_F b$. (I)
(i): Suppose that $0 < A(m, a_1) < 1$ holds. Hence by (ii) of Proposition 2.1, we have $A(m \vee_F b, a_1 \vee_F b) > 0$.
By (I) we get $A(a \vee_F b, a_1 \vee_F b) > 0$. (II)
As $A(a_1, a) > 0$ so by (ii) of Proposition 2.1, we have $A(a_1 \vee_F b, a \vee_F b) > 0$. (III)
From (II) and (III) by fuzzy antisymmetry of A we have $a_1 \vee_F b = a \vee_F b$. (IV)
By (I) and (IV) we have $a_1 \vee_F b = a \vee_F b = m \vee_F b$.
As $A(a_1, a) > 0$ by (ii) of Proposition 2.1, we have $A(a_1 \wedge_F b, a \wedge_F b) > 0$.
So, we have $A(a_1 \wedge_F b, \perp) > 0$ as $a \wedge_F b = \perp$ and $A(\perp, a_1 \wedge_F b) > 0$ always holds.
Therefore, by fuzzy antisymmetry of A we get $a_1 \wedge_F b = \perp$.

$$(m \vee_F b) \wedge_F a_1 = (a_1 \vee_F b) \wedge_F a_1 = a_1 \text{ and } m \vee_F (b \wedge_F a_1) = m \vee_F \perp = m \neq a_1.$$

Hence $(b, a_1)_F M_m$ does not hold.
(ii): Suppose that $A(m, a_1) > 0$ and $0 < A(a_1, a) < 1$ hold.
As $0 < A(a_1, a) < 1$ so by (ii) of Proposition 2.1, we have $A(a_1 \vee_F b, a \vee_F b) > 0$. (I)
As $A(m, a_1) > 0$ by (ii) of Proposition 2.1, we get $A(m \vee_F b, a_1 \vee_F b) > 0$.
But $m \vee_F b = a \vee_F b$. So, we have $A(a \vee_F b, a_1 \vee_F b) > 0$. (II)
From (I) and (II) by fuzzy antisymmetry of A we have $a_1 \vee_F b = a \vee_F b$.
We have

$$(a \wedge_F b) \vee_F a_1 = \perp \vee_F a_1 = a_1 \text{ and } a \wedge_F (b \vee_F a_1) = a \wedge_F (b \vee_F a) = a \neq a_1.$$

Hence $(b, a_1)_F M_j$ does not hold.
(iii): Assume that $\mathcal{L} = (X, A)$ is a FM-symmetric fuzzy lattice and let $A(m, a_1) > 0$,

$A(a_1, a) > 0$ by (ii) of Proposition 2.1, we have $A(m \vee_F b, a_1 \vee_F b) > 0$, (I)
and $A(a_1 \vee_F b, a \vee_F b) > 0$.
As $a <|_{(m)} b$ we have $a \wedge_F b = \perp$. There exists a fuzzy modular element $m \in X$ such that $A(m, a) > 0$ and $m \vee_F b = a \vee_F b$. So, (I) reduces to $A(a \vee_F b, a_1 \vee_F b) > 0$.
Therefore, we have $A(a \vee_F b, a_1 \vee_F b) > 0$ and $A(a_1 \vee_F b, a \vee_F b) > 0$.
By fuzzy antisymmetry of A we get $a \vee_F b = a_1 \vee_F b$.
So, we have $a \vee_F b = a_1 \vee_F b = m \vee_F b$.
Let $A(b, c) > 0$ since m is a fuzzy modular element we have $(c, m)_F M_m$.
But $\mathcal{L} = (X, A)$ is a FM-symmetric fuzzy lattice so, we have $(m, c)_F M_m$.
Consider $a_1 \vee_F b = m \vee_F b$.
We have

$$\begin{aligned} c \wedge_F (a_1 \vee_F b) &= c \wedge_F (m \vee_F b), \\ &= b \vee_F (m \wedge_F c), \quad \text{because } (m, c)_F M_m. \end{aligned}$$

Therefore, $c \wedge_F (a_1 \vee_F b) = b \vee_F (m \wedge_F c)$. (II)
We have $A(m, a_1) > 0$.
By applying (ii) of Proposition 2.1, repeatedly we have
$A(m \wedge_F c, a_1 \wedge_F c) > 0$ and $A(b \vee_F (m \wedge_F c), b \vee_F (a_1 \wedge_F c)) > 0$.
From (II) we have $A(c \wedge_F (a_1 \vee_F b), (c \wedge_F a_1) \vee_F b) > 0$.
Therefore, $(a_1, b)_F M_j$ holds. □

Lemma 5.2. *Let $\mathcal{L} = (X, A)$ be a fuzzy lattice with $\perp$. Let $a, b \in X - \{\perp\}$. Then $a \parallel_{(m,n)} b$ if and only if $a \wedge_F b = \perp$ and there exist fuzzy modular elements $m, n \in X$ such that $A(m, a) > 0$, $A(n, b) > 0$ and $a \vee_F n = b \vee_F m$.*

