TWMS J. App. and Eng. Math. V.12, N.3, 2022, pp. 931-939

THE MEAN REVERTING ORNSTEIN-UHLENBECK PROCESSES WITH NONLINEAR AUTOREGRESSIVE DRIFT TERM INNOVATIONS

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ABSTRACT. The main purpose of this paper is to present a new approach for energy markets governed by a two-factor Ornstein-Uhlenbeck process with a stochastic nonlinear autoregressive drift term innovation and an unknown diffusion coefficient. This model has interesting characteristics: since the drift is stochastic, it allows for price to fluctuate around a level that is not fixed. A semiparametric method is proposed to estimate the nonlinear regression function based on the conditional least square method for parametric estimation and the nonparametric kernel approach for the AR adjustment estimation. For estimating the diffusion coefficient of the Ornstein-Uhlenbeck process from discretely observed data a semiparametric approach based on the least-squares estimator is carried out. Finally, numerical simulations are performed using Matlab programming to show efficiency and the accuracy of the present work.

Keywords: Ornstein-Uhlenbeck processes; Nonlinear autoregressive model; Semiparametric estimation; Smooth kernel approach; Conditional nonlinear least-squares method.

AMS Subject Classification: 60H10; 62G05; 62G08; 62M10.

1. Introduction

The diffusion process has played an important role in modeling and dynamic financial variables, such as pricing derivative securities that formulated in continuous time as solutions to stochastic differential equations (SDE's). SDEs have been used to model option prices, interest rates, and exchange rates. Some applications of SDEs are described in Nabati et al. [16]. For modeling first, we have to specify the drift and diffusion coefficients of an SDE, $\mu(.)$ and $\sigma(.)$ as given below:

$$dX_t = \mu(X_t, \theta)dt + \sigma(X_t, \theta)dB_t, \tag{1}$$

where B_t is a standard Brownian motion defined on the probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}$ is a sigma-algebra. Mean reverting price processes of the diffusion type relevant to energy prices are considered in this paper. In general one to three factor models exists that motivating the idea behind the notion of mean reversion. A one-factor mean reverting

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§ Manuscript received: June 13, 2020; accepted: October 15, 2020.

Ornstein-Uhlenbech (OU) process is proposed for the spot price of energy commodity by Pilipovic as follows [17]:

$$dP_t = \beta(\nu - P_t)dt + \sigma dB_t, \tag{2}$$

where β, ν and σ are constants. It is easy to show that as $t \to \infty$, $\bar{P}_t \to \nu$. In other words, the mean $\bar{P}_t$ revert in the long run to ν. Generalization of it to the case where ν is made time-dependent to capture seasonality is discussed in Tifenbach [20].

Two-factor models are achieved by either allowing the long-run mean ν or the volatility σ governed by an SDE. The first model assumes the stochastics long-run mean as:

$$\begin{cases} dP_t = \beta(\nu_t - P_t)dt + \sigma dB_t \\ d\nu_t = \mu\nu_t dt + \xi\nu_t dW_t \end{cases} \tag{3}$$

where μ and ξ are constants and dB_t and dW_t are uncorrelated Brownian motions. Some numerical implementation of this model can be found in Tifenbach, Lari-Lavasani et al. and Hernandez et al., [20, 15, 12]. The second generalization is the two-factor model where volatility is allowed to be stochastic that is discussed in Lari-Lavasani et al. as follows [15]:

$$\begin{cases} dP_t = \beta(\nu_- P_t)dt + \sqrt{\sigma_t} dB_t \\ d\sigma_t = \mu(\sigma_0 - \sigma_t)dt + \xi\sigma_t dW_t \end{cases} \tag{4}$$

