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SOME NEW RESULTS ON NONLINEAR FRACTIONAL ITERATIVE VOLTERRA-FREDHOLM INTEGRO DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we establish some new results concerning the existence and uniqueness of the solutions of iterative nonlinear Volterra-Fredholm integro differential equations subject to the initial conditions. The fractional derivatives are considered in the Caputo sense. Also these new results are obtained by applying the Gronwall-Bellman's inequality and the Banach contraction fixed point theorem. Moreover, the results of references [16, 17, 27] appear as a special case of our results.

Keywords: Volterra-Fredholm integro-differential equation, Caputo sense, Gronwall-Bellman's inequality, Banach contraction fixed point theorem.

AMS Subject Classification: 45J05, 34A12, 46B80.

1. Introduction

In recent years, there has been a growing interest in the linear and nonlinear integro-differential equations which are a combination of differential and integral equations [4, 6, 17, 19]. The nonlinear integro-differential equations play an important role in many branches of nonlinear functional analysis and their applications in the theory of engineering, mechanics, physics, electrostatics, biology, chemistry and economics [8] and signal processing [25].

The challenging work is to find the solution while dealing with Volterra-Fredholm fractional integro-differential equations. Therefore, many researchers have tried their best to use different techniques to find the analytical and numerical solutions of these problems [1, 2, 3, 5, 9, 10, 11, 15, 22, 23, 28].

The study of iterative differential and integro-differential equations is linked to the wide applications of calculus in mathematical sciences. These equations are vital in the study of infection models. Many papers have dealt with the existence, uniqueness and other properties of solutions of special forms of the iterative differential equations and integro-differential equations [16, 17, 20, 21].

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Recently, Cheng et al. (2009), in [7, 21] investigated analytic and exact solutions of an iterative functional differential equation

$$\begin{aligned} &u'(x) = f(x, u(h(x) + g(u(x)))),\\ &u(x_0) = x_0 \end{aligned}$$

Lauran (2011) [20], investigated the existence and uniqueness results for first order differential and iterative differential equations with deviating argument of the type

$$\begin{aligned} &u'(t) = f(t, u(t), u(u(t)), u(\lambda u(t))),\\ &u(t_0) = x_0 \end{aligned}$$

In [16], Ibrahim (2013) investigated the existence and uniqueness of solution for iterative differential equations of the type

$$\begin{aligned} &D^\alpha u(t) = f(t, u(u(t))),\\ &u(0) = u_0. \end{aligned}$$

Kendre et al. (2015), [17] investigated the existence of solution for iterative integro-differential equations of the type

$$\begin{aligned} &u'(t) = f(t, u(u(t)), \int_{t_0}^{t} k(t, s)u(u(s))ds),\\ &u(t_0) = x_0 \end{aligned}$$

Unhale and Kendre (2019), in [27] established the existence and uniqueness of solution for iterative integro-differential equations of the type

$$\begin{aligned} &D^\alpha u(t) = f(t) + \int_{0}^{t} h(t, s)u(\lambda u(s))ds,\\ &u(0) = u_0, \end{aligned}$$

Motivated by the above work, in this paper we discuss new existence and uniqueness results for nonlinear fractional Volterra-Fredholm integro-differential equation with deviating argument of the type

$$D^\alpha u(t) = f(t) + \int_{0}^{t} h(t, s)u(\lambda u(s))ds + \int_{0}^{T} k(t, s)u(\lambda u(s))ds,\ t, s \in J := [0, T], \tag{1}$$

with the initial condition

$$u(0) = u_0, \tag{2}$$

where $D^\alpha(.)$, $0 < \alpha < 1$, is the Caputo fractional derivative, $f(t)$, h(t, s) and k(t, s) are given continuous functions, $u(x)$ is the unknown function to be determined, $u_0 \in J$ and $\lambda \in (0, 1)$.

The main objective of the present paper is to study the new existence and uniqueness results for iterative nonlinear fractional Volterra-Fredholm integro-differential equation with deviating argument.

The rest of the paper is organized as follows: In Section 2, some essential notations, definitions and Lemmas related to fractional calculus are recalled. In Section 3, the new existence and uniqueness results of the solution for nonlinear fractional Volterra-Fredholm integro-differential equation have been proved. In Section 4, focuses on an example to illustrate the theory. Finally, we will give a report on our paper and a brief conclusion is given in Section 5.

2. Preliminaries

The mathematical definitions of fractional derivative and fractional integration are the subject of several different approaches. The most frequently used definitions of the fractional calculus involves the Riemann-Liouville fractional derivative, Caputo derivative [9, 11, 12, 13, 18, 19, 24, 26, 28].

Definition 2.1. [17] *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function f is defined as*

$$\begin{aligned} J^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \qquad x>0, \quad \alpha \in \mathbb{R}^+, \\ J^0 f(x) &= f(x), \end{aligned} \tag{3}$$

where $\mathbb{R}^+$ is the set of positive real numbers.

Definition 2.2. [17] *The Riemann-Liouville derivative of order α with the lower limit zero for a function $f : [0,1) \longrightarrow \mathbb{R}$ can be written as*

$$^{L}D^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{f(t)}{(x-t)^{\alpha}} dt, \; x>0, \; 0<\alpha<1. \tag{4}$$

Definition 2.3. [14] *The Caputo derivative of order α for a function $f : [0,1) \longrightarrow \mathbb{R}$ can be written as*

$$D^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{f'(t)}{(x-t)^{\alpha}} dt, \; x>0, \; 0<\alpha<1.$$

Definition 2.4. [14] *The fractional derivative of $f(x)$ in the Caputo sense is defined by*

$$\begin{aligned} ^{c}D^{\alpha} f(x) &= J^{n-\alpha} D^n f(x) \\ &= \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} \frac{d^n f(t)}{dt^n} dt, & n-1<\alpha<n, \\ \frac{d^n f(x)}{dx^n}, & \alpha = n, \end{cases} \end{aligned} \tag{5}$$

where the parameter α is the order of the derivative, in general it is real or even complex.

Definition 2.5. [14] *The Riemann-Liouville fractional derivative of order $\alpha > 0$ is normally defined as*

$$D^{\alpha} f(x) = D^m J^{m-\alpha} f(x), \qquad m-1<\alpha \le m. \tag{6}$$

Lemma 2.1. [14] *(Gronwall-Bellman's Inequality). Let $u(x)$ and $f(x)$ be nonnegative continuous functions defined on $J = [\alpha, \alpha+h]$ and c be a nonnegative constant. If*

$$u(x) \le c + \int_{\alpha}^x f(s)u(s)ds, \; x \in J,$$

then

$$u(x) \le c \; \exp\Big(\int_{\alpha}^x f(s)ds \Big), \; x \in J,$$

Theorem 2.1. [26] *(Banach contraction principle). Let (X,d) be a complete metric space, then each contraction mapping $\mathcal{T} : X \longrightarrow X$ has a unique fixed point x of $\mathcal{T}$ in X i.e. $\mathcal{T}x = x$.*

Theorem 2.2. [26] *(Schauder's fixed point theorem). Let X be a Banach space and let A a convex, closed subset of X. If $T : A \longrightarrow A$ be the map such that the set $\{Tu : u \in A\}$ is relatively compact in X (or T is continuous and completely continouous). Then T has at least one fixed point $u^* \in A : Tu^* = u^*$.*

3. Main Results

In this section, we shall give an existence and uniqueness results of Eq.(1), with the initial condition (2). Let $B = C(J, J)$ be the Banach space equipped with the norm $\|u\| = \max_{x\in[0,T]} |u(x)|$. For convenience, we are listing the following hypotheses used in our further discussion:

(A1) There exists constants β_h and β_k such that

$$\beta_h = \sup\{|h(t, s)| : 0 \leq s \leq t \leq T\}.$$

$$\beta_k = \sup\{|k(t, s)| : 0 \leq s \leq t \leq T\}.$$

(A2) There exists a constant $M > 0$ such that

$$|u(x_1) - u(x_2)| \leq M|x_1 - x_2|^\alpha, \text{ for } u \in B,\ x_1, x_2 \in J,\ x_1 \leq x_2.$$

(A3) There exists a constant $L > 0$ such that $L = \sup\{|f(t)| : 0 \leq t \leq T\}$.