Proof. If $a \parallel_{(m,n)} b$, then we have $a <|_{(m)} b$ and $b <|_{(n)} a$. Hence we have

$$a \wedge_F b = \perp,\ A(m, a) > 0,\ A(n, b) > 0 \ \text{ and} b \vee_F m = a \vee_F b = n \vee_F a.$$

Conversely, suppose that $a \wedge_F b = \perp$ and there exist fuzzy modular elements $m, n \in X$ such that $A(m, a) > 0$, $A(n, b) > 0$ and $a \vee_F n = b \vee_F m$ so, we have $A(a, b \vee_F m) > 0$.
By (ii) of Proposition 2.1, we have $A(a \vee_F b, b \vee_F m) > 0$. (I)
Also, we have $A(m, a) > 0$ so, by (ii) of Proposition 2.1,we have
$A(m \vee_F b, a \vee_F b) > 0$. (II)
From (I) and (II) by fuzzy antisymmetry of A we have $m \vee_F b = a \vee_F b$.
Hence we get $a <|_{(m)} b$. Similarly, we have $b <|_{(n)} a$.
Thus $a \parallel_{(m,n)} b$ holds. □

Lemma 5.3. *In a fuzzy lattice $\mathcal{L} = (X, A)$ with $\perp$, if $a <|_{(m)} b$, $A(m, a_1) > 0$ and $0 < A(a_1, a) < 1$, then $a_1 <|_{(m)} b$.*

Proof. Suppose that $a <|_{(m)} b$ holds. Then we have $a \wedge_F b = \perp$ and there exists a fuzzy modular element $m \in X$ such that $m \vee_F b = a \vee_F b$ and $A(m, a) > 0$.
Given $0 < A(a_1, a) < 1$ so by (ii) of Proposition 2.1, we have $A(a_1 \wedge_F b, a \wedge_F b) > 0$, that is, $A(a_1 \wedge_F b, \perp) > 0$.
Since $A(\perp, a_1 \wedge_F b) > 0$ always holds, by fuzzy antisymmetry of A we get $a_1 \wedge_F b = \perp$.
Also, given that $A(m, a) > 0$ and $m \vee_F b = a \vee_F b$. (I)
To show that $m \vee_F b = a_1 \vee_F b$ holds.
As $A(m, a_1) > 0$ and $A(a_1, a) > 0$ so, by (ii) of Proposition 2.1, we have
$A(m \vee_F b, a_1 \vee_F b) > 0$ and $A(a_1 \vee_F b, a \vee_F b) > 0$.
By (I) we have $A(a \vee_F b, a_1 \vee_F b) > 0$ and $A(a_1 \vee_F b, a \vee_F b) > 0$.
So, by fuzzy antisymmetry of A we have $a_1 \vee_F b = a \vee_F b$. (II)

Therefore, from (I) and (II) we get $a_1 \vee_F b = m \vee_F b$.
Hence we have $a_1 \wedge_F b = \perp$, $a_1 \vee_F b = m \vee_F b$ and $A(m, a_1) > 0$.
Thus, $a_1 <|_{(m)} b$ holds. □

Lemma 5.4. *Let $\mathcal{L} = (X, A)$ be a fuzzy lattice with $\perp$ and $a, b, b_1 \in X$. Suppose that $a <|_{(m)} b$ holds and $b \neq \perp$ is such that $A(b, b_1) > 0$.*
(i) If $a \wedge_F b_1 = \perp$, then $a <|_{(m)} b_1$;
(ii) If $m \neq \perp$ and $A(m, b_1) > 0$, then $0 < A(a, b_1) < 1$.

Proof. As $a <|_{(m)} b$ holds we have $a \wedge_F b = \perp$ and m be a fuzzy modular element such that $m \vee_F b = a \vee_F b$ and $A(m, a) > 0$.
(i): Suppose that $a \wedge_F b_1 = \perp$. To show that $a <|_{(m)} b_1$ holds.
We have

$$\begin{aligned} m \vee_F b_1 &= m \vee_F b \vee_F b_1, \quad \text{because } A(b, b_1) > 0 \\ &= a \vee_F b \vee_F b_1, \\ &= a \vee_F b_1, \quad \text{because } A(b, b_1) > 0. \end{aligned}$$

Therefore, we get $m \vee_F b_1 = a \vee_F b_1$. (I)
Hence if $a \wedge_F b_1 = \perp$ holds, then $a <|_{(m)} b_1$.
(ii): Suppose that $m \neq \perp$ and $A(m, b_1) > 0$. To show that $A(a, b_1) > 0$.
From (I) we have $a \vee_F b_1 = m \vee_F b_1 = b_1$. Therefore, $a \vee_F b_1 = b_1$, that is, $A(a, b_1) > 0$. □

6. Conclusion

In this paper, we have introduced the notion of 'del' relation in a fuzzy lattice and have presented a novel approach to parallelism and atom free parallelism in fuzzy lattices. Also, we have studied some properties. We have proposed a new notion and notation of distributive and neutral elements in fuzzy lattices.

7. Acknowledgement

The authors are thankful to the referee for fruitful suggestions, which enhanced the quality of the paper.

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