Our innovation in this paper is to assume a two factor OU process with a nonlinear autoregressive drift term. Identification, estimation and hence the investigation of the coefficients of this process have proved to be quite difficult. The nonlinear autoregressive model is studied by many researchers in a few last years. Zhuoxi et al. introduced a semiparametric estimation for regression function in autoregressive models [21]. Farnoosh and Mortazavi extend the Zhuoxi method for the autoregressive models with a dependent error [7]. Hajrajabi and Mortazavi proposed the semiparametric estimation for nonlinear autoregressive models with skew normal error terms [10]. Hajrajabi and Maleki investigate the autoregressive model with finite mixtures od scale mixtures of skew normal error term. They use the expectation maximization algorithm to perform the maximum likelihood inference of unknown parameters of the model [9]. Finally, Hajrajabi et al. proposed the nonlinear autoregressive models with stochastic volatility. They used the optimal filtering technique based on a sequential Monte Carlo perspective for estimation of the hidden log-volatility in the model [11]. The nonparametric approach for these models is considered in [3, 8]. The parameter estimation of SDE's has been considered in the literature for many years. For instance, the nonparametric drift and diffusion function estimators have been proposed by Fan and Yao, Jacod and Fan and Zhang, [5, 13, 6]. These studies basically an unknown function by a polynomial. This localization is done by using some kernel function so that the method is also called the kernel regression. The performance of this method depends on the choice of a kernel function and its bandwidth, the details can be seen in Fan and Gijbles, [4]. Bandi and Philips investigate a functional estimation procedure for homogeneous SDE based on a discrete sample of observation. They show how to identify the drift and diffusion function in situations when one or the other function is considered a nuisance parameter [1]. Reno introduces nonparametric estimators of the drift and diffusion coefficient of stochastic volatility models that exploit techniques for estimating integrated volatility with high-frequency data [18]. Bonsoo and linton propose a class of locally stationary diffusion processes. Their model is semiparametric because they allow the functions to be unknown and the innovation process is parametrically specified indeed completely known [2]. Shoji provides a semiparametric model to estimate the diffusion coefficient of an SDE from discretely observed data without assuming any functional form of the diffusion coefficient [19]. Kanaya and Kristensen present a two-step estimation model of SDE. Their strategy applies to both parametric and nonparametric stochastic volatility models and can handle both jumps and market microstructure noise [14]. The outline of this paper is as follows. In section 2, the mathematical modeling for the OU process with autoregressive drift term innovation is presented. A least square estimation is proposed to estimate the parameter vector of the regression function and a semiparametric regression estimator is presented with the local L2 fitting criteria. The parameter estimation for both the drift and diffusion terms of the OU process are discussed in section 3. Section 4 provides computational examples and numerical simulations illustrating the use of the method. In the last section, we give the concluding remarks.

2. OU process with nonlinear autoregressive drift term

The modeling of commodity prices has been attractive over the past few years, owing to many factors including the dominant volatility and the deregulation of markets for some commodities (e.g. electricity). Historically, mean reversion has been viewed as a key component of commodities prices [12]. Now consider the system of stochastic differential equations in a two factor model for energy and commodity spot price processes as:

$$\begin{cases} dP_t = \beta(\nu_t - P_t)dt + \sigma(P_t)dB_t \\ \nu_t = f(\nu_{t-1}) + \xi_t \end{cases} \tag{5}$$

where $\beta > 0$, $\sigma(P_t)$ is an unknown function of P_t that should be estimated. $f(.)$ is the unknown nonlinear autoregressive function and ξ_t is a sequence of i.i.d random variables with mean zero and variance σ_ξ^2. The model has interesting characteristics: since the drift is stochastic, it allows for price to fluctuate around a level that is not fixed. Traditionally, the parametric or nonparametric approach can be adapted to estimate the autoregressive function $f(.)$. A semiparametric approach based on the work of Zhouxi et al. and Farnoosh and Mortazavi is used in this paper [21, 7]. Suppose that $f(.)$ has a parametric framework, a parametric model ,

$$f(x) \in \{h(x,\theta), \theta \in \Theta\} \tag{6}$$

is presented as a prior selection where $\Theta \in \mathbf{R}^p$ is the parameter space. In this case the estimator of the regression functions is replaced by the estimator of the unknown parameter vector Θ, then the regression function $f(.)$ is estimated by $\hat{f}(x) = h(x, \hat{\theta})$ where $\hat{\theta}$ is an estimator of θ. The estimator θ is obtanied via conditional nonlinear least square errors method as follows:

$$\begin{aligned} \hat{\theta} &= argmin_{\theta\in\Theta} \textstyle\sum_{t=1}^{n} \left(\nu_t - E(\nu_t|\nu_{t-1})\right)^2 \\ &= argmin_{\theta\in\Theta} \textstyle\sum_{t=1}^{n} \left(\nu_t - h(\nu_t - 1, \theta)\right)^2 \end{aligned} \tag{7}$$