(A4) Let $\rho := u_0 + \frac{T^\alpha(L+T^3(\beta_h+\beta_k))}{\Gamma(\alpha+1)} \leq T \leq M$.

Lemma 3.1. *If a function $u \in C[0, T]$ satisfies (1)-(2) in the closed interval $[0, T]$, then the problems (1)-(2) are equivalent to the problem of finding a continuous solution of the integral equation*

$$u(t) = u_0 + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t) + \int_0^t h(t, s)u(\lambda u(s))ds + \int_0^T k(t, s)u(\lambda u(s))ds\Big)dt.$$

Proof. Applying I^α on both sides of equation (1) and using initial condition, we get

$$u(t) - u_0 = I^\alpha\Big(f(t) + \int_0^t h(t, s)u(\lambda u(s))ds + \int_0^T k(t, s)u(\lambda u(s))ds\Big)$$

$$u(t) = u_0 + I^\alpha\Big(f(t) + \int_0^t h(t, s)u(\lambda u(s))ds + \int_0^T k(t, s)u(\lambda u(s))ds\Big)$$

$$u(t) = u_0 + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t) + \int_0^t h(t, s)u(\lambda u(s))ds + \int_0^T k(t, s)u(\lambda u(s))ds\Big)dt.$$

□

Theorem 3.1. *Suppose that the hypotheses (A1)–(A4) are satisfied and*

$$\frac{T^{\alpha+1}\lambda((\beta_h + \beta_k)(M + 1)}{\Gamma(\alpha + 1)} < 1.$$

Then there is a unique solution to the problems (1)-(2).

Proof. Let $S(\rho) = \{u \in B :\ 0 \leq u \leq \rho, |u(x_1) - u(x_2)| \leq M|x_1 - x_2|^\alpha\}$.

To apply Banach contraction principle, we define an operator $\Psi : S(\rho) \longrightarrow S(\rho)$ by

$$(\Psi u)(t) = u_0 + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t) + \int_0^t h(t, s)u(\lambda u(s))ds + \int_0^T k(t, s)u(\lambda u(s))ds\Big)dt.$$

So, we have

$$\begin{aligned} 0 \leq (\Psi u) &= \Big|u_0 + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t) + \int_0^t h(t,s)u(\lambda u(s))ds + \int_0^T k(t,s)u(\lambda u(s))ds\Big)dt\Big| \\ &\leq u_0 + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(\Big|f(t) + \int_0^t h(t,s)u(\lambda u(s))ds + \int_0^T k(t,s)u(\lambda u(s))ds\Big|\Big)dt \\ &\leq u_0 + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(\Big|f(t)\Big| + \int_0^t \Big|h(t,s)\Big|\Big|u(\lambda u(s))\Big|ds + \int_0^T \Big|k(t,s)\Big|\Big|u(\lambda u(s))\Big|ds\Big)dt \\ &\leq u_0 + L\frac{T^\alpha}{\Gamma(\alpha+1)} + \beta_h\frac{T^\alpha T^3}{\Gamma(\alpha+1)} + \beta_k\frac{T^\alpha T^3}{\Gamma(\alpha+1)} \\ &\leq u_0 + \frac{T^\alpha(L + (\beta_h+\beta_k)T^3)}{\Gamma(\alpha+1)} = \rho. \end{aligned}$$

Also, for each $0 \leq x_1 \leq x_2 \leq T$, we have

$$\begin{aligned} &\Psi u(x_2) - \Psi u(x_1) \\ = &\int_0^{x_1} \frac{(x_2-t)^{\alpha-1} - (x_1-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t) + \int_0^t h(t,s)u(\lambda u(s))ds + \int_0^T k(t,s)u(\lambda u(s))ds\Big)dt \\ + &\int_{x_1}^{x_2} \frac{(x_2-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t) + \int_0^t h(t,s)u(\lambda u(s))ds + \int_0^T k(t,s)u(\lambda u(s))ds\Big)dt \\ = &\frac{-1}{\Gamma(\alpha)}\int_0^{x_1}\Big[(x_1-t)^{\alpha-1} - (x_2-t)^{\alpha-1}\Big]\Big(f(t) + \int_0^t h(t,s)u(\lambda u(s))ds \\ + &\int_0^T k(t,s)u(\lambda u(s))ds\Big)dt + \frac{1}{\Gamma(\alpha)}\int_{x_1}^{x_2}(x_2-t)^{\alpha-1}\Big(f(t) + \int_0^t h(t,s)u(\lambda u(s))ds \\ + &\int_0^T k(t,s)u(\lambda u(s))ds\Big)dt. \end{aligned}$$

Hence,

$$\begin{aligned} &\Big|\Psi u(x_2) - \Psi u(x_1)\Big| \\ \leq &\Big|\frac{1}{\Gamma(\alpha)}\int_0^{x_1}\Big[(x_1-t)^{\alpha-1} - (x_2-t)^{\alpha-1}\Big]\Big(f(t) \\ + &\int_0^t h(t,s)u(\lambda u(s))ds + \int_0^T k(t,s)u(\lambda u(s))ds\Big)dt\Big| + \Big|\frac{1}{\Gamma(\alpha)}\int_{x_1}^{x_2}(x_2-t)^{\alpha-1} \\ \times &\Big(f(t) + \int_0^t h(t,s)u(\lambda u(s))ds + \int_0^T k(t,s)u(\lambda u(s))ds\Big)dt\Big| \\ \leq &\frac{1}{\Gamma(\alpha)}\int_0^{x_1}\Big[(x_1-t)^{\alpha-1} - (x_2-t)^{\alpha-1}\Big]\Big(\Big|f(t)\Big| + \int_0^t \Big|h(t,s)\Big|\Big|u(\lambda u(s))\Big|ds \\ + &\int_0^T \Big|k(t,s)\Big|\Big|u(\lambda u(s))\Big|ds\Big)dt + \frac{1}{\Gamma(\alpha)}\int_{x_1}^{x_2}(x_2-t)^{\alpha-1}\Big(\Big|f(t)\Big| + \int_0^t \Big|h(t,s)\Big|\Big|u(\lambda u(s))\Big|ds \\ + &\int_0^T \Big|k(t,s)\Big|\Big|u(\lambda u(s))ds\Big|\Big)dt \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{x_1}\Big[(x_1-t)^{\alpha-1}-(x_2-t)^{\alpha-1}\Big]\Big(L+(\beta_h+\beta_k)T^3\Big)dt\\
&+ \frac{1}{\Gamma(\alpha)}\int_{x_1}^{x_2}(x_2-t)^{\alpha-1}\Big(L+(\beta_h+\beta_k)T^3\Big)dt\\
&\leq \frac{\Big(L+(\beta_h+\beta_k)T^3\Big)}{\Gamma(\alpha)}\int_0^{x_1}\Big[(x_1-t)^{\alpha-1}-(x_2-t)^{\alpha-1}\Big]dt\\
&+ \frac{\Big(L+(\beta_h+\beta_k)T^3\Big)}{\Gamma(\alpha)}\int_{x_1}^{x_2}(x_2-t)^{\alpha-1}dt\\
&\leq \frac{\Big(L+(\beta_h+\beta_k)T^3\Big)}{\Gamma(\alpha+1)}\Big[x_1^\alpha-x_2^\alpha+2(x_2-x_1)^\alpha\Big]\\
&\leq \frac{2\Big(L+(\beta_h+\beta_k)T^3\Big)}{\Gamma(\alpha+1)}|x_2-x_1|^\alpha.
\end{aligned}
$$

This shows that Ψ maps from $S(\rho)\longrightarrow S(\rho)$, Now, for all $u,v\in S(\rho)$ we have

$$
\begin{aligned}
&\Big|\Psi u(x)-\Psi v(x)\Big|\\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}\Big(\int_0^t\Big|h(t,s)\Big|\Big|u(\lambda u(s))-v(\lambda v(s))\Big|ds\\
&+ \int_0^T\Big|k(t,s)\Big|\Big|u(\lambda u(s))-v(\lambda v(s))\Big|ds\Big)dt\\
&\leq \frac{(\beta_h+\beta_k)}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}\Big(\int_0^t\Big|u(\lambda u(s))-u(\lambda v(s))\Big|+\Big|u(\lambda v(s))-v(\lambda v(s))\Big|ds\\
&+ \int_0^T\Big|u(\lambda u(s))-u(\lambda v(s))\Big|+\Big|u(\lambda v(s))-v(\lambda v(s))\Big|ds\Big)dt
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\lambda(\beta_h+\beta_k)}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}\Big(\int_0^t(M\Big|u(s)-v(s)\Big|+\Big|u(s)-v(s)\Big|)ds\\
&+ \int_0^T(M\Big|u(s)-v(s)\Big|+\Big|u(s)-v(s)\Big|)ds\Big)dt\\
&\leq \frac{\lambda(\beta_h+\beta_k)}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}\Big(\int_0^t(M+1)\Big|u(s)-v(s)\Big|ds+\int_0^T(M+1)\Big|u(s)-v(s)\Big|ds\Big)dt\\
&\leq \frac{T^{\alpha+1}\lambda(\beta_h+\beta_k)(M+1)}{\Gamma(\alpha+1)}\Big|\Big|u-v\Big|\Big|.
\end{aligned}
$$

Since

$$\frac{T^{\alpha+1}\lambda(\beta_h+\beta_k)(M+1)}{\Gamma(\alpha+1)}<1,$$

by the Banach contraction principle, Ψ has a unique fixed point. This means that the problems (1)-(2) has unique solution.

□

The above theorem shows that there exists a unique solution to the problems (1)-(2). However, it does not tell us how to find this solution. To find the solution of the problems

(1)-(2), we will define the following sequence

$$\begin{aligned} u_{n+1}(t) &= u_0 + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t) + \int_0^t h(t,s)u_n(\lambda u_n(s))ds \\ &+ \int_0^T k(t,s)u_n(\lambda u_n(s))ds\Big)dt. \end{aligned} \tag{7}$$

where $n = 0, 1, 2, ...$ and $u_0(x)$ is fixed functions of the class C^1 mapping $[0, T] \longrightarrow [0, T]$ such that $|u_0(x)| \leq T$. For this, we have the following theorem.

Theorem 3.2. *If the assumptions of the Theorem 3.1 are satisfied then the sequences defined in* (7) *converges uniformly to the unique solution of the problems* (1)-(2).

Proof. Let $U_k = \max_{x\in J} |u_k(x) - u_{k-1}(x)|$. Then

$$\begin{aligned} U_1 &= \max_{x\in J} |u_1(x) - u_0(x)| \\ &= \max_{x\in J} \Big| u_0 + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t) + \int_0^t h(t,s)u_0(\lambda u_0(s))ds \\ &\quad + \int_0^T k(t,s)u_0(\lambda u_0(s))ds\Big)dt - u_0(x)\Big| \\ &\leq \frac{T^\alpha(L + T^3(\beta_h + \beta_k))}{\Gamma(\alpha+1)} \\ &\leq T. \end{aligned}$$

Since $u_0 : [0, T] \longrightarrow [0, T]$, we have $U_1 \leq T$.

$$\begin{aligned} U_2 &= \max_{x\in J} |u_2(x) - u_1(x)| \\ &= \max_{x\in J} \Big| \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^t h(t,s)\Big(u_1(\lambda u_1(s)) - u_0(\lambda u_0(s))\Big)ds \\ &\quad + \int_0^T k(t,s)\Big(u_1(\lambda u_1(s)) - u_0(\lambda u_0(s))\Big)ds\Big)dt\Big| \\ &\leq \max_{x\in J} \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^t \Big|h(t,s)\Big(u_1(\lambda u_1(s)) - u_0(\lambda u_0(s))\Big)ds\Big| \\ &\quad + \int_0^T \Big|k(t,s)\Big(u_1(\lambda u_1(s)) - u_0(\lambda u_0(s))\Big)ds\Big|\Big)dt \\ &\leq TU_1 \leq T^2. \end{aligned}$$

Assume that result is true for n i.e. $U_n \leq TU_{n-1} \leq T^n$. Now, we show that result holds for $n+1$

$$\begin{aligned} U_{n+1} &= \max_{x\in J} |u_{n+1}(x) - u_n(x)| \\ &= \max_{x\in J} \Big| \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(\int_0^t h(t,s)\Big(u_n(\lambda u_n(s)) - u_{n-1}(\lambda u_{n-1}(s))\Big) ds \\ &\quad + \int_0^T k(t,s)\Big(u_n(\lambda u_n(s)) - u_{n-1}(\lambda u_{n-1}(s))\Big) ds \Big) dt \Big| \\ &\leq \max_{x\in J} \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(\int_0^t \Big| h(t,s)\Big(u_n(\lambda u_n(s)) - u_{n-1}(\lambda u_{n-1}(s))\Big) ds \Big| \\ &\quad + \int_0^T \Big| k(t,s)\Big(u_n(\lambda u_n(s)) - u_{n-1}(\lambda u_{n-1}(s))\Big) ds \Big| \Big) dt \\ &\leq TU_n \leq T^{n+1}. \end{aligned}$$

Thus by induction, we have $U_k \leq T^k$. Since

$$u_0 + \frac{T^\alpha(L + T^3(\beta_h + \beta_k))}{\Gamma(\alpha+1)} \leq T < 1, \text{ when } u_0 \geq 0.$$

Hence U_k tends to zero as k tends to infinity. Since the family $\{U_k\}$ is the Arzelà-Ascoli family thus for every subsequence $\{u_{kj}\}$ of $\{U_k\}$ there exists a subsequence $\{u_{kj}\}$ uniformly convergent and the limit needs to be a solution of the problem (1)-(2). Thus, the sequence $\{u_k\}$ tends uniformly to the unique solution of the problem (1)-(2). □

Theorem 3.3. *Suppose that the hypotheses of the Theorem 3.1 hold. Let u_1 and u_2 satisfy the equation* (1) *for $0 \leq x \leq T$, $M > 0$ with $u_1(0) = u_0^*$ and $u_2(0) = u_0^{**}$ respectively then*

$$\|u_1(x) - u_2(x)\| \leq \exp \frac{\lambda(\beta_h + \beta_k)(M+1)T^{\alpha+1}}{\Gamma(\alpha+1)} \|u_0^* - u_0^{**}\|.$$

Proof. From Theorem 3.1, we have

$$u_1(t) = u_0^* + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(f(t) + \int_0^t h(t,s)u_1(\lambda u_1(s))ds + \int_0^T k(t,s)u_1(\lambda u_1(s))ds \Big) dt.$$

and

$$u_2(t) = u_0^{**} + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(f(t) + \int_0^t h(t,s)u_2(\lambda u_2(s))ds + \int_0^T k(t,s)u_2(\lambda u_2(s))ds \Big) dt.$$

Then,

$$
\begin{aligned}
&\Big|u_1(t)-u_2(t)\Big| \\
\leq\ & \Big|u_0^*-u_0^{**}\Big| + \Big|\int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^t h(t,s)\Big(u_1(\lambda u_1(s))-u_2(\lambda u_2(s))\Big)ds \\
&+\int_0^T k(t,s)\Big(u_1(\lambda u_1(s))-u_2(\lambda u_2(s))\Big)ds\Big)dt\Big| \\
\leq\ & \Big|u_0^*-u_0^{**}\Big| + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(\beta_h\int_0^t \Big|u_1(\lambda u_1(s))-u_2(\lambda u_2(s))\Big|ds \\
&+\beta_k\int_0^T \Big|u_1(\lambda u_1(s))-u_2(\lambda u_2(s))\Big|ds\Big)dt \\
\leq\ & \Big|u_0^*-u_0^{**}\Big| + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(\beta_h\int_0^t \Big[\Big|u_1(\lambda u_1(s))-u_1(\lambda u_2(s))\Big|+\Big|u_1(\lambda u_2(s))-u_2(\lambda u_2(s))\Big|\Big]ds \\
&+\beta_k\int_0^T \Big[\Big|u_1(\lambda u_1(s))-u_1(\lambda u_2(s))\Big|+\Big|u_1(\lambda u_2(s))-u_2(\lambda u_2(s))\Big|\Big]ds\Big)dt \\
\leq\ & \Big|u_0^*-u_0^{**}\Big| + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(\lambda\beta_h\int_0^t (M+1)\Big|u_1(s)-u_2(s)\Big|ds \\
&+\lambda\beta_k\int_0^T (M+1)\Big|u_1(s)-u_2(s)\Big|ds\Big)dt \\
\leq\ & \Big|u_0^*-u_0^{**}\Big| + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(\lambda\beta_h\int_s^x (M+1)\Big|u_1(s)-u_2(s)\Big|dt \\
&+\lambda\beta_k\int_0^T (M+1)\Big|u_1(s)-u_2(s)\Big|dt\Big)ds \\
\leq\ & \Big|u_0^*-u_0^{**}\Big| + \frac{\lambda(M+1)(\beta_h+\beta_k)T^\alpha}{\Gamma(\alpha+1)}\int_0^x \Big|u_1(s)-u_2(s)\Big|ds
\end{aligned}
$$

Using Gronwall-Bellman's inequality, we get

$$
\begin{aligned}
\Big|u_1(s)-u_2(s)\Big| &\leq \Big|u_0^*-u_0^{**}\Big|\exp\Big(\int_0^x \frac{\lambda(M+1)(\beta_h+\beta_k)T^\alpha}{\Gamma(\alpha+1)}ds\Big) \\
&\leq \Big|u_0^*-u_0^{**}\Big|\exp\Big(\frac{\lambda(M+1)(\beta_h+\beta_k)T^{\alpha+1}}{\Gamma(\alpha+1)}\Big)
\end{aligned}
$$

Hence, we have

$$
\Big|\Big|u_1(s)-u_2(s)\Big|\Big| \leq \exp\Big(\frac{\lambda(M+1)(\beta_h+\beta_k)T^{\alpha+1}}{\Gamma(\alpha+1)}\Big)\Big|\Big|u_0^*-u_0^{**}\Big|\Big|.
$$

This completes the proof.

□

4. An Example

We consider the nonlinear iterative fractional integro-differential equation (1) with

$$
u(0)=0.25,\ T=0.5, L=0.2, M=1, \beta_h=\beta_k=0.4,\ \lambda=\frac{2}{3}, \text{and } \alpha=0.5.
$$

New, we have

$$\begin{aligned}
u_0 + \frac{T^\alpha(L + T^3(\beta_h + \beta_k))}{\Gamma(\alpha+1)} &= 0.25 + \frac{0.5^{0.5}(0.2 + 0.5^3(0.4 + 0.4))}{\Gamma(0.5+1)} \\
&= 0.25 + \frac{0.71(0.2 + 0.125(0.8))}{\Gamma(1.5)} \\
&= 0.25 + \frac{0.71(0.3)}{0.88} \\
&= 0.25 + \frac{0.71(0.3)}{0.88} \\
&= 0.492045 \\
&< 0.5 = T.
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{T^{\alpha+1}(M+1)\lambda(\beta_h + \beta_k)}{\Gamma(\alpha+1)} &= \frac{0.5^{0.5+1}(1+1)\frac{2}{3}(0.4+0.4)}{\Gamma(0.5+1)} \\
&= \frac{0.355(2)\frac{2}{3}(0.8)}{0.88} \\
&= 0.4303 \\
&< 1.
\end{aligned}$$

Since all the hypotheses of Theorem 3.1 are fulfilled, then there exists a unique solution of the given equation.

5. Conclusion

The main purpose of this paper is to present new existence and uniqueness results of the solution for Caputo fractional iterative Volterra-Fredholm integro-differential. The techniques used to prove our results are a variety of tools such as the the Gronwall-Bellman's inequality, some properties of fractional calculus and the Banach contraction fixed point theorem. Moreover, the results of references [16, 17, 27] appear as a special case of our results.

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