TWMS J. App. and Eng. Math. V.12, N.3, 2022, pp. 1020-1031

ON PROPER HAMILTONIAN-CONNECTION NUMBER OF GRAPHS

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ABSTRACT. A graph G is Hamiltonian-connected if every two vertices of G are connected by a Hamilton path. A bipartite graph H is Hamiltonian-laceable if any two vertices from different partite sets of H are connected by a Hamilton path. An edge-coloring (adjacent edges may receive the same color) of a Hamiltonian-connected (respectively, Hamiltonian-laceable) graph G (resp. H) is a proper Hamilton path coloring if every two vertices u and v of G (resp. H) are connected by a Hamilton path P_{uv} such that no two adjacent edges of P_{uv} are colored the same. The minimum number of colors in a proper Hamilton path coloring of G (resp. H) is the proper Hamiltonian-connection number of G (resp. H). In this paper, proper Hamiltonian-connection numbers are determined for some classes of Hamiltonian-connected graphs and that of Hamiltonian-laceable graphs.

Keywords: Hamiltonian-connected graph, Hamiltonian-laceable graph, proper Hamilton path coloring, proper Hamiltonian-connection number.

AMS Subject Classification: 05C15, 05C45.

1. Hamiltonian-connected graphs

We refer the book [1] for graph theory notation and terminology not described here. A Hamilton path in a graph G is a path containing every vertex of G. A graph G is Hamiltonian-connected if for every pair u, v of distinct vertices of G, there is a Hamilton u-v path in G. Let G be an edge-colored connected graph, where adjacent edges may be colored the same. A path P in G is properly colored or P is a proper path in G if no two adjacent edges of P are colored the same.

For a Hamiltonian-connected graph G, an edge-coloring $c : E(G) \to \{1, 2, \ldots, k\}$ is a proper Hamilton path k-coloring if any two vertices of G are connected by a proper Hamilton path in G. An edge-coloring c is a proper Hamilton path coloring if c is a proper Hamilton path k-coloring for some positive integer k. The minimum number of colors in a proper Hamilton path coloring of G is the proper Hamiltonian-connection number of G, denoted by $\mathrm{hpc}(G)$.

Since every proper edge-coloring of a Hamiltonian-connected graph G is a proper Hamilton path coloring of G and there is no proper Hamilton path 1-coloring of G, we have $2 \leq \mathrm{hpc}(G) \leq \chi'(G)$, where G is of order at least 3 and $\chi'(G)$ is the chromatic index of G.

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§ Manuscript received: April 24, 2020; accepted: June 22, 2020.

In [2], Bi, Byers and Zhang introduced the concept of proper Hamiltonian-connection number for Hamiltonian-connected graphs and proved that: for every integer $n \geq 4$, $hpc(K_n) = 2$, where K_n is the complete graph on n vertices; for each odd integer $n \geq 3$, $hpc(C_n \Box K_2) = 3$, where C_n is the cycle on n vertices and $\Box$ denotes the Cartesian product. Also, they conjectured that: if G is a Hamiltonian-connected graph, then $hpc(G) \leq 3$.

Let G be a Hamiltonian-connected graph of order $n \geq 4$. Then, G is 3-connected, and so $\delta(G) \geq 3$, where $\delta(G)$ is the minimum degree of G. This implies that the minimum possible size of G is $\left\lfloor \frac{3n+1}{2} \right\rfloor$. In [4], Moon proved that for each integer $n \geq 4$, there exists a Hamiltonian-connected graph of order n and size $\left\lfloor \frac{3n+1}{2} \right\rfloor$.

For each integer $k \geq 2$, consider $P_k \Box K_2$. The two disjoint paths in $P_k \Box K_2$ of order k with $x_i y_i \in E(P_k \Box K_2)$ for $i \in \{1, 2, \ldots, k\}$ are $P_k = x_1 x_2 \ldots x_k$ and $P'_k = y_1 y_2 \ldots y_k$. Let H_k be the cubic graph of order $2k + 2$ obtained by adding two adjacent vertices u and v to $P_k \Box K_2$ and joining the vertex u to x_1 and y_1; the vertex v to x_k and y_k. Graph H_k is Hamiltonian-connected and has the minimum size $3(k+1)$ among the Hamiltonian-connected graphs of even order $2k + 2$. For $k \geq 3$, the graph F_k of odd order $2k + 1$ is constructed from $P_k \Box K_2$ by adding a new vertex u and joining u to each vertex in $\{x_1, x_k, y_1, y_k\}$. Graph F_k has $2k$ vertices of degree 3 and one vertex of degree 4; it is a Hamiltonian-connected graph and has the minimum size $3k + 2$ among the Hamiltonian-connected graphs of order $2k + 1$. In [2], Bi et al. proved that, for each integer $k \geq 2$, $hpc(H_k) = 3$ and for each integer $k \geq 3$, $hpc(F_k) = 3$.

A *circulant graph*, denoted by $Circ(n : \{a_1, a_2, \ldots, a_k\})$, where $0 < a_1 < a_2 < \cdots < a_k \leq \left\lfloor \frac{n}{2} \right\rfloor$, has vertices $v_0, v_1, v_2, \ldots, v_{n-1}$ and edge $v_i v_j$ if, and only if, $\mid j - i \mid \equiv a_t \pmod{n}$ for some t, $t \in \{1, 2, \ldots, k\}$. If '$n$ is even and $a_k \neq \frac{n}{2}$' or 'n is odd', then it is $2k$-regular; otherwise, it is ($2k$-1)-regular. In circulants, subscripts in v_i are reduced modulo n.

2. Graphs with $hpc = 2$

The only known graph with $hpc = 2$ is K_n, where $n \geq 4$. Let G be a Hamiltonian-connected graph of order at least 4. To show that $hpc(G) = 2$, we must show that G has a proper Hamilton path 2-coloring; that is, a 2-edge-coloring of G with the property that for every two vertices u and v of G, there is a proper Hamilton u-v path in G. In this section, we find more graphs in the class of graphs with $hpc = 2$.

Lemma 2.1. *For every integer* $n \geq 7$, $hpc(Circ(n : \{1, 2, 3\})) = 2$.

Proof. We consider two cases, according to whether n is even or odd.
Case 1. n is even.

Let $n = 2k$, $k \geq 4$, $G = Circ(2k : \{1, 2, 3\})$ and $F = \{v_i v_{i+1} : i \in \{1, 3, 5, \ldots, 2k-1\}\}$, where $v_{2k} = v_0$. Then, F is a 1-factor of G. Define an edge-coloring c of G by assigning color blue to each edge of F and color red to the remaining edges of G. We show that for every two vertices v_i and v_j of G, there is a proper Hamilton v_i-v_j path in G. As the edge-colored G is vertex-transitive, we verify for $i = 0$.

(Observe that, in the following paths, the first and the last edges are colored blue.)

v_0-v_1 path: $v_0 v_{2k-1} v_{2k-2} v_{2k-3} \ \cdots \ v_4 v_3 v_2 v_1$;

v_0-v_2 path: $v_0 v_{2k-1} v_{2k-2} v_{2k-3} \ \cdots \ v_4 v_3 v_1 v_2$;

v_0-v_3 path: for $k \geq 5$, $v_0 v_{2k-1} v_{2k-2} v_{2k-3} \ \cdots \ v_5 v_2 v_1 v_4 v_3$; for $k = 4$, $v_0 v_7 v_6 v_5 v_2 v_1 v_4 v_3$;

v_0-v_4 path: for $k \geq 5$, $v_0 v_{2k-1} v_{2k-2} v_{2k-3} \ \cdots \ v_5 v_2 v_1 v_3 v_4$; for $k = 4$, $v_0 v_7 v_6 v_5 v_2 v_1 v_3 v_4$;

v_0-v_{2i-1} path: $v_0 v_{2k-1} \ v_1 v_2 v_3 v_4 \ldots v_{2i-2} \ v_{2i+1} v_{2i+2} \ v_{2i+5} v_{2i+6} \ v_{2i+9} v_{2i+10} \cdots$
$v_{2k-13} v_{2k-12} \ v_{2k-9} v_{2k-8} \ v_{2k-5} v_{2k-4} \ v_{2k-2} \ v_{2k-3} \ v_{2k-6} v_{2k-7} \ v_{2k-10} v_{2k-11} \ v_{2k-14} v_{2k-15}$
$\cdots v_{2i+12} v_{2i+11} \ v_{2i+8} v_{2i+7} \ v_{2i+4} v_{2i+3} \ v_{2i} v_{2i-1}$

if '$k \geq 10$ is even and $i \in \{3, 5, 7, \ldots, k-7, k-5, k-3\}$' or '$k \geq 11$ is odd and $i \in \{4, 6, 8,$

$\ldots, k-7, k-5, k-3\}$';

for $k = 9$, $v_0v_{17}v_1v_2v_3v_4v_5v_6v_9v_{10}v_{13}v_{14}v_{16}v_{15}v_{12}v_{11}v_8v_7$, $v_0v_{17}v_1v_2v_3v_4v_5v_6v_7v_8v_9v_{10}v_{13}v_{14}v_{16}v_{15}v_{12}v_{11}$;

for $k = 8$, $v_0v_{15}v_1v_2v_3v_4v_7v_8v_{11}v_{12}v_{14}v_{13}v_{10}v_9v_6v_5$, $v_0v_{15}v_1v_2v_3v_4v_5v_6v_7v_8v_{11}v_{12}v_{14}v_{13}v_{10}v_9$;

for $k = 7$, $v_0v_{13}v_1v_2v_3v_4v_5v_6v_9v_{10}v_{12}v_{11}v_8v_7$;

for $k = 6$, $v_0v_{11}v_1v_2v_3v_4v_7v_8v_{10}v_9v_6v_5$; and

$v_0v_{2k-1}\ v_1v_2v_3v_4 \ldots v_{2i-2}\ v_{2i+1}v_{2i+2}\ v_{2i+5}v_{2i+6}\ v_{2i+9}v_{2i+10} \cdots\ v_{2k-15}v_{2k-14}\ v_{2k-11}v_{2k-10}\ v_{2k-7}v_{2k-6}\ v_{2k-3}\ v_{2k-2}\ v_{2k-4}v_{2k-5}\ v_{2k-8}v_{2k-9}\ v_{2k-12}v_{2k-13} \cdots v_{2i+12}v_{2i+11}\ v_{2i+8}v_{2i+7}\ v_{2i+4}v_{2i+3}\ v_{2i}v_{2i-1}$

if '$k \geq 10$ is even and $i \in \{4, 6, 8, \ldots, k-6, k-4, k-2\}$' or '$k \geq 9$ is odd and $i \in \{3, 5, 7, \ldots, k-6, k-4, k-2\}$';

for $k = 8$, $v_0v_{15}v_1v_2v_3v_4v_5v_6v_9v_{10}v_{13}v_{14}v_{12}v_{11}v_8v_7$, $v_0v_{15}v_1v_2v_3v_4v_5v_6v_7v_8v_9v_{10}v_{13}v_{14}v_{12}v_{11}$;

for $k = 7$, $v_0v_{13}v_1v_2v_3v_4v_7v_8v_{11}v_{12}v_{10}v_9v_6v_5$, $v_0v_{13}v_1v_2v_3v_4v_5v_6v_7v_8v_{11}v_{12}v_{10}v_9$;

for $k = 6$, $v_0v_{11}v_1v_2v_3v_4v_5v_6v_9v_{10}v_8v_7$;

for $k = 5$, $v_0v_9v_1v_2v_3v_4v_7v_8v_6v_5$; and

v_0-v_{2i} path: $v_0v_{2k-1}v_{2k-2}v_{2k-3} \cdots v_{2i+1}\ v_{2i-2}v_{2i-3}\ v_{2i-6}v_{2i-7}\ v_{2i-10}v_{2i-11}\ \cdots\ v_{13}\ v_{10}v_9\ v_6v_5\ v_2v_1\ v_3v_4\ v_7v_8\ v_{11}v_{12} \ldots v_{2i-13}v_{2i-12}\ v_{2i-9}v_{2i-8}\ v_{2i-5}v_{2i-4}\ v_{2i-1}v_{2i}$

if '$k \geq 10$ is even and $i \in \{4, 6, 8, \ldots, k-6, k-4, k-2\}$' or '$k \geq 11$ is odd and $i \in \{4, 6, 8, \ldots, k-7, k-5, k-3\}$';

for $k = 9$, $v_0v_{17}v_{16}v_{15}v_{14}v_{13}v_{12}v_{11}v_{10}v_9v_6v_5v_2v_1v_3v_4v_7v_8$, $v_0v_{17}v_{16}v_{15}v_{14}v_{13}v_{10}v_9v_6v_5v_2v_1v_3v_4v_7v_8v_{11}v_{12}$;

for $k = 8$, $v_0v_{15}v_{14}v_{13}v_{12}v_{11}v_{10}v_9v_6v_5v_2v_1v_3v_4v_7v_8$, $v_0v_{15}v_{14}v_{13}v_{10}v_9v_6v_5v_2v_1v_3v_4v_7v_8v_{11}v_{12}$;

for $k = 7$, $v_0v_{13}v_{12}v_{11}v_{10}v_9v_6v_5v_2v_1v_3v_4v_7v_8$;

for $k = 6$, $v_0v_{11}v_{10}v_9v_6v_5v_2v_1v_3v_4v_7v_8$; and

$v_0v_{2k-1}v_{2k-2}v_{2k-3} \cdots v_{2i+1}\ v_{2i-2}v_{2i-3}\ v_{2i-6}v_{2i-7}\ v_{2i-10}v_{2i-11}\ \cdots\ v_{15}\ v_{12}v_{11}\ v_8v_7\ v_4v_3\ v_1v_2\ v_5v_6\ v_9v_{10}\ v_{13}v_{14} \ldots v_{2i-13}v_{2i-12}\ v_{2i-9}v_{2i-8}\ v_{2i-5}v_{2i-4}\ v_{2i-1}v_{2i}$

if '$k \geq 10$ is even and $i \in \{3, 5, 7, \ldots, k-7, k-5, k-3\}$' or '$k \geq 9$ is odd and $i \in \{3, 5, 7, \ldots, k-6, k-4, k-2\}$';

for $k = 8$, $v_0v_{15}v_{14}v_{13}v_{12}v_{11}v_{10}v_9v_8v_7v_4v_3v_1v_2v_5v_6$, $v_0v_{15}v_{14}v_{13}v_{12}v_{11}v_8v_7v_4v_3v_1v_2v_5v_6v_9v_{10}$;

for $k = 7$, $v_0v_{13}v_{12}v_{11}v_{10}v_9v_8v_7v_4v_3v_1v_2v_5v_6$, $v_0v_{13}v_{12}v_{11}v_8v_7v_4v_3v_1v_2v_5v_6v_9v_{10}$;

for $k = 6$, $v_0v_{11}v_{10}v_9v_8v_7v_4v_3v_1v_2v_5v_6$;

for $k = 5$, $v_0v_9v_8v_7v_4v_3v_1v_2v_5v_6$;

v_0-v_{2k-3} path: for $k \geq 5$, $v_0v_{2k-1}v_1v_2v_3\ \ldots\ v_{2k-5}v_{2k-4}v_{2k-2}v_{2k-3}$;
for $k = 4$, $v_0v_7v_1v_2v_3v_4v_6v_5$;

v_0-v_{2k-2} path: for $k \geq 5$, $v_0v_{2k-1}v_1v_2v_3\ \ldots\ v_{2k-5}v_{2k-4}v_{2k-3}v_{2k-2}$;
for $k = 4$, $v_0v_7v_1v_2v_3v_4v_5v_6$;

(Observe that, in the following path, the first and the last edges are colored red.)

v_0-v_{2k-1} path: $v_0v_1v_2v_3\ \ldots\ v_{2k-4}v_{2k-3}v_{2k-2}v_{2k-1}$.

Case 2. n is odd.

Let $n = 2k-1$, $k \geq 4$, $G = Circ(2k-1 : \{1, 2, 3\})$ and $C = v_0v_1v_2 \ldots v_{2k-2}v_0$. Then, C is a Hamilton cycle of G. Define an edge-coloring c of G by assigning color red to each edge of C and color blue to the remaining edges of G. As the edge-colored G is vertex-transitive, we show that for every vertex v_j, $j \neq 0$, of G, there is a proper Hamilton v_0-v_j path in G.

v_0-v_1 path: for $k \geq 7$, $v_0\ v_{2k-3}v_{2k-2}\ v_{2k-5}v_{2k-4}\ v_{2k-7}v_{2k-6}\ \ldots\ v_5v_6\ v_3v_4\ v_2v_1$;
for $k = 6$, $v_0v_9v_{10}v_7v_8v_5v_6v_3v_4v_2v_1$;

for $k = 5$, $v_0v_7v_8v_5v_6v_3v_4v_2v_1$;
for $k = 4$, $v_0v_5v_6v_3v_4v_2v_1$;
v_0-v_2 path: for $k \geq 7$, $v_0\ v_{2k-3}v_{2k-2}\ v_{2k-5}v_{2k-4}\ v_{2k-7}v_{2k-6}\ \cdots\ v_5v_6\ v_3v_4\ v_1v_2$;
for $k = 6$, $v_0v_9v_{10}v_7v_8v_5v_6v_3v_4v_1v_2$;
for $k = 5$, $v_0v_7v_8v_5v_6v_3v_4v_1v_2$;
for $k = 4$, $v_0v_5v_6v_3v_4v_1v_2$;
v_0-v_3 path: for $k \geq 8$, $v_0\ v_2v_1\ v_{2k-3}v_{2k-2}\ v_{2k-5}v_{2k-4}\ v_{2k-7}v_{2k-6}\ \cdots\ v_7v_8\ v_5v_6\ v_4v_3$;
for $k = 7$, $v_0v_2v_1v_{11}v_{12}v_9v_{10}v_7v_8v_5v_6v_4v_3$;
for $k = 6$, $v_0v_2v_1v_9v_{10}v_7v_8v_5v_6v_4v_3$;
for $k = 5$, $v_0v_2v_1v_7v_8v_5v_6v_4v_3$;
for $k = 4$, $v_0v_2v_1v_5v_6v_4v_3$;
v_0-v_4 path: for $k \geq 8$, $v_0\ v_2v_1\ v_{2k-3}v_{2k-2}\ v_{2k-5}v_{2k-4}\ v_{2k-7}v_{2k-6}\ \cdots\ v_7v_8\ v_5v_6\ v_3v_4$;
for $k = 7$, $v_0v_2v_1v_{11}v_{12}v_9v_{10}v_7v_8v_5v_6v_3v_4$;
for $k = 6$, $v_0v_2v_1v_9v_{10}v_7v_8v_5v_6v_3v_4$;
for $k = 5$, $v_0v_2v_1v_7v_8v_5v_6v_3v_4$;
for $k = 4$, $v_0v_2v_1v_5v_6v_3v_4$;
v_0-v_{2i-1} path: $v_0v_{2k-2}\ v_2v_1\ v_4v_3\ v_6v_5\ \cdots\ v_{2i-6}v_{2i-7}\ v_{2i-4}v_{2i-5}\ v_{2i-2}v_{2i-3}\ v_{2i}v_{2i+1}$
$v_{2i+4}v_{2i+5}\ v_{2i+8}v_{2i+9}\ \cdots\ v_{2k-12}v_{2k-11}\ v_{2k-8}v_{2k-7}\ v_{2k-4}v_{2k-3}\ v_{2k-5}v_{2k-6}$
$v_{2k-9}v_{2k-10}\ v_{2k-13}v_{2k-14}\ \cdots v_{2i+11}v_{2i+10}\ v_{2i+7}v_{2i+6}\ v_{2i+3}v_{2i+2}\ v_{2i-1}$
if '$k \geq 10$ is even and $i \in \{4, 6, 8, \ldots, k-6, k-4, k-2\}$' or '$k \geq 11$ is odd and $i \in \{3, 5, 7, \ldots, k-6, k-4, k-2\}$';
for $k = 9$, $v_0v_{16}v_2v_1v_4v_3v_6v_7v_{10}v_{11}v_{14}v_{15}v_{13}v_{12}v_9v_8v_5$,
$v_0v_{16}v_2v_1v_4v_3v_6v_5v_8v_7v_{10}v_{11}v_{14}v_{15}v_{13}v_{12}v_9$,
$v_0v_{16}v_2v_1v_4v_3v_6v_5v_8v_7v_{10}v_9v_{12}v_{11}v_{14}v_{15}v_{13}$;
for $k = 8$, $v_0v_{14}v_2v_1v_4v_3v_6v_5v_8v_9v_{12}v_{13}v_{11}v_{10}v_7$, $v_0v_{14}v_2v_1v_4v_3v_6v_5v_8v_7v_{10}v_9v_{12}v_{13}v_{11}$;
for $k = 7$, $v_0v_{12}v_2v_1v_4v_3v_6v_7v_{10}v_{11}v_9v_8v_5$, $v_0v_{12}v_2v_1v_4v_3v_6v_5v_8v_7v_{10}v_{11}v_9$;
for $k = 6$, $v_0v_{10}v_2v_1v_4v_3v_6v_5v_8v_9v_7$; and
$v_0v_{2k-2}\ v_2v_1\ v_4v_3\ v_6v_5 \cdots\ v_{2i-6}v_{2i-7}\ v_{2i-4}v_{2i-5}\ v_{2i-2}v_{2i-3}\ v_{2i}v_{2i+1}$
$v_{2i+4}v_{2i+5}\ v_{2i+8}v_{2i+9} \cdots\ v_{2k-14}v_{2k-13}\ v_{2k-10}v_{2k-9}\ v_{2k-6}v_{2k-5}\ v_{2k-3}v_{2k-4}$
$v_{2k-7}v_{2k-8}\ v_{2k-11}v_{2k-12} \cdots\ v_{2i+11}v_{2i+10}\ v_{2i+7}v_{2i+6}\ v_{2i+3}v_{2i+2}\ v_{2i-1}$
if '$k \geq 10$ is even and $i \in \{3, 5, 7, \ldots, k-7, k-5, k-3\}$' or '$k \geq 11$ is odd and $i \in \{4, 6, 8, \ldots, k-7, k-5, k-3\}$';
for $k = 9$, $v_0v_{16}v_2v_1v_4v_3v_6v_5v_8v_9v_{12}v_{13}v_{15}v_{14}v_{11}v_{10}v_7$,
$v_0v_{16}v_2v_1v_4v_3v_6v_5v_8v_7v_{10}v_9v_{12}v_{13}v_{15}v_{14}v_{11}$;
for $k = 8$, $v_0v_{14}v_2v_1v_4v_3v_6v_7v_{10}v_{11}v_{13}v_{12}v_9v_8v_5$, $v_0v_{14}v_2v_1v_4v_3v_6v_5v_8v_7v_{10}v_{11}v_{13}v_{12}v_9$;
for $k = 7$, $v_0v_{12}v_2v_1v_4v_3v_6v_5v_8v_9v_{11}v_{10}v_7$;
for $k = 6$, $v_0v_{10}v_2v_1v_4v_3v_6v_7v_9v_8v_5$;
v_0-v_{2i} path: $v_0v_1\ v_3v_2\ v_5v_4\ v_7v_6\ \cdots\ v_{2i-7}v_{2i-8}\ v_{2i-5}v_{2i-6}\ v_{2i-3}v_{2i-4}\ v_{2i-1}v_{2i-2}$
$v_{2i+1}v_{2i+2}\ v_{2i+5}v_{2i+6}\ v_{2i+9}v_{2i+10}\ \cdots\ v_{2k-11}v_{2k-10}\ v_{2k-7}v_{2k-6}\ v_{2k-3}v_{2k-2}$
$v_{2k-4}v_{2k-5}\ v_{2k-8}v_{2k-9}\ v_{2k-12}v_{2k-13} \cdots\ v_{2i+12}v_{2i+11}\ v_{2i+8}v_{2i+7}\ v_{2i+4}v_{2i+3}\ v_{2i}$
if '$k \geq 10$ is even and $i \in \{4, 6, 8, \ldots, k-6, k-4, k-2\}$' or '$k \geq 11$ is odd and $i \in \{3, 5, 7, \ldots, k-6, k-4, k-2\}$;'
for $k = 9$, $v_0v_1v_3v_2v_5v_4v_7v_8v_{11}v_{12}v_{15}v_{16}v_{14}v_{13}v_{10}v_9v_6$,
$v_0v_1v_3v_2v_5v_4v_7v_6v_9v_8v_{11}v_{12}v_{15}v_{16}v_{14}v_{13}v_{10}$,
$v_0v_1v_3v_2v_5v_4v_7v_6v_9v_8v_{11}v_{10}v_{13}v_{12}v_{15}v_{16}v_{14}$;
for $k = 8$, $v_0v_1v_3v_2v_5v_4v_7v_6v_9v_{10}v_{13}v_{14}v_{12}v_{11}v_8$, $v_0v_1v_3v_2v_5v_4v_7v_6v_9v_8v_{11}v_{10}v_{13}v_{14}v_{12}$;
for $k = 7$, $v_0v_1v_3v_2v_5v_4v_7v_8v_{11}v_{12}v_{10}v_9v_6$, $v_0v_1v_3v_2v_5v_4v_7v_6v_9v_8v_{11}v_{12}v_{10}$;
for $k = 6$, $v_0v_1v_3v_2v_5v_4v_7v_6v_9v_{10}v_8$; and
$v_0v_1\ v_3v_2\ v_5v_4\ v_7v_6 \cdots\ v_{2i-5}v_{2i-6}\ v_{2i-3}v_{2i-4}\ v_{2i-1}v_{2i-2}\ v_{2i+1}v_{2i+2}$

$v_{2i+5}v_{2i+6}\ v_{2i+9}v_{2i+10}\ \cdots\ v_{2k-13}v_{2k-12}\ v_{2k-9}v_{2k-8}\ v_{2k-5}v_{2k-4}\ v_{2k-2}v_{2k-3}$
$v_{2k-6}v_{2k-7}\ v_{2k-10}v_{2k-11}\ \cdots\ v_{2i+12}v_{2i+11}\ v_{2i+8}v_{2i+7}\ v_{2i+4}v_{2i+3}\ v_{2i}$
if '$k \geq 10$ is even and $i \in \{3, 5, 7, \ldots, k-7, k-5, k-3\}$' or '$k \geq 11$ is odd and $i \in \{4, 6, 8, \ldots, k-7, k-5, k-3\}$';

for $k = 9$, $v_0v_1v_3v_2v_5v_4v_7v_6v_9v_{10}v_{13}v_{14}v_{16}v_{15}v_{12}v_{11}v_8$,
$v_0v_1v_3v_2v_5v_4v_7v_6v_9v_8v_{11}v_{10}v_{13}v_{14}v_{16}v_{15}v_{12}$;
for $k = 8$, $v_0v_1v_3v_2v_5v_4v_7v_8v_{11}v_{12}v_{14}v_{13}v_{10}v_9v_6$, $v_0v_1v_3v_2v_5v_4v_7v_6v_9v_8v_{11}v_{12}v_{14}v_{13}v_{10}$;
for $k = 7$, $v_0v_1v_3v_2v_5v_4v_7v_6v_9v_{10}v_{12}v_{11}v_8$;
for $k = 6$, $v_0v_1v_3v_2v_5v_4v_7v_8v_{10}v_9v_6$;

v_0-v_{2k-3} path: for $k \geq 7$, $v_0\ v_2v_1\ v_4v_3\ v_6v_5\ \ldots\ v_{2k-6}v_{2k-7}\ v_{2k-4}v_{2k-5}\ v_{2k-2}v_{2k-3}$;
for $k = 6$, $v_0v_2v_1v_4v_3v_6v_5v_8v_7v_{10}v_9v_{12}v_{11}$;
for $k = 5$, $v_0v_2v_1v_4v_3v_6v_5v_8v_7$;
for $k = 4$, $v_0v_2v_1v_4v_3v_6v_5$;

v_0-v_{2k-2} path: for $k \geq 7$, $v_0\ v_2v_1\ v_4v_3\ v_6v_5\ \ldots\ v_{2k-6}v_{2k-7}\ v_{2k-4}v_{2k-5}\ v_{2k-3}v_{2k-2}$;
for $k = 6$, $v_0v_2v_1v_4v_3v_6v_5v_8v_7v_9v_{10}$;
for $k = 5$, $v_0v_2v_1v_4v_3v_6v_5v_7v_8$;
for $k = 4$, $v_0v_2v_1v_4v_3v_5v_6$. This completes the proof. □

It follows from Lemma 2.1 that

Theorem 2.1. *If G is a graph with n vertices, $n \geq 7$, such that $Circ(n : \{1, 2, 3\}) \subseteq G$, then $hpc(G) = 2$.*

Corollary 2.1. *(Bi, Byers and Zhang [2]) For $n \geq 7$, $hpc(K_n) = 2$.*

Lemma 2.2. *For any odd integer $k \geq 5$, $hpc(Circ(2k : \{1, 2, k\})) = 2$.*

Proof. Let $G = Circ(2k : \{1, 2, k\})$ and $F = \{v_iv_{i+1} : i \in \{1, 3, 5, \ldots, 2k-1\}\}$, where $v_{2k} = v_0$. Then F is a 1-factor of G. Define an edge-coloring c of G by assigning color blue to each edge of F and color red to the remaining edges of G. As the edge-colored G is vertex-transitive, we show that for every vertex v_j, $j \neq 0$, of G, there is a proper Hamilton v_0-v_j path in G.

(Observe that, in the following paths, the first and the last edges are colored blue.)

v_0-v_1 path: $v_0v_{2k-1}\ v_{2k-2}v_{2k-3}\ v_{2k-4}v_{2k-5}\ \cdots\ v_6v_5\ v_4v_3\ v_2v_1$;

v_0-v_2 path: $v_0v_{2k-1}\ v_{2k-2}v_{2k-3}\ v_{2k-4}v_{2k-5}\ \cdots\ v_6v_5\ v_4v_3\ v_1v_2$;

v_0-v_{2i-1} path, $i \in \{2, 3, 4, \ldots, \frac{k-3}{2}\}$: for $k \geq 13$,
$v_0v_{2k-1}\ v_{k-1}v_{k-2}\ v_{2k-2}v_{2k-3}\ v_{k-3}v_{k-4}\ v_{2k-4}v_{2k-5}\ v_{k-5}v_{k-6}\ v_{2k-6}v_{2k-7}$
$\cdots\ v_{2i+6}v_{2i+5}\ v_{k+2i+5}v_{k+2i+4}\ v_{2i+4}v_{2i+3}\ v_{k+2i+3}v_{k+2i+2}\ v_{2i+2}v_{2i+1}\ v_{k+2i+1}v_{k+2i}$
$v_{k+2i-1}v_{k+2i-2}\ v_{k+2i-3}v_{k+2i-4}\ \cdots\ v_{k+7}v_{k+6}\ v_{k+5}v_{k+4}\ v_{k+3}v_{k+2}\ v_kv_{k+1}$
$v_1v_2\ v_3v_4\ v_5v_6\ \cdots\ v_{2i-7}v_{2i-6}\ v_{2i-5}v_{2i-4}\ v_{2i-3}v_{2i-2}\ v_{2i}v_{2i-1}$;
for $k = 11$, $v_0v_{21}v_{10}v_9v_{20}v_{19}v_8v_7v_{18}v_{17}v_6v_5v_{16}v_{15}v_{14}v_{13}v_{11}v_{12}v_1v_2v_4v_3$,
$v_0v_{21}v_{10}v_9v_{20}v_{19}v_8v_7v_{18}v_{17}v_{16}v_{15}v_{14}v_{13}v_{11}v_{12}v_1v_2v_3v_4v_6v_5$,
$v_0v_{21}v_{10}v_9v_{20}v_{19}v_{18}v_{17}v_{16}v_{15}v_{14}v_{13}v_{11}v_{12}v_1v_2v_3v_4v_5v_6v_8v_7$;
for $k = 9$, $v_0v_{17}v_8v_7v_{16}v_{15}v_6v_5v_{14}v_{13}v_{12}v_{11}v_9v_{10}v_1v_2v_4v_3$,
$v_0v_{17}v_8v_7v_{16}v_{15}v_{14}v_{13}v_{12}v_{11}v_9v_{10}v_1v_2v_3v_4v_6v_5$;
for $k = 7$, $v_0v_{13}v_6v_5v_{12}v_{11}v_{10}v_9v_7v_8v_1v_2v_4v_3$;

v_0-v_{2i} path, $i \in \{2, 3, 4, \ldots, \frac{k-3}{2}\}$: for $k \geq 13$,
$v_0v_{2k-1}\ v_{k-1}v_{k-2}\ v_{2k-2}v_{2k-3}\ v_{k-3}v_{k-4}\ v_{2k-4}v_{2k-5}\ v_{k-5}v_{k-6}\ v_{2k-6}v_{2k-7}$
$\cdots\ v_{2i+6}v_{2i+5}\ v_{k+2i+5}v_{k+2i+4}\ v_{2i+4}v_{2i+3}\ v_{k+2i+3}v_{k+2i+2}\ v_{2i+2}v_{2i+1}\ v_{k+2i+1}v_{k+2i}$
$v_{k+2i-1}v_{k+2i-2}\ v_{k+2i-3}v_{k+2i-4}\ \cdots\ v_{k+7}v_{k+6}\ v_{k+5}v_{k+4}\ v_{k+3}v_{k+2}\ v_kv_{k+1}$
$v_1v_2\ v_3v_4\ v_5v_6\ \cdots\ v_{2i-7}v_{2i-6}\ v_{2i-5}v_{2i-4}\ v_{2i-3}v_{2i-2}\ v_{2i-1}v_{2i}$;
for $k = 11$, $v_0v_{21}v_{10}v_9v_{20}v_{19}v_8v_7v_{18}v_{17}v_6v_5v_{16}v_{15}v_{14}v_{13}v_{11}v_{12}v_1v_2v_3v_4$,

$v_0v_{21}v_{10}v_9v_{20}v_{19}v_8v_7v_{18}v_{17}v_{16}v_{15}v_{14}v_{13}v_{11}v_{12}v_1v_2v_3v_4v_5v_6$,
$v_0v_{21}v_{10}v_9v_{20}v_{19}v_{18}v_{17}v_{16}v_{15}v_{14}v_{13}v_{11}v_{12}v_1v_2v_3v_4v_5v_6v_7v_8$;
for $k = 9$, $v_0v_{17}v_8v_7v_{16}v_{15}v_6v_5v_{14}v_{13}v_{12}v_{11}v_9v_{10}v_1v_2v_3v_4$,
$v_0v_{17}v_8v_7v_{16}v_{15}v_{14}v_{13}v_{12}v_{11}v_9v_{10}v_1v_2v_3v_4v_5v_6$;
for $k = 7$, $v_0v_{13}v_6v_5v_{12}v_{11}v_{10}v_9v_7v_8v_1v_2v_3v_4$;

v_0-v_{k-2} path: for $k \geq 13$, $v_0v_{2k-1}\ v_{2k-2}v_{2k-3}\ v_{2k-4}v_{2k-5}\ \cdots\ v_{k+5}v_{k+4}\ v_{k+3}v_{k+2}$ $v_kv_{k+1}\ v_1v_2\ v_3v_4\ v_5v_6\ \cdots\ v_{k-6}v_{k-5}\ v_{k-4}v_{k-3}\ v_{k-1}v_{k-2}$;
for $k = 11$, $v_0v_{21}v_{20}v_{19}v_{18}v_{17}v_{16}v_{15}v_{14}v_{13}v_{11}v_{12}v_1v_2v_3v_4v_5v_6v_7v_8v_{10}v_9$;
for $k = 9$, $v_0v_{17}v_{16}v_{15}v_{14}v_{13}v_{12}v_{11}v_9v_{10}v_1v_2v_3v_4v_5v_6v_8v_7$;
for $k = 7$, $v_0v_{13}v_{12}v_{11}v_{10}v_9v_7v_8v_1v_2v_3v_4v_6v_5$;

v_0-v_{k-1} path: for $k \geq 13$, $v_0v_{2k-1}\ v_{2k-2}v_{2k-3}\ v_{2k-4}v_{2k-5}\ \cdots\ v_{k+5}v_{k+4}\ v_{k+3}v_{k+2}$ $v_kv_{k+1}\ v_1v_2\ v_3v_4\ v_5v_6\ \cdots\ v_{k-6}v_{k-5}\ v_{k-4}v_{k-3}\ v_{k-2}v_{k-1}$;
for $k = 11$, $v_0v_{21}v_{20}v_{19}v_{18}v_{17}v_{16}v_{15}v_{14}v_{13}v_{11}v_{12}v_1v_2v_3v_4v_5v_6v_7v_8v_9v_{10}$;
for $k = 9$, $v_0v_{17}v_{16}v_{15}v_{14}v_{13}v_{12}v_{11}v_9v_{10}v_1v_2v_3v_4v_5v_6v_7v_8$;
for $k = 7$, $v_0v_{13}v_{12}v_{11}v_{10}v_9v_7v_8v_1v_2v_3v_4v_5v_6$;

v_0-v_k path: for $k \geq 13$, $v_0v_{2k-1}\ v_{2k-2}v_{2k-3}\ v_{2k-4}v_{2k-5}\ \cdots\ v_{k+5}v_{k+4}\ v_{k+3}v_{k+2}$ $v_2v_1\ v_3v_4\ v_5v_6\ v_7v_8\ \cdots\ v_{k-4}v_{k-3}\ v_{k-2}v_{k-1}\ v_{k+1}v_k$;
for $k = 11$, $v_0v_{21}v_{20}v_{19}v_{18}v_{17}v_{16}v_{15}v_{14}v_{13}v_2v_1v_3v_4v_5v_6v_7v_8v_9v_{10}v_{12}v_{11}$;
for $k = 9$, $v_0v_{17}v_{16}v_{15}v_{14}v_{13}v_{12}v_{11}v_2v_1v_3v_4v_5v_6v_7v_8v_{10}v_9$;
for $k = 7$, $v_0v_{13}v_{12}v_{11}v_{10}v_9v_2v_1v_3v_4v_5v_6v_8v_7$;

v_0-v_{k+1} path: for $k \geq 13$, $v_0v_{2k-1}\ v_{2k-2}v_{2k-3}\ v_{2k-4}v_{2k-5}\ \cdots\ v_{k+5}v_{k+4}\ v_{k+3}v_{k+2}$ $v_2v_1\ v_3v_4\ v_5v_6\ v_7v_8\ \cdots\ v_{k-4}v_{k-3}\ v_{k-2}v_{k-1}\ v_kv_{k+1}$;
for $k = 11$, $v_0v_{21}v_{20}v_{19}v_{18}v_{17}v_{16}v_{15}v_{14}v_{13}v_2v_1v_3v_4v_5v_6v_7v_8v_9v_{10}v_{11}v_{12}$;
for $k = 9$, $v_0v_{17}v_{16}v_{15}v_{14}v_{13}v_{12}v_{11}v_2v_1v_3v_4v_5v_6v_7v_8v_9v_{10}$;
for $k = 7$, $v_0v_{13}v_{12}v_{11}v_{10}v_9v_2v_1v_3v_4v_5v_6v_7v_8$;

v_0-v_{k+2} path: for $k \geq 13$, $v_0v_{2k-1}\ v_{k-1}v_{k-2}\ v_kv_{k+1}\ v_1v_2\ v_3v_4\ v_5v_6\ \cdots\ v_{k-6}v_{k-5}\ v_{k-4}v_{k-3}$ $v_{2k-3}v_{2k-2}\ v_{2k-4}v_{2k-5}\ v_{2k-6}v_{2k-7}\ v_{2k-8}v_{2k-9}\ \cdots\ v_{k+5}v_{k+4}\ v_{k+3}v_{k+2}$;
for $k = 11$, $v_0v_{21}v_{10}v_9v_{11}v_{12}v_1v_2v_3v_4v_5v_6v_7v_8v_{19}v_{20}v_{18}v_{17}v_{16}v_{15}v_{14}v_{13}$;
for $k = 9$, $v_0v_{17}v_8v_7v_9v_{10}v_1v_2v_3v_4v_5v_6v_{15}v_{16}v_{14}v_{13}v_{12}v_{11}$;
for $k = 7$, $v_0v_{13}v_6v_5v_7v_8v_1v_2v_3v_4v_{11}v_{12}v_{10}v_9$;

v_0-v_{2i-1} path, $i \in \{\frac{k+5}{2}, \frac{k+7}{2}, \frac{k+9}{2}, \ldots, k-2\}$: for $k \geq 15$,
$v_0v_{2k-1}\ v_{k-1}v_{k-2}\ v_{2k-2}v_{2k-3}\ v_{k-3}v_{k-4}\ v_{2k-4}v_{2k-5}\ v_{k-5}v_{k-6}\ v_{2k-6}v_{2k-7}$ $\cdots\ v_{2i+6}v_{2i+5}\ v_{2i+5-k}v_{2i+4-k}\ v_{2i+4}v_{2i+3}\ v_{2i+3-k}v_{2i+2-k}\ v_{2i+2}v_{2i+1}$ $v_{2i+1-k}v_{2i-k}v_{2i-k-1}v_{2i-k-2}\ v_{2i-k-3}v_{2i-k-4}\ \cdots\ v_6v_5\ v_4v_3\ v_2v_1\ v_{k+1}v_k$ $v_{k+2}v_{k+3}\ v_{k+4}v_{k+5}\ v_{k+6}v_{k+7}\ \cdots\ v_{2i-7}v_{2i-6}\ v_{2i-5}v_{2i-4}\ v_{2i-3}v_{2i-2}\ v_{2i}v_{2i-1}$;
for $k = 13$, $v_0v_{25}v_{12}v_{11}v_{24}v_{23}v_{10}v_9v_{22}v_{21}v_8v_7v_{20}v_{19}v_6v_5v_4v_3v_2v_1v_{14}v_{13}v_{15}v_{16}v_{18}v_{17}$,
$v_0v_{25}v_{12}v_{11}v_{24}v_{23}v_{10}v_9v_{22}v_{21}v_8v_7v_6v_5v_4v_3v_2v_1v_{14}v_{13}v_{15}v_{16}v_{17}v_{18}v_{20}v_{19}$,
$v_0v_{25}v_{12}v_{11}v_{24}v_{23}v_{10}v_9v_8v_7v_6v_5v_4v_3v_2v_1v_{14}v_{13}v_{15}v_{16}v_{17}v_{18}v_{19}v_{20}v_{22}v_{21}$;
for $k = 11$, $v_0v_{21}v_{10}v_9v_{20}v_{19}v_8v_7v_{18}v_{17}v_6v_5v_4v_3v_2v_1v_{12}v_{11}v_{13}v_{14}v_{16}v_{15}$,
$v_0v_{21}v_{10}v_9v_{20}v_{19}v_8v_7v_6v_5v_4v_3v_2v_1v_{12}v_{11}v_{13}v_{14}v_{15}v_{16}v_{18}v_{17}$;
for $k = 9$, $v_0v_{17}v_8v_7v_{16}v_{15}v_6v_5v_4v_3v_2v_1v_{10}v_9v_{11}v_{12}v_{14}v_{13}$;

v_0-v_{2i} path, $i \in \{\frac{k+3}{2}, \frac{k+5}{2}, \frac{k+7}{2}, \ldots, k-2\}$: for $k \geq 15$,
$v_0v_{2k-1}\ v_{k-1}v_{k-2}\ v_{2k-2}v_{2k-3}\ v_{k-3}v_{k-4}\ v_{2k-4}v_{2k-5}\ v_{k-5}v_{k-6}\ v_{2k-6}v_{2k-7}$ $\cdots\ v_{2i+6}v_{2i+5}\ v_{2i+5-k}v_{2i+4-k}\ v_{2i+4}v_{2i+3}\ v_{2i+3-k}v_{2i+2-k}\ v_{2i+2}v_{2i+1}$ $v_{2i+1-k}v_{2i-k}\ v_{2i-k-1}v_{2i-k-2}\ v_{2i-k-3}v_{2i-k-4}\ \cdots\ v_6v_5\ v_4v_3\ v_2v_1\ v_{k+1}v_k$ $v_{k+2}v_{k+3}\ v_{k+4}v_{k+5}\ v_{k+6}v_{k+7}\ \cdots\ v_{2i-7}v_{2i-6}\ v_{2i-5}v_{2i-4}\ v_{2i-3}v_{2i-2}\ v_{2i-1}v_{2i}$;
for $k = 13$, $v_0v_{25}v_{12}v_{11}v_{24}v_{23}v_{10}v_9v_{22}v_{21}v_8v_7v_{20}v_{19}v_6v_5v_4v_3v_2v_1v_{14}v_{13}v_{15}v_{16}v_{17}v_{18}$,
$v_0v_{25}v_{12}v_{11}v_{24}v_{23}v_{10}v_9v_{22}v_{21}v_8v_7v_6v_5v_4v_3v_2v_1v_{14}v_{13}v_{15}v_{16}v_{17}v_{18}v_{19}v_{20}$,
$v_0v_{25}v_{12}v_{11}v_{24}v_{23}v_{10}v_9v_8v_7v_6v_5v_4v_3v_2v_1v_{14}v_{13}v_{15}v_{16}v_{17}v_{18}v_{19}v_{20}v_{21}v_{22}$;

for $k = 11$, $v_0v_{21}v_{10}v_9v_{20}v_{19}v_8v_7v_{18}v_{17}v_6v_5v_4v_3v_2v_1v_{12}v_{11}v_{13}v_{14}v_{15}v_{16}$,
$v_0v_{21}v_{10}v_9v_{20}v_{19}v_8v_7v_6v_5v_4v_3v_2v_1v_{12}v_{11}v_{13}v_{14}v_{15}v_{16}v_{17}v_{18}$;
for $k = 9$, $v_0v_{17}v_8v_7v_{16}v_{15}v_6v_5v_4v_3v_2v_1v_{10}v_9v_{11}v_{12}v_{13}v_{14}$;
v_0-v_{2k-3} path: $v_0v_{2k-1}\ v_1v_2\ v_3v_4\ v_5v_6\ \ldots\ v_{2k-7}v_{2k-6}\ v_{2k-5}v_{2k-4}\ v_{2k-2}v_{2k-3}$;
v_0-v_{2k-2} path: $v_0v_{2k-1}\ v_1v_2\ v_3v_4\ v_5v_6\ \ldots\ v_{2k-7}v_{2k-6}\ v_{2k-5}v_{2k-4}\ v_{2k-3}v_{2k-2}$;
(Observe that, in the following path, the first and the last edges are colored red.)
v_0-v_{2k-1} path: $v_0\ v_1v_2\ v_3v_4\ v_5v_6\ \ldots\ v_{2k-7}v_{2k-6}\ v_{2k-5}v_{2k-4}\ v_{2k-3}v_{2k-2}\ v_{2k-1}$.
This completes the proof. □

From Lemma 2.2, we have the following result.

Theorem 2.2. *If G is a graph with n vertices, $n \geq 10$, $n \equiv 2\ (mod\ 4)$, such that $Circ(n : \{1, 2, \frac{n}{2}\}) \subseteq G$, then $hpc(G) = 2$.*

Theorem 2.2 is open for $n \equiv 0 \pmod 4$. We show that it is true for $n = 8$, i.e.,

$$hpc(Circ(8 : \{1, 2, 4\})) = 2.$$

Let $G = Circ(8 : \{1, 2, 4\})$ and $F = \{v_iv_{i+1} : i \in \{1, 3, 5, 7\}\}$, where $v_8 = v_0$. Then F is a 1-factor of G. Define an edge-coloring c of G by assigning color red to each edge of F and color blue to the remaining edges of G. We show that, for every vertex v_j, $j \neq 0$, of G, there is a proper Hamilton v_0-v_j path in G. v_0-v_1 path: $v_0v_7\ v_6v_5\ v_4v_3\ v_2v_1$;
v_0-v_2 path: $v_0v_7\ v_6v_5\ v_4v_3\ v_1v_2$; v_0-v_3 path: $v_0v_7\ v_1v_2\ v_6v_5\ v_4v_3$;
v_0-v_4 path: $v_0v_7\ v_1v_2\ v_6v_5\ v_3v_4$; v_0-v_5 path: $v_0v_7\ v_1v_2\ v_3v_4\ v_6v_5$;
v_0-v_6 path: $v_0v_7\ v_1v_2\ v_3v_4\ v_5v_6$; v_0-v_7 path: $v_0v_1\ v_2v_3\ v_4v_5\ v_6v_7$. □

Suppose that $G_0 = (V_0, E_0)$ and $G_1 = (V_1, E_1)$ are two disjoint graphs with $|V_0| = |V_1|$. A $1-1$ *connection* between G_0 and G_1 is defined as an edge set $E_c = \{(v, \overline{v}) \mid v \in V_0, \overline{v} = \phi(v) \in V_1$ and $\phi : V_0 \to V_1$ is a bijection$\}$. $G_0 \oplus G_1$ denotes the graph $G = (V_0 \cup V_1, E_0 \cup E_1 \cup E_c)$. Thus, ϕ induces a 1-factor in $G_0 \oplus G_1$.

Theorem 2.3. *(See Theorem 9.15 of* [3]*) $G_0 \oplus G_1$ is Hamiltonian-connected if both G_0 and G_1 are Hamiltonian-connected and $|V(G_0)| = |V(G_1)| \geq 3$.*

Theorem 2.4. *Suppose that $G_0 = (V_0, E_0)$ and $G_1 = (V_1, E_1)$ are two disjoint Hamiltonian-connected graphs with an even number $|V_0| = |V_1| \geq 4$ of vertices. If, for each $i \in \{0, 1\}$, there is a proper Hamilton path 2-coloring c_i of G_i with colors blue and red such that for any two vertices u and v of G_i, there is a proper Hamilton u-v path in G_i with the first and the last edges colored blue, then there is a proper Hamilton path 2-coloring c of $G_0 \oplus G_1$ with colors blue and red such that for any two vertices x and y of $G_0 \oplus G_1$, there is a proper Hamilton x-y path in $G_0 \oplus G_1$ with the first and the last edges colored blue. So, $hpc(G_0 \oplus G_1) = 2$.*

Proof. Define c so that c restricted to E_0 is c_0, c restricted to E_1 is c_1, and the edges of E_c are colored red. Without loss of generality, we have the following two cases: (1) both x and y are in G_0; (2) x is in G_0 and y is in G_1.

First, assume that both x and y are in G_0. By hypothesis, there exists a proper Hamilton path P of G_0 joining x and y with the first and the last edges colored blue. The path P can be written as (x, P_1, w, z, P_2, y) with $c_0(wz) =$ red. Obviously, $\overline{w} \neq \overline{z}$ and, by hypothesis, there exists a proper Hamilton path Q of G_1 joining $\overline{w}$ and $\overline{z}$ with the first and the last edges colored blue. Thus, $(x, P_1, w, \overline{w}, Q, \overline{z}, z, P_2, y)$ forms a proper Hamilton path of $G_0 \oplus G_1$ joining x and y with the first and the last edges colored blue.

Next, assume that x is in G_0 and y is in G_1. Since $|V(G_0)| = |V(G_1)| \geq 4$, there exists a vertex z in G_0 such that $x \neq z$ and $\overline{z} \neq y$. Thus, there exists a proper Hamilton path P of G_0 joining x and z with the first and the last edges colored blue and there exists a proper Hamilton path Q of G_1 joining $\overline{z}$ and y with the first and the last edges colored blue. Obviously, $(x, P, z, \overline{z}, Q, y)$ forms a proper Hamilton path of $G_0 \oplus G_1$ joining x and y with the first and the last edges colored blue. This completes the proof. □

Next, we observe that, for any integer $k \geq 5$, $G_0 = Circ(2k : \{1, 2, 3, 4\})$ satisfies the hypothesis of the previous theorem. By the proof of Case 1 of Lemma 2.1, it is enough if we define c to the edges of length 4 and to find a proper Hamilton v_0-v_{2k-1} path. Color the edges of length 4 by blue and the required path is: $v_0v_{2k-4}\ v_{2k-3}v_{2k-2}v_1v_2v_3 \ldots v_{2k-7}v_{2k-6}\ \ v_{2k-5}v_{2k-1}$. Also, we observe that, for any odd integer $k \geq 5$, $Circ(2k : \{1, 2, 3, k-1, k\})$ satisfies the hypothesis of the previous theorem. By the proof of Lemma 2.2, it is enough if we define c to the edges of lengths 3 and $k-1$ so that we have a proper Hamilton v_0-v_{2k-1} path. Color the edges of length 3 by red and length $k-1$ by blue and the required path is: $v_0v_{k+1}\ v_{k+2}v_{k+3}\ v_{k+4}v_{k+5} \ldots v_{2k-5}v_{2k-4}\ v_{2k-3}v_{2k-2}\ v_1v_2\ v_3v_4 \ldots v_{k-2}v_{k-1}\ v_kv_{2k-1}$.

3. Graphs with $hpc = 3$

I. Known graphs G with $hpc(G) = 3 = \chi'(G)$ are: $C_{2n+1}\Box K_2$ and H_k. Let G be a Hamiltonian-connected graph with $\chi'(G) = 3$. To show that $hpc(G) = 3$, we must show that G has no proper Hamilton path 2-coloring.

II. Known graph G with $\chi'(G) \geq 4$ and $hpc(G) = 3$ is: F_k.

Theorem 3.1. *For $k \geq 2$, $hpc(Circ(4k : \{1, 2k\})) = 3$.*

Proof. Let $G = Circ(4k : \{1, 2k\})$. Consider the proper 3-edge-coloring $(\{v_iv_{i+1} : i \in \{0, 2, 4, \ldots, 4k-2\}\}, \{v_iv_{i+1} : i \in \{1, 3, 5, \ldots, 4k-1\}\}, \{v_iv_{i+2k} : i \in \{0, 1, 2, \ldots, 2k-1\}\})$ of the 3-regular graph G. Thus $\chi'(G) = 3$. It remains to show that G has no proper Hamilton path 2-coloring. Assume, to the contrary, that there is a proper Hamilton path 2-coloring c of G.

Claim 1. The Hamilton paths from v_0 to v_{2k} are

$P_1 := v_0v_1v_2v_3 \ldots v_{2k-2}v_{2k-1} - v_{4k-1}v_{4k-2}v_{4k-3} \ldots v_{2k+1}\,v_{2k}$ and

$P_2 := v_0v_{4k-1}v_{4k-2}v_{4k-3} \ldots v_{2k+2}v_{2k+1} - v_1v_2v_3 \ldots v_{2k-1}\,v_{2k}$.

Assume, by symmetry, the edge v_0v_1 is in P, a Hamilton path from v_0 to v_{2k}. Then, $v_0v_{4k-1} \notin E(P)$ and so $v_{4k-1}v_{4k-2} \in E(P)$ and $v_{4k-1}v_{2k-1} \in E(P)$. Suppose $v_{2k-1}v_{2k} \in E(P)$, then $P := v_0v_1 \ldots v_{4k-2}v_{4k-1}v_{2k-1}v_{2k}$; it follows that $P^{-1} := v_{2k}v_{2k-1}v_{4k-1}v_{4k-2}\ v_{2k-2}v_{2k-3}v_{4k-3}v_{4k-4}v_{2k-4}v_{2k-5}v_{4k-5}v_{4k-6} \cdots$; now the vertex $v_{2k+1} \notin P$, a contradiction. Hence, $v_{2k-1}v_{2k} \notin E(P)$. So, $v_{2k}v_{2k+1} \in E(P)$. Thus $P := v_0v_1 \ldots - \ldots v_{2k+1}v_{2k}$. Consequently, $P := v_0v_1v_2 \ldots - \ldots v_{2k+2}v_{2k+1}v_{2k}$ and therefore, $P = P_1$.

Claim 2. The Hamilton paths from v_0 to v_2 are

$Q_1 := v_0v_1v_{2k+1} - v_{2k}v_{2k-1} - v_{4k-1}v_{4k-2} - v_{2k-2}v_{2k-3} - v_{4k-3}v_{4k-4} - v_{2k-4}v_{2k-5}$
$-v_{4k-5}v_{4k-6} - \cdots - v_6v_5 - v_{2k+5}v_{2k+4} - v_4v_3 - v_{2k+3}v_{2k+2} - v_2$ and

$Q_2 := v_0 - v_{2k}v_{2k-1} - v_{4k-1}v_{4k-2} - v_{2k-2}v_{2k-3} - v_{4k-3}v_{4k-4} - v_{2k-4}v_{2k-5}$
$-v_{4k-5}v_{4k-6} - \cdots - v_6v_5 - v_{2k+5}v_{2k+4} - v_4v_3 - v_{2k+3}v_{2k+2} - v_{2k+1}v_1v_2$.

Since $N(v_1) = \{v_0, v_2, v_{2k+1}\}$, any Hamilton path Q from v_0 to v_2 contains $v_0v_1v_{2k+1}$ or $v_{2k+1}v_1v_2$ but not both. Assume, by symmetry, $Q := v_0v_1v_{2k+1} \ldots v_2$. Edge $v_0v_{4k-1} \notin E(Q)$ implies $v_{4k-2}v_{4k-1}v_{2k-1}$ is in Q and $v_0v_{2k} \notin E(Q)$ implies $v_{2k-1}v_{2k}v_{2k+1}$ is in Q. Hence, $Q := v_0v_1v_{2k+1} - v_{2k}v_{2k-1} - v_{4k-1}v_{4k-2} - \cdots - v_2$. Now, $v_{2k-1}v_{2k-2} \notin E(Q)$ implies $v_{2k-3}v_{2k-2}v_{4k-2}$ is in Q. Proceeding in this way, we get $Q = Q_1$.

We have four possibilities. If the paths required for c are P_1 and Q_1, then we have a contradiction, since $c(v_0v_1) \neq c(v_{2k-1}v_{4k-1})$ in P_1 and $c(v_0v_1) = c(v_{2k-1}v_{4k-1})$ in Q_1. If the paths required for c are P_1 and Q_2, then also we have a contradiction, since $c(v_{2k-3}v_{2k-2}) = c(v_{2k-1}v_{4k-1})$ in P_1 and $c(v_{2k-3}v_{2k-2}) \neq c(v_{2k-1}v_{4k-1})$ in Q_2. Similarly, the reason for P_2 and Q_1 is $c(v_1v_{2k+1}) \neq c(v_{2k-1}v_{2k})$ in P_2 and $c(v_1v_{2k+1}) = c(v_{2k-1}v_{2k})$ in Q_1; and the same for P_2 and Q_2 is $c(v_{2k+2}v_{2k+3}) \neq c(v_3v_4)$ in P_2 and $c(v_{2k+2}v_{2k+3}) = c(v_3v_4)$ in Q_2. This completes the proof. $\square$

Conclusion The conjecture '*if G is a Hamiltonian-connected graph, then $hpc(G) \leq 3$*' of Bi, Byers and Zhang [2] is verified for some classes of graphs (see Theorems 2.1, 2.2 and 3.1). Also, Theorem 2.4 generates more graphs that serve as a support to the conjecture.

We pose the following problems.

Problem 3.1. *Find $a_1 < a_2 < a_3$ such that for every integer $n \geq 2a_3 + 1$, $hpc(Circ(n : \{a_1, a_2, a_3\})) = 2$.*

If $(a_1, a_2, a_3) = (1, 2, 3)$, then we have Lemma 2.1.

Problem 3.2. *Find $a_1 < a_2$ such that for every odd integer $k \geq 2a_2 + 1$, $hpc(Circ(2k : \{a_1, a_2, k\})) = 2$.*

If $(a_1, a_2) = (1, 2)$, then we have Lemma 2.2.

In the next two sections, we consider Hamiltonian-laceable graphs and apply the *hpc -Conjecture.*

4. Hamiltonian-laceable graphs

A bipartite graph with bipartition (X, Y) is *Hamiltonian-laceable* if there exists a Hamilton path joining any two vertices from different partite sets; that is, one in X and one in Y. For a Hamiltonian-laceable graph G, an edge-coloring $c : E(G) \to \{1, 2, \ldots, k\}$ is a *proper Hamilton path k-coloring* if every two vertices from different partite sets of G are connected by a proper Hamilton path in G. The minimum number k of colors in a proper Hamilton path k-coloring of G is also called the *proper Hamiltonian-connection number* of G, but it is denoted by $\mathrm{hpc}_b(G)$.

5. Graphs with $\mathrm{hpc}_b = 2$

Let G be a Hamiltonian-laceable graph with bipartition (X, Y). To show that $hpc_b(G) = 2$, we must show that G has a 2-edge-coloring with the property that for every two vertices $u \in X$ and $v \in Y$ of G, there is a proper Hamilton u-v path in G.

Lemma 5.1. *For every integer $n \geq 5$, $hpc_b(Circ(2n : \{1, 3, 5\})) = 2$.*

Proof. Let $G = Circ(2n : \{1, 3, 5\})$ and $F = \{v_iv_{i+1} : i \in \{1, 3, 5, \ldots, 2n-1\}\}$, where $v_{2n} = v_0$. Then, F is a 1-factor of G. Let $X = \{v_i : i \in \{0, 2, 4, \ldots, 2n-2\}\}$ and $Y = \{v_i : i \in \{1, 3, 5, \ldots, 2n-1\}\}$. Define an edge-coloring c of G by assigning color blue to each edge of F and color red to the remaining edges of G. As the edge-colored G is vertex-transitive, we show that for every vertex $v_j \in Y$ of G, there is a proper Hamilton v_0-v_j path in G.

(Observe that, in the following paths, the first and the last edges are colored blue.)

v_0-v_1 path: $v_0v_{2n-1}v_{2n-2}v_{2n-3} \ldots v_4v_3v_2v_1$;

v_0-v_3 path: for $n \geq 6$, $v_0v_{2n-1}v_{2n-2}v_{2n-3} \ldots v_8v_7v_6v_5v_2v_1v_4v_3$;
for $n = 5$, $v_0v_9v_8v_7v_6v_5v_2v_1v_4v_3$;

v_0-v_5 path: for $n \geq 6$, $v_0v_{2n-1}v_{2n-2}v_{2n-3} \ldots v_8v_7v_2v_1v_4v_3v_6v_5$;
for $n = 5$, $v_0v_9v_8v_7v_2v_1v_4v_3v_6v_5$;

v_0-v_7 path: for $n \geq 7$, $v_0v_{2n-1}v_{2n-2}v_{2n-3} \ldots v_{10}v_9v_6v_5v_2v_1v_4v_3v_8v_7$;
for $n = 6$, $v_0v_{11}v_{10}v_9v_6v_5v_2v_1v_4v_3v_8v_7$; for $n = 5$, $v_0v_9v_6v_5v_2v_1v_4v_3v_8v_7$;

Assume $n \geq 6$ and $j \in \{5, 6, 7, \ldots, n-1\}$:

v_0-v_{2j-1} path, if $j \equiv 0 \pmod 2$: for $n \geq 10$,

$$v_0v_{2n-1}v_{2n-2}v_{2n-3} \ldots v_{2j+2}v_{2j+1}\ v_{2j-2}v_{2j-3}\ v_{2j-6}v_{2j-7}\ v_{2j-10}v_{2j-11} \ldots v_{10}v_9\ v_6v_5\ v_2v_1\ v_4v_3\ v_8v_7\ v_{12}v_{11} \ldots v_{2j-8}v_{2j-9}\ v_{2j-4}v_{2j-5}\ v_{2j}v_{2j-1};$$

for $n = 9$, $v_0v_{17}v_{16}v_{15}v_{14}v_{13}v_8v_7v_2v_1v_4v_3v_6v_5v_{10}v_9v_{12}v_{11}$,
$v_0v_{17}v_{14}v_{13}v_{10}v_9v_6v_5v_2v_1v_4v_3v_8v_7v_{12}v_{11}v_{16}v_{15}$;
for $n = 8$, $v_0v_{15}v_{14}v_{13}v_8v_7v_2v_1v_4v_3v_6v_5v_{10}v_9v_{12}v_{11}$;
for $n = 7$, $v_0v_{13}v_8v_7v_2v_1v_4v_3v_6v_5v_{10}v_9v_{12}v_{11}$;
v_0-v_{2j-1} path, if $j \equiv 1 \pmod 2$: for $n \geq 10$, $v_0v_{2n-1}v_{2n-2}v_{2n-3} \ \cdots \ v_{2j+2}v_{2j+1}$
$v_{2j-2}v_{2j-3} \ v_{2j-6}v_{2j-7} \ v_{2j-10}v_{2j-11} \ \cdots \ v_{12}v_{11} \ v_8v_7 \ v_2v_1$
$v_4v_3 \ v_6v_5 \ v_{10}v_9v_{14}v_{13} \ \cdots \ v_{2j-8}v_{2j-9} \ v_{2j-4}v_{2j-5} \ v_{2j}v_{2j-1}$;
for $n = 9$, $v_0v_{17}v_{16}v_{15}v_{14}v_{13}v_{12}v_{11}v_8v_7v_2v_1v_4v_3v_6v_5v_{10}v_9$,
$v_0v_{17}v_{16}v_{15}v_{12}v_{11}v_8v_7v_2v_1v_4v_3v_6v_5v_{10}v_9v_{14}v_{13}$;
for $n = 8$, $v_0v_{15}v_{14}v_{13}v_{12}v_{11}v_8v_7v_2v_1v_4v_3v_6v_5v_{10}v_9$,
$v_0v_{15}v_{12}v_{11}v_8v_7v_2v_1v_4v_3v_6v_5v_{10}v_9v_{14}v_{13}$;
for $n = 7$, $v_0v_{13}v_{12}v_{11}v_8v_7v_2v_1v_4v_3v_6v_5v_{10}v_9$;
for $n = 6$, $v_0v_{11}v_8v_7v_2v_1v_4v_3v_6v_5v_{10}v_9$;
(Observe that, in the following path, the first and the last edges are colored red.)
v_0-v_{2n-1} path: $v_0 \ v_1v_2 \ v_3v_4 \ v_5v_6 \ldots v_{2n-4}v_{2n-3} \ v_{2n-2}v_{2n-1}$. This completes the proof. □

Theorem 5.1. *Let G be a bipartite graph with $n \geq 5$ vertices in each partite set. If $Circ(2n : \{1, 3, 5\}) \subseteq G$, then $hpc_b(G) = 2$.*

Corollary 5.1. *For $n \geq 5$, $hpc_b(K_{n,n}) = 2$.*

Theorem 5.2. *(See Theorem 9.17 of* [3]*) Assume that G_0, G_1, and $G_0 \oplus G_1$ are bipartite graphs such that $|V(G_0)| = |V(G_1)| \geq 2$. Then $G_0 \oplus G_1$ is Hamiltonian-laceable if both G_0 and G_1 are Hamiltonian-laceable.*

Theorem 5.3. *Suppose that $G_0 = (V_0^0 \cup V_0^1, E_0)$ and $G_1 = (V_1^0 \cup V_1^1, E_1)$ are two disjoint Hamiltonian-laceable graphs with $|V_0^0| = |V_0^1| = |V_1^0| = |V_1^1| \geq 2$, where (V_i^0, V_i^1) is a bipartition of G_i, $i \in \{0, 1\}$. If, for each $i \in \{0, 1\}$, there is a proper Hamilton path 2-coloring c_i of G_i with colors blue and red such that for every two vertices $u \in V_i^0$ and $v \in V_i^1$ of G_i, there is a proper Hamilton u-v path in G_i with the first and the last edges colored blue, then there is a proper Hamilton path 2-coloring c of $G_0 \oplus G_1$ with colors blue and red such that for every two vertices x and y of $G_0 \oplus G_1$, there is a proper Hamilton x-y path in $G_0 \oplus G_1$ with the first and the last edges colored blue. So, $hpc_b(G_0 \oplus G_1) = 2$.*

Proof. Define c so that c restricted to E_0 is c_0, c restricted to E_1 is c_1, and the edges of E_c are colored red. By the symmetric property of $G_0 \oplus G_1$, without loss of generality we can assume the following two cases:
Case 1. $x \in V_0^0$ and $y \in V_0^1$. By hypothesis, there exists a proper Hamilton path P of G_0 joining x and y with the first and the last edges colored blue. The path P can be written as (x, P_1, w, z, P_2, y) with $c_0(wz) = \text{red}$, $w \in V_0^1$ and $z \in V_0^0$. Obviously, $\overline{w} \in V_1^1$ and $\overline{z} \in V_1^0$. Thus, there exists a proper Hamilton path Q of G_1 joining $\overline{w}$ and $\overline{z}$ with the first and the last edges colored blue. Thus, $(x, P_1, w, \overline{w}, Q, \overline{z}, z, P_2, y)$ forms a proper Hamilton path of $G_0 \oplus G_1$ with the first and the last edges colored blue.
Case 2. $x \in V_0^0$ and $y \in V_1^1$. Then, there exists a vertex z in V_0^1. Obviously, $\overline{z} \in V_1^0$. Thus, there exists a proper Hamilton path P of G_0 joining x to z with the first and the last edges colored blue and there exists a proper Hamilton path Q of G_1 joining $\overline{z}$ to y with the first and the last edges colored blue. Obviously, $(x, P, z, \overline{z}, Q, y)$ forms a proper Hamilton path of $G_0 \oplus G_1$ with the first and the last edges colored blue. This completes the proof. □

Next, we observe that, for any even integer $n \geq 10$, $Circ(2n : \{1, 3, 5, 7, 9\})$ satisfies the hypothesis of the previous theorem. By the proof of Lemma 5.1, it is enough if we define c

to the edges of lengths 7 and 9 so that we have a proper Hamilton v_0-v_{2k-1} path. Color the edges of lengths 7 and 9 by blue, the required path is v_0-v_{2n-1} path: $v_0v_{2n-9}\ v_{2n-10}v_{2n-11}\ v_{2n-12}v_{2n-13}\cdots v_4v_3\ v_2v_1\ v_{2n-2}v_{2n-3}v_{2n-4}v_{2n-5}\ v_{2n-6}v_{2n-7}\ v_{2n-8}v_{2n-1}$.

6. Graphs with $\mathrm{HPC}_b = 3$

Let G be a Hamiltonian-laceable graph with $\chi'(G) = 3$. To show that $hpc_b(G) = 3$, we must show that G has no proper Hamilton path 2-coloring.

Theorem 6.1. *For each integer $n \geq 2$, $hpc_b(C_{2n}\square K_2) = 3$.*

Proof. Construct $G = C_{2n}\square K_2$ from the two $2n$-cycles $u_1u_2u_3 \ldots u_{2n-1}u_{2n}u_1$ and $v_1v_2v_3 \ldots v_{2n-1}v_{2n}v_1$ by adding the $2n$ edges u_iv_i for $i \in \{1, 2, \ldots, 2n\}$. Let $X = \{u_1, u_3, u_5, \ldots, u_{2n-3}, u_{2n-1}\} \cup \{v_2, v_4, v_6, \ldots, v_{2n-2}, v_{2n}\}$ and $Y = \{u_2, u_4, u_6, \ldots, u_{2n-2}, u_{2n}\} \cup \{v_1, v_3, v_5, \ldots, v_{2n-3}, v_{2n-1}\}$. Then (X, Y) is a bipartition of G. Note that $\chi'(G) = 3$. Assume, to the contrary, that there is a proper Hamilton path 2-coloring c of G.

First, consider a Hamilton u_1-v_1 path P in G. P begins with u_1u_2 or u_1u_{2n} and ends with v_2v_1 or $v_{2n}v_1$. Assume, by symmetry, P begins with u_1u_2.

If P ends with v_2v_1, then, as $u_1u_{2n} \notin E(P)$ and $v_1v_{2n} \notin E(P)$, we have the subpath $u_{2n-1}u_{2n}v_{2n}v_{2n-1}$ in P. Again, as $u_{2n-1}v_{2n-1} \notin E(P)$, we have $u_{2n-2}u_{2n-1}, v_{2n-2}v_{2n-1} \in E(P)$. Proceeding in this way, we get $P = u_1u_2u_3 \ldots u_{2n-2}u_{2n-1}u_{2n}v_{2n}v_{2n-1}v_{2n-2} \ldots v_3v_2v_1 = P_1$.

If P ends with $v_{2n}v_1$, then, as $v_1v_2 \notin E(P)$, we have the subpath $u_2v_2v_3$ in P. Since $u_2u_3 \notin E(P)$, the subpath $v_3u_3u_4$ in P. As $v_3v_4 \notin E(P)$, the subpath $u_4v_4v_5$ in P. Proceeding in this way, we get $P = u_1u_2v_2v_3u_3u_4v_4v_5 \ldots v_{2n-1}u_{2n-1}u_{2n}v_{2n}v_1 = P_2$.
Next, consider Hamilton u_3-v_3 paths in G. By the above argument, the paths are:

$Q_1 = u_3u_4u_5 \ldots u_{2n-2}u_{2n-1}u_{2n}u_1u_2v_2v_1v_{2n}v_{2n-1}v_{2n-2} \ldots v_5v_4v_3$,
$Q_2 = u_3u_4v_4v_5u_5u_6v_6v_7 \ldots v_{2n-1}u_{2n-1}u_{2n}v_{2n}v_1u_1u_2v_2v_3$,
$Q_3 = u_3u_2u_1u_{2n}u_{2n-1}u_{2n-2} \ldots u_5u_4v_4v_5v_6 \ldots v_{2n-1}v_{2n}v_1v_2v_3$, and
$Q_4 = u_3u_2v_2v_1u_1u_{2n}v_{2n}v_{2n-1}u_{2n-1}u_{2n-2}v_{2n-2}v_{2n-3} \ldots v_5u_5u_4v_4v_3$.

If the paths required in c are P_1 and Q_2, then, we have a contradiction, since $c(u_{2n}v_{2n}) = c(v_{2n-1}v_{2n-2})$ in P_1 and $c(u_{2n}v_{2n}) \neq c(v_{2n-1}v_{2n-2})$ in Q_2.

If the paths required in c are P_1 and Q_4, then, we have a contradiction, since $c(u_{2n}v_{2n}) = c(u_{2n-2}u_{2n-1})$ in P_1 and $c(u_{2n}v_{2n}) \neq c(u_{2n-2}u_{2n-1})$ in Q_4.

If the paths required in c are P_2 and Q_1, then, we have a contradiction, since $c(u_{2n-1}u_{2n}) = c(v_{2n-2}v_{2n-1})$ in P_2 and $c(u_{2n-1}u_{2n}) \neq c(v_{2n-2}v_{2n-1})$ in Q_1.

If the paths required in c are P_2 and Q_3, then, we have a contradiction, since $c(u_{2n-1}u_{2n}) = c(v_{2n-2}v_{2n-1})$ in P_2 and $c(u_{2n-1}u_{2n}) \neq c(v_{2n-2}v_{2n-1})$ in Q_3.

If the paths required in c are P_1 and Q_1, then, there is no proper Hamilton u_1-v_3 path in G. To see this, consider the first edge of this path. If it is either u_1v_1 or u_1u_{2n}, then the edges v_2u_2 and u_2u_3 with same color are in the path. Otherwise, it is u_1u_2, and the edges $u_{2n}v_{2n}$ and $v_{2n}v_1$ with same color are in the path. A contradiction.

If the paths required in c are P_1 and Q_3, then, there is no proper Hamilton u_1-v_3 path in G. To see this, consider the first edge of this path. If it is u_1u_2, then the edges $u_{2n}v_{2n}$ and $v_{2n}v_1$ with same color are in the path. If it is u_1u_{2n}, then we have the subpath $v_1v_2u_2u_3$ in the path; now the edge v_2u_2 has no color. If it is u_1v_1, then we have the subpath $v_2u_2u_3$, with color $1, 2$ in order, in the path; now there is no second edge for this path. A contradiction.

If the paths required in c are P_2 and Q_4, then, each of the edges in the two $2n$-cycles $u_1u_2u_3 \ldots u_{2n-1}u_{2n}u_1$ and $v_1v_2v_3 \ldots v_{2n-1}v_{2n}v_1$ are of one color, say 1, and each of the $2n$ edges u_iv_i, $i \in \{1, 2, \ldots, 2n\}$, are of another color, say 2. Now, there is no proper Hamilton u_1-v_3 path in G, a contradiction.

If the paths required in c are P_2 and Q_2, then, there is no proper Hamilton u_1-u_4 path R in G. To see this, consider the first edge of R. If it is u_1u_{2n}, then we have the subpath $u_2v_2v_1v_{2n}$; as the edges u_2v_2 and v_1v_{2n} are of different colors, there is no color for the edge v_2v_1. So it is either u_1u_2 or u_1v_1. First, assume that it is u_1u_2. If $R = u_1u_2u_3\ldots$, then $R = u_1u_2u_3u_4$. So, $R = u_1u_2v_2\ldots$ and therefore $R = u_1u_2v_2\ldots v_3u_3u_4$. As $R \neq u_1u_2v_2v_3u_3u_4$, $R = u_1u_2v_2v_1\ldots v_3u_3u_4$. Thus $R = u_1u_2v_2v_1u_1$. Next, assume that it is u_1v_1. By symmetry, assume that the last edge of R is v_4u_4. As u_1u_2 and u_3u_4 are not in R, $R = u_1v_1\ldots v_2u_2u_3v_3\ldots v_4u_4$. Since v_2v_3 is not in R, $R = u_1v_1v_2u_2u_3v_3v_4u_4$. A contradiction. This completes the proof. □

Using the following two facts, we have:

If $n \geq 2$, then, for any edge e in $C_{2n}\square K_2$, $\chi'((C_{2n}\square K_2) - e) = 3$, and it is known that (see Lemma 9.3 of [3]), $(C_{2n}\square K_2) - e$ is Hamiltonian-laceable.

If H is a Hamiltonian-laceable spanning subgraph of a Hamiltonian-laceable graph G, then $\text{hpc}_b(G) \leq \text{hpc}_b(H)$.

Corollary 6.1. *For $n \geq 2$ and for any edge e in $C_{2n}\square K_2$, $hpc_b((C_{2n}\square K_2) - e) = 3$.*

We pose the following problem.

Problem 6.1. *Find odd integers $a_1 < a_2 < a_3$ such that for every integer $n \geq a_3$, $hpc_b(Circ(2n : \{a_1, a_2, a_3\})) = 2$.*

If $(a_1, a_2, a_3) = (1, 3, 5)$, then we have Lemma 5.1.

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