Zhouxi et al. earn the strong consistency of $\hat{\theta}$ under a variaty of conditions [21]. $h(x, \theta)$ is a crude guess of $f(x)$, then for adjusting the initial approximation, the semiparametric form $h(x, \theta)\eta(x)$ is used where $\eta(x)$ is the adjustment factor. For determining $\eta(x)$ the local L2-fitting criterion is defined as:

$$q(x, \eta) = \frac{1}{b_n} \sum_{t=1}^{n} K(\frac{v_{t-1} - x}{b_n})\{f(\nu_{t-1}) - h(\nu_{t-1}, \hat{\theta}).\eta(x)\}^2, \tag{8}$$

where K and b_n are the kernel and bandwidth respectivly. The estimator $\hat{\eta}(x)$ is obtained by minimizing the criterion in equation (8) with respect to $\eta(x)$. The estimator is calculated as,

$$\hat{\eta}(x) = \frac{\sum_{t=1}^{n} f(\nu_{t-1}) K(\frac{\nu_{t-1}-x}{b_n}) h(\nu_{t-1}, \hat{\theta})}{\sum_{t=1}^{n} K(\frac{\nu_{t-1}-x}{b_n}) h^2(\nu_{t-1}, \hat{\theta})} \tag{9}$$

since the equation(9) containe the unknown function $f(x)$, therefore using $\xi_t = \nu_t - f(\nu_{t-1})$ and with considering the fact that $E(\xi_t) = 0$, we can write $\nu_t \simeq f(\nu_{t-1})$ then,

$$\tilde{\eta}(x) = \frac{\sum_{t=1}^{n} \nu_t K(\frac{\nu_{t-1}-x}{b_n}) h(\nu_{t-1}, \hat{\theta})}{\sum_{t=1}^{n} K(\frac{\nu_{t-1}-x}{b_n}) h^2(\nu_{t-1}, \hat{\theta})} \tag{10}$$

Finally the autoregressive estimator is obtained by

$$\tilde{f}(x) = h(x, \hat{\theta}).\tilde{\eta}(x) \tag{11}$$

The consistence of this estimator is studied in [21].

3. Estimation

consider the one dimentional diffusion process P_t that satisfies in the stochastic differential equation,

$$dP_t = \beta(\nu_t - P_t)dt + \sigma(P_t)dB_t, \tag{12}$$

Assume that β is an unknown parameter and $\sigma(P_t)$ is completely unknown function. Suppose the process P_t is observed at discrete times $0 = t_0,\ t_1, ..., t_n = T$ over the time span $[0, T]$ with $\Delta t = \frac{T}{n}$. Using the conditional expectation we have:

$$\begin{aligned} E(P_{t+1}|P_t) &= P_t + \beta\big(E(\nu_t) - P_t\big)\Delta t \\ &= P_t + \beta\big(f(\nu_{t-1}) - P_t\big)\Delta t \end{aligned} \tag{13}$$

then the parameter β can be estimated by minimizing the local constant objective function,

$$Q = \sum_{t=0}^{n-1} (P_{t+1} - E(P_{t+1}|P_t))^2.K(\frac{x - \frac{t}{T}}{h}), \tag{14}$$

with respect to β. This yeild the estimator,

$$\hat{\beta} = \frac{\sum_{t=0}^{n-1} (P_{t+1} - P_t).(f(\nu_{t-1}) - P_t).K(\frac{x-\frac{t}{T}}{h})}{\big(\sum_{t=0}^{n-1} (\nu_t - P_t)^2.K(\frac{x-\frac{t}{T}}{h})\big)\Delta t} \tag{15}$$

where $K(.)$ is a real valued kernel function concentrated around the origin and h is the bandwidth parameter.

