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# NEW RESULTS ON ODD HARMONIOUS LABELING OF GRAPHS

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Abstract. Let $G = (V, E)$ be a graph with $p$ vertices and $q$ edges. A graph $G$ is said to be odd harmonious if there exists an injection $f : V(G) \to \{0, 1, 2, \cdots, 2q - 1\}$such that the induced function $f^* : E(G) \to \{1, 3, \cdots, 2q - 1\}$ defined by $f^*(uv) = f(u) + f(v)$ is a bijection. If $f(V(G)) = \{0, 1, 2, \cdots, q\}$ then $f$ is called strongly odd harmonious labeling and the graph is called strongly odd harmonious graph. In this paper we prove that $Spl(C_{bn})$ and $Spl(B(m)_{(n)})$, slanting ladder $SL_n$, $mG_n$, H-super subdivision of path $P_n$ and cycle $C_n$, $n \equiv 0(mod\ 4)$ admit odd harmonious labeling. In addition we observe that all strongly odd harmonious graphs admit mean labeling, odd mean labeling, odd sequential labeling and all odd sequential graphs are odd harmonious and all odd harmonious graphs are even sequential harmonious.

Keywords: Odd harmonious labeling; Strongly odd harmonious labeling; Odd sequential labeling; Even sequential harmonious labeling; Mean labeling; Odd mean labeling.

AMS Subject Classification: 05C78.

## 1. Introduction

Throughout this paper, by a graph, we mean a finite, simple and undirected one. For standard terminology and notation we follow Harary [11]. One of the major themes in graph theory is graph labeling, introduced by Alex Rosa in 1967. The graph labeling is an assignment of integers to the set of vertices or edges or both, subject to certain conditions. During the last five decades nearly 300 graph labeling techniques have been studied which are beautifully classified by Gallian [7] in his survey under seven headings. One of such classifications is harmonious labeling, introduced by Graham and Sloane [9]. The concept of odd harmonious labeling (one of the variations of harmonious labeling) was due to Liang and Bai [23] who proved the following results:
1. If $G$ is an odd harmonious graph, then $G$ is a bipartite graph. Hence any graph that contains an odd cycle is not an odd harmonious.
2. If a $(p, q)$ − graph $G$ is odd harmonious, then $2\sqrt{q} \leq p \leq (2q - 1)$.
3. If $G$ is an odd harmonious Eulerian graph with $q$ edges, then $q \equiv 0, 2(mod\ 4)$.

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Followed by this, some authors have also proved several results on odd harmonious labeling. For example, Vaidya and Shah [29], [30], have proved that shadow and splitting of $P_n$, $K_{1,n}$, $B_{n,n}$ are odd harmonious. Also they established that the arbitrary super subdivision of path, join sum of two copies of cycle and $H_{n,n}$ are odd harmonious. Abdel- Aal [1] -[3] has proved that cyclic snakes, $m$-shadow of path and complete bipartite graph, $n$-splitting of path and star, symmetric product between paths and null graphs and two copies of even cycles sharing a common edge and a common vertex are odd harmonious. Gustri Suptri and Sugeng [10] have established that dumbbell graph $D_{n,k,2}$ is odd harmonious if and only if $n, k \equiv 2(mod\ 4)$. Selvaraju et. al [25] have proved that quadrilateral snake and k-regular caterpillars are odd harmonious. Fery Firmansah [5], [6] has constructed the odd harmonious labeling of pleated of the dutch windmill graphs and the variation of the double quadrilateral windmill graph.

Motivated by the above results, we have [12] - [22] further studied and proved that several graphs are odd harmonious. In this paper, we establish some new results on odd harmonious labeling. In order to prove our results we use the following definitions.

**Definition 1.** *A graph $G$ is said to be harmonious if there exists an injection $f : V(G) \to Z_q$ such that the induced function $f^* : E(G) \to Z_q$ defined by $f^*(uv) = (f(u) + f(v))$ (mod $q$) is a bijection and $f$ is called harmonious labeling of $G$.*

**Definition 2.** *A graph $G$ is said to be odd harmonious if there exists an injection $f : V(G) \to \{0, 1, 2, \cdots, 2q-1\}$ such that the induced function $f^* : E(G) \to \{1, 3, \cdots, 2q-1\}$ defined by $f^*(uv) = f(u) + f(v)$ is a bijection. If $f : V(G) \to \{0, 1, 2, \cdots, q\}$ then $f$ is called as strongly odd harmonious labeling and $G$ is called a strongly odd harmonious graph.*

**Definition 3.** *The Corona of a graph $G$ on $p$ vertices $v_1, v_2, \cdots, v_p$ is obtained from $G$ by adding $p$ new vertices $u_1, u_2, \cdots, u_p$ and new edges $u_iv_i$ for $1 \le i \le p$, denoted by $G \circ K_1$. The graph $P_n \circ K_1$ is called a comb $C_{bn}$.*

**Definition 4.** *The m-splitting graph $Spl_m(G)$ is obtained by adding to each vertex $v$ of $G$ new $m$ vertices, say $v^1, v^2, ...., v^m$ such that $v_i, 1 \le i \le m$ is adjacent to every vertex that is adjacent to $v$ in $G$.*

**Definition 5.** [31] *The graph obtained by attaching $m$ pendant vertices to each vertex of a path of length $2n-1$ is denoted by $B(m)_{(n)}$.*

**Definition 6.** *The slanting ladder $SL_n$ is obtained from two paths $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_n$ by joining each $u_i$ with $v_{i+1}$, $1 \le i \le n-1$.*

**Definition 7.** [28] *The graph $\langle K_{1,n} : K_{1,m} \rangle$ is obtained by joining the center $u$ of the star $K_{1,n}$ and the center $v$ of another star $K_{1,m}$ to a new vertex $w$. The number of vertices is $n + m + 3$ and the number of edges is $n + m + 2$.*

**Definition 8.** [28] *The graph $mG_n$ is obtained from $m$ copies of $\langle K_{1,n} : K_{1,n} \rangle$ by joining one leaf of $i^{th}$ copy of $\langle K_{1,n} : K_{1,n} \rangle$ with the center of $(i+1)^{th}$ copy of $\langle K_{1,n} : K_{1,n} \rangle$ where $1 \le i \le m-1$.*

**Definition 9.** [4] *A graph obtained from $G$ by replacing each edge $e_i$ by a $H$- graph in such a way that the ends of $e_i$ are merged with a pendent vertex in $p_2$ and a pendