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QUENCHING FOR A REACTION-DIFFUSION EQUATION WITH WEAK SINGULARITIES

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ABSTRACT. This paper studies the following reaction-diffusion equation with a weak singular boundary condition. The primary objective for this problem is to analyze the quenching properties. It is obtained that finite time quenching occurs on the left boundary, the time derivative of the solution blows up at the same time and also quenching rate estimates of the solution of the eqaution $k_t(x,t) = k_{xx}(x,t) + \ln \alpha k(x,t)$, $(x,t) \in (0,1) \times (0,T)$ with $k_x(0,t) = -\ln \beta k(0,t)$, $k_x(1,t) = 0$, $t \in (0,T)$ and initial function $k(x,0) = k_0(x)$ with $[0,1] \to (0,1)$ where $0 < \alpha, \beta < 1$ and T is a finite time.

Keywords: Reaction-diffusion equation, Singular boundary condition, Quenching, Maximum principles.

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1. Introduction

This paper studies quenching of solutions in a reaction-diffusion equation with weak singularities:

$$\begin{cases} k_t = k_{xx} + \ln(\alpha k), \ x \in (0,1), \ t \in (0,T), \\ k_x(0,t) = -\ln(\beta k(0,t)), \ k_x(1,t) = 0, \ t \in (0,T), \\ k(x,0) = k_0(x), \ x \in [0,1], \end{cases} \tag{1}$$

where $0 < \alpha, \beta < 1$ and $T \in (0,\infty)$. The initial function $k_0 : [0,1] \to (0,1)$ satisfies the compatibility conditions

$$k_0'(0) = -\ln(\beta k_0), \ k_0'(1) = 0.$$

The problem (1) arises in this paper of the Micro-Electro Mechanical System devices coming out of a thin dielectric elastic membrane. The dynamic solution characterizes the dynamic deflection of the elastic membrane in these models (see [1],[5],[14]). The primary objective of this paper is to analyze the quenching properties of (1). Now, we define quenching phenomena.

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Definition 1.1 The solution of the problem (1) is said to quench if there exists a finite time $T = T(k_0) < \infty$ such that

$$\lim_{t \to T^-} \min\{k(x,t) : 0 \leq x \leq 1\} \to 0.$$

Lately, the authors considered to get quenching properties of various reaction-diffusion equations with two singular boundary conditions ([3],[4],[8],[9],[10],[11],[14]). In literature, quenching problem with weak singularities of logarithmic type is less studied ([2],[6],[7],[12],[13]). In [6], the author studied the following diffusion problem

$$\begin{cases} k_t = k_{xx} + \log(\alpha k), \text{ in } (-l,l) \times (0,T), \\ k(\pm l, t) = 1, \ t \in [0,T), \\ k(x,0) = k_0(x), \ x \in [-l,l]. \end{cases} \tag{2}$$

He showed that quenching points are finite under specific conditions on the initial function. Also, he derived quenching rate

$$\lim_{t \to T} \left(1 + \frac{1}{T-t} \int_0^{k(x,t)} \frac{ds}{\log(\alpha s)} \right)$$

uniformly for $|x| < C\sqrt{T-t}$. In [12], the authors considered a parabolic system with Neumann boundary conditions:

$$\begin{cases} k_t = k_{xx} + \log(\alpha j), j_t = j_{xx} + \log(\beta k), \ x \in (0,1), t \in (0,T), \\ k_x(0,t) = 0 = k_x(1,t), \ t \in (0,T), \\ j_x(0,t) = 0 = j_x(1,t), \ t \in (0,T), \\ 0 < k(x,0) = u_0(x) \leq 1, x \in [0,1], \\ 0 < j(x,0) = v_0(x) \leq 1, x \in [0,1], \end{cases} \tag{3}$$

where $0 < \alpha, \beta < 1$. They showed that quenching is always non-simultaneous. Further, they also gave the quenching rate estimate for this non-simultaneous quenching.

Until now in literature, the quenching problem with two weak singularities of logarithmic type have not been studied. Here, we deal with the quenching character of the problem (1) motivated by problems (2) and (3). Here, we suppose initial function k_0 satisfies

$$k_{xx}(x,0) + \ln(\alpha k(x,0)) \leq 0, \tag{4}$$

$$k_x(x,0) \geq 0. \tag{5}$$

This paper is organized as follows. In Section 2, we obtain that single quenching point is $x = 0$ in finite time and k_t blows up at the same time by using the certain assumptions in (1). Also, we obtain quenching rate estimates.

2. Quenching Properties

Firstly, We can easily prove the existence of positive local solution of the problem (1) for some $T > 0$ in [9]. This problem have two nonlinear sources which are emissions. Therefore, it can be predicted that the solution of the problem goes to zero for positive initial functions that do not take too large value. The conditions on the initial functions determine at what point and at what time (finite or infinite) the quenching event will occur.

Also, we can easily show that $k_0(x)$ providing (4), (5) and compatibility conditions. Indeed, we assume that the conditions (4), (5) are proper.

Remark 2.1. $k_0(x) = 1.1 - (1-x)^{3.912}$ satisfies compatibility conditions, (4) and (5) where any value $\alpha \in (0,1)$ and $\beta = 0.2$. ($\ln 0.02 = -3.912$)

Lemma 2.1. If k_0 satisfies (4) and (5), then we get $k_t < 0$ and $k_x > 0$ in $(0,1) \times (0,T)$, respectively.
Proof. Let $G = k_t$, $H = k_x$, we have

$$\begin{aligned} G_t - G_{xx} - G/k &= 0, x \in (0,1), t \in (0,T), \\ G_x(0,t) + G(0,t)/k(0,t) &= 0, t \in (0,T), \\ G_x(1,t) &= 0, t \in (0,T), \\ G(x,0) &= k_{xx}(x,0) + \ln(\alpha k(x,0)) \le 0, x \in [0,1], \end{aligned}$$

and

$$\begin{aligned} H_t - H_{xx} - H/k &= 0, x \in (0,1), t \in (0,T), \\ H(0,t) &= -\ln(\beta k(0,t)), t \in (0,T), \\ H(1,t) &= 0, t \in (0,T), \\ H(x,0) &= k_x(x,0) \ge 0, x \in [0,1]. \end{aligned}$$

where $0 < \beta < 1$ and $0 < k(0,t) < e$. From the maximum principle, it is obtained that $G = k_t \le 0$ and $H = k_x \ge 0$ in $[0,1] \times [0,T)$. From the strong maximum principle, it is also obtained that $G = k_t < 0$ and $H = k_x > 0$ in $(0,1) \times (0,T)$.

Theorem 2.1. k quenches in a finite time T_β if $k_0(x)$ satisfies (4).
Proof. Suppose that $k_0(x)$ satisfies (4). Thus, it is obtained by integration that

$$Z = \ln(\beta k(0,0)) + \int_0^1 \ln(\alpha k(x,0))\, dx < 0.$$

Let's define an auxliary function; $\Theta(t) = \int_0^1 k(x,t)\, dx, 0 < t < T$. In that case, it is obtained that

$$\Theta'(t) = \ln(\beta k(0,t)) + \int_0^1 \ln(\alpha k(x,t))\, dx \le \ln(\beta k(0,0)) + \int_0^1 \ln(\alpha k(x,0))\, dx \ = Z$$

from $k_t < 0$ by the previous lemma. Thus, $\Theta(t) \le \Theta(0) + Zt$. Namely, $\Theta(T_\beta) = 0$ for some $T_\beta = -\Theta(0)/Z$, which means quenching occurs in finite time T_β for $0 < T \le T_\beta$.

The proof of Corollary 2.1 is a trivial modification of Lemma 2.1.

Corollary 2.1. Let $\rho \in (0,T)$, $\epsilon > 0$ and $k_0 \ge \epsilon$. If we define $H = k_x - \epsilon$ for $(x,t) \in [0,1] \times [\rho,T)$, then we have $k_x \ge \epsilon$ for $(x,t) \in [0,1] \times [\rho,T)$.

Theorem 2.2. $x = 0$ is the single quenching point if k_0 satisfies (4) and (5).
Proof. Let $\rho \in (0,T)$, $M = ||k_0(x)||_\infty \le 1$ and $\delta > 0$. Define

$$\Phi(x,t) = k_x + \delta(1-x)\ln(\alpha M)$$

where $(x,t) \in [0,1] \times [\rho,T)$. $\Theta(x,t)$ supplies

$$\Phi_t - \Phi_{xx} - \frac{1}{k}\Phi = -\frac{\delta(1-x)\ln(\alpha M)}{k} > 0 \text{ in } (0,1) \times [\rho,T),$$

in $(0,1) \times (\rho,T)$. From the maximum principle, it is obtained that $\Theta(x,t)$ can't acquire a negative interior minimum. Let δ be a sufficiently small constant. It is obtained $\Theta(x,\rho) > 0$ from Corollary 2.1. On the other hand, it is achieved that

$$\begin{aligned} \Phi(0,t) &= -\ln(\beta k(0,t)) + \delta \ln(\alpha M) > 0, \\ \Phi(1,t) &= 0, \end{aligned}$$

where δ be a sufficiently small constant for $t \in (\rho, T)$. From the maximum principle, it is achieved that $\Phi(x,t) \geq 0$, namely $k_x \geq -\delta(1-x)\ln(\alpha M)$ where $(x,t) \in [0,1] \times [\rho, T)$. Taking integral with respect to x from 0 to x, it is achieve that

$$k(x,t) \geq k(0,t) - \delta \frac{2x - x^2}{2} \ln(\alpha M)$$

So k does not quench in $(0,1]$.

Theorem 2.3. If k_0 satisfies (4) and (5), then time derivative of k blows up at the finite time.

Proof. Let $\delta > 0$ and $\rho \in (0,T)$. Define

$$\chi(x,t) = k_t + \delta k_x$$

where $(x,t) \in [0,1] \times [\rho, T)$. Thus, $\chi(x,t)$ supplies

$$\chi_t - \chi_{xx} - \frac{1}{k}\chi = 0$$

in $(0,1) \times (\rho, T)$. $\chi(x,\rho) \leq 0$ if δ is small enough and by Lemma 2.1. Further, if δ is small enough,

$$\begin{aligned} \chi(0,t) &= k_t(0,t) + \delta k_x(0,t) < 0, \\ \chi(1,t) &= k_t(1,t) < 0 \end{aligned}$$

for $t \in (\rho, T)$. From the maximum principle, we obtain that $\chi(x,t) \leq 0$ for $(x,t) \in [0,1] \times [\rho, T)$. Namely, $k_t \leq -\delta k_x$ for $(x,t) \in [0,1] \times [\rho, T)$. For $x = 0$, we infer that

$$k_t(0,t) \leq \delta \ln(\beta k(0,t)), \tag{6}$$

and

$$\lim_{t \to T^-} k_t(0,t) \leq \lim_{t \to T^-} \delta \ln(\beta k(0,t)) = -\infty.$$

The theorem is proved.

Remark 2.2. By integrating (6), we infer that a quenching rate is

$$\int_{k(0,t)}^{0} \frac{d\eta}{\ln(\beta\eta)} \geq \delta(T-t). \tag{7}$$

Theorem 2.4. If k_0 satisfies (4)-(5) and $\alpha \geq \beta$, then there exists a positive constant c such that

$$\int_{k(0,t)}^{0} \frac{d\eta}{\ln(\beta\eta)} \leq c(T-t).$$

for t sufficiently close to T.

Proof. Define $\Omega(x,t) = k_x + \Psi(x)\ln(\beta k)$ in $[0,1] \times [0,T)$, where the function Ψ is defined in [13] and is nonnegative, nonincreasing, convex, C^2 function such that $\Psi(0) = 1$, $\Psi(1) = 0$, $\Psi(x) \leq -\frac{k_0'(x)}{\ln(\beta k_0(x))}$.

Let $\Omega(x,t) = k_x + \Psi(x)\ln(\beta k)$. It is achieved that

$$\Omega_t - \Omega_{xx} - \frac{1}{k}\Omega = \Psi(x)\frac{\ln(\alpha/\beta)}{k} - \Psi''(x)\ln(\beta k) - 2\Psi'(x)\frac{k_x}{k} + \Psi(x)\frac{k_x^2}{k^2} > 0$$

since $\alpha \geq \beta$ and $k_x > 0$, $J(x,t)$ cannot acquire a negative interior minimum. On the other hand, $\Omega(x,0) \geq 0$ from definition of $\Psi(x)$ and

$$\Omega(0,t) = 0, \ \Omega(1,t) = 0,$$

for $t \in (0,T)$. From the maximum principle, it is achieved that $\Omega(x,t) \geq 0$ for $(x,t) \in [0,1] \times [0,T)$. Therefore

$$\Omega_x(0,t) = \lim_{h \to 0^+} \frac{\Omega(h,t) - \Omega(0,t)}{h} = \lim_{h \to 0^+} \frac{\Omega(h,t)}{h} \geq 0.$$

It is infered that

$$\begin{aligned}
\Omega_x(0,t) &= k_{xx}(0,t) + \Psi'(0)\ln(\beta k(0,t)) + \Psi(0)\frac{k_x(0,t)}{k(0,t)} \\
&= k_{xx}(0,t) + \Psi'(0)\ln(\beta k(0,t)) + \frac{k_x(0,t)}{k(0,t)} \\
&= k_t(0,t) - \ln(\alpha k(0,t)) + \Psi'(0)\ln(\beta k(0,t)) - \frac{\ln(\beta k(0,t))}{k(0,t)} \geq 0
\end{aligned}$$

and

$$k_t(0,t) \geq \ln(\alpha k(0,t)) - \Psi'(0)\ln(\beta k(0,t)) + \frac{\ln(\beta k(0,t))}{k(0,t)} \geq c\ln(\beta k(0,t)).$$

from $\alpha \geq \beta$ and definition of function Ψ . Namely, $k_t(0,t) \geq c\ln(\beta k(0,t))$. Integrating for t from 0 to T, it is obtained that

$$\int_{k(0,t)}^{0} \frac{d\eta}{\ln(\beta\eta)} \leq c(T-t). \tag{8}$$

The theorem is proved.

Corollary 2.2. From (7) and (8), it is obtained that

$$\int_{k(0,t)}^{0} \frac{d\eta}{\ln(\beta\eta)} \sim (T-t).$$

since k_0 satisfies (4)-(5) and $\alpha \geq \beta$.

3. Conclusion

The main results in (1) are the following;
(i) the single quenching point is $x = 0$ and time derivative of k blows up at the finite quenching time since k_0 satisfies (4) and (5).
(ii) the quenching rate is

$$\int_{k(0,t)}^{0} \frac{d\eta}{\ln(\beta\eta)} \sim (T-t),$$

since k_0 satisfies (4)-(5) and $\alpha \geq \beta$..

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