3.1. Estimation of volatility process.

From equation (12) we can write,

$$dP_t.dP_t = \sigma^2(P_t).dt, \tag{16}$$

hence $\sigma^2(p_{t_k}) = \dfrac{(P_{t_{k+1}} - P_{t_k})^2}{\Delta t}$. In order to estimate $\sigma(P_t)$ as a function of P_t, suppose that $\sigma(.)$ has a parametric framework, namely a parametric model as $\sigma(P_t) \in \{g(P_t, \gamma), \gamma \in \Gamma\}$, then $\sigma(.)$ can be estimated as $\hat{\sigma}(P_t) = g(P_t, \hat{\gamma})$ where $\hat{\gamma}$ is an estimator of γ. The coefficient γ is obtain by minimizing the objective function

$$\Psi = \sum_{k=1}^{n-1} \big(Z^*_{t_k} - g(P_{t_k}, \gamma)\big)^2.K_h(P_{t_k} - P_0) \tag{17}$$

Here $Z^*_{t_k} = \frac{(P_{t_{k+1}} - P_{t_k})^2}{\Delta t}$ and $K_h(.) = \frac{K(./h)}{h}$. K is a kernel function and h is a bandwidth. We use the Epanechnikov kernel defined by $K(u) = \frac{3}{4}(1 - u^2)I(|u| \leq 1)$ where $I(.)$ is the indicator function.

Sample size	h_n	$a = 0.1$	$a = 0.3$
	0.04	0.003439	0.025912
$n = 100$	0.08	0.003343	0.026119
	0.12	0.003549	0.026191
	0.04	0.003752	0.02624
$n = 1000$	0.08	0.003516	0.02615
	0.12	0.003688	0.02595

Table 1. MSE for estimating the model $f_1(x) = 5\exp(-x^2) + a\cos x$

Sample size	h_n	$a = 0.1$	$a = 0.3$
	0.04	0.002938	0.003392
$n = 100$	0.08	0.002773	0.003611
	0.12	0.003357	0.003234
	0.04	0.002973	0.003458
$n = 1000$	0.08	0.003011	0.003249
	0.12	0.002897	0.003371

Table 2. MSE for estimating the model $f_2(x) = \exp(-3x) + a\sin x$

4. Simulation study

In this section, we present a simulation study corresponding to the two factors mean reverting OU process (5) that is divided into two parts to illustrate the performance of our proposed model. At first, by simulating from the nonlinear autoregressive drift term the asymptotic properties of the estimator is studied in term of mean square error (MSE) of the estimated parameters. Second, the diffusion coefficient is estimated using the proposed model (17). The Bias and MSE are presented for this estimator.

4.1. **Simulation study 1.** We choice two different autoregressive drift term functions as follows,

$$f_1(x) = 5e^{-x^2} + a\cos(x), \text{ by assuming } h_1(x,\theta) = \theta_{11}e^{-x^2}$$
$$f_2(x) = 2e^{-3x} + a\sin(x), \text{ by assuming } h_2(x,\theta) = \theta_{21}e^{\theta_{22}x}$$

Tables 1 and 2 shows the mean square error (MSE) for semiparametric estimation with 1000 itaration. The sample size of simulation, bandwidth and values of a are chosen. Figures 1 and 2 shows the curves of $f(x)$ and $\tilde{f}(x)$ for selected bandwidth respectively. The solid line is the the function $f(x)$ and the broken line is its estimator. The simulation results show that the semiparametric estimator performs well.

4.2. **Simulation study 2.** This simulation study is performed according the model (12) with 1000 sample path. we consider $\beta = 3$ and $\sigma_1(P_t) = \sqrt{P_t}$, $\sigma_2(P_t) = P_t^{\frac{3}{4}}$ be two different functions for diffusion process. The Bias and MSE for two autoregressive functions $(f_1(x), f_2(x))$ are presented in tables 3 and 4 respectively. As we see, the results show the justification of the proposed estimator based on the MSE. The simulated functions for

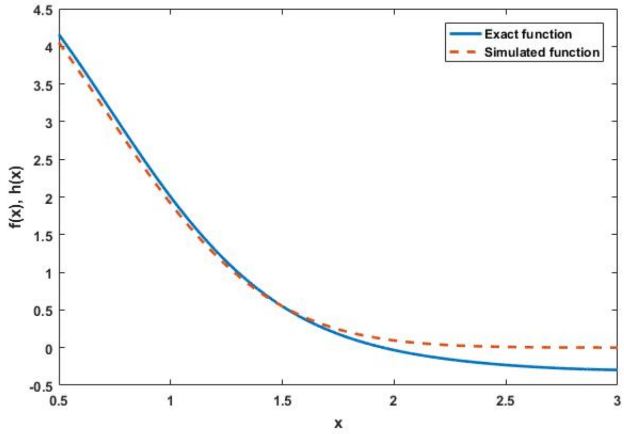

FIGURE 1. Exact and simulated functions for nonlinear autoregressive function $f_1(x)$.

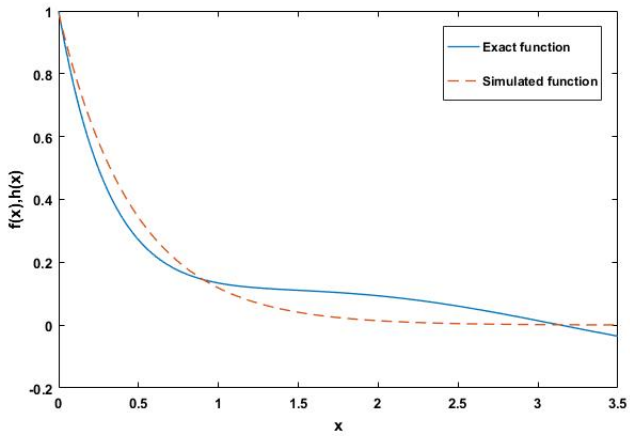

FIGURE 2. Exact and simulated functions for nonlinear autoregressive function $f_2(x)$.

Exact diffusion	Simulated diffusion	Coefficients with 95% confidence interval	Bias	MSE
$\sqrt{P_t}$	$\gamma_{11} P_t^{\gamma_{12}}$	$\gamma_{11} = 0.6487$ (0.232,1.065)	0.04561	0.013927
		$\gamma_{12} = 0.4489$ (-0.08249,0.9803)		
$P_t^{\frac{3}{4}}$	$\gamma_{21} P_t^{\gamma_{22}}$	$\gamma_{21} = 0.9484$ (0.2919,1.605)	0.01171	0.02246
		$\gamma_{22} = 0.1467$ (-0.439,0.7324)		

TABLE 3. Bias and MSE of the drift and diffusion estimators for the OU process with drift term $f_1(x)$

different diffusion processes are presented in Figures 3 and 5. Figures 4 and 6 show the residuals for these estimators.

Exact diffusion	Simulated diffusion	Coefficients with 95% confidence interval	Bias	MSE
$\sqrt{P_t}$	$\gamma_{11} P_t^{\gamma_{12}}$	$\gamma_{11} = 0.2779$ (0.2257,0.3302) $\gamma_{12} = 1.201$ (1.054,1.349)	0.040191	0.34661
$P_t^{\frac{3}{4}}$	$\gamma_{21} P_t^{\gamma_{22}}$	$\gamma_{21} = 0.2444$ (0.1884,0.3003) $\gamma_{22} = 1.321$ (1.143,1.499)	0.01966	0.74798

TABLE 4. Bias and MSE of the drift and diffusion estimators for the OU process with drift term $f_2(x)$

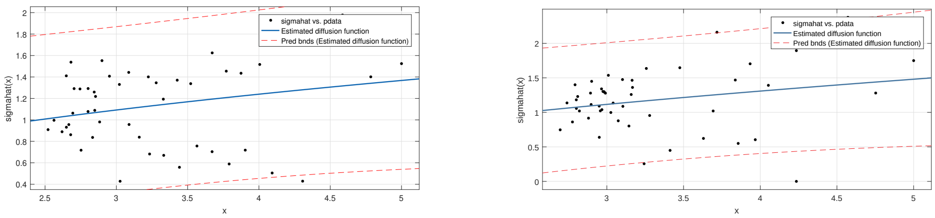

FIGURE 3. The estimated function with 95% confidence interval for OU process with nonlinear autoregressive function $f_1(x)$, Left: for $\sigma(P_t) = \sqrt{P_t}$, Right: for $\sigma(P_t) = P_t^{\frac{3}{4}}$.

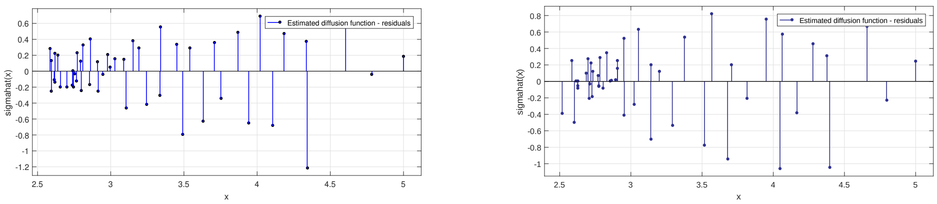

FIGURE 4. The residuals for OU process with drift term $f_1(x)$, Left: for $\sigma(P_t) = \sqrt{P_t}$, Right: for $\sigma(P_t) = P_t^{\frac{3}{4}}$.

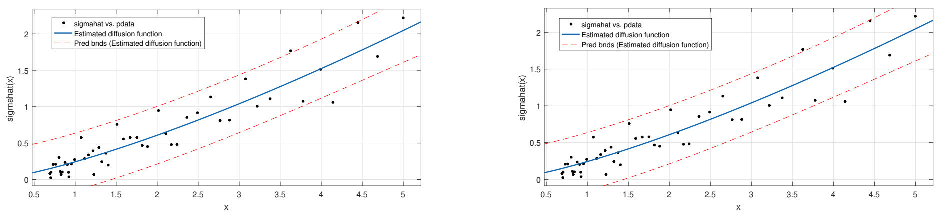

FIGURE 5. The estimated function with 95% confidence interval for OU process with nonlinear autoregressive function $f_2(x)$, Left: for $\sigma(P_t) = \sqrt{P_t}$, Right: for $\sigma(P_t) = P_t^{\frac{3}{4}}$.

5. CONCLUSION

The two factors mean reverting OU process with nonlinear autoregressive drift term innovation and an unknown diffusion coefficient is investigated, to our knowledge for the first time, in this paper. Since parametric methods are not very efficient to estimate regression function, the semiparametric method is used. MSE criterion is also applied to

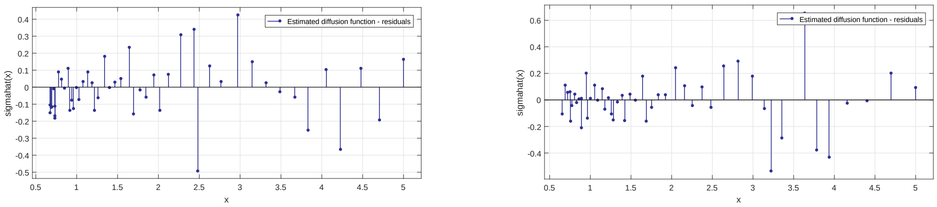

FIGURE 6. The residuals for OU process with drift term $f_2(x)$, Left: for $\sigma(P_t) = \sqrt{P_t}$, Right: for $\sigma(P_t) = P_t^{\frac{3}{4}}$.

verify the accuracy of the suggested model. For estimating the diffusion coefficient of a stochastic differential equation a semiparametric approach is presented. The simulation study demonstrates the efficiency of the present work. The asymptotic behavior of the estimators and conditions for their consistency are investigated in future work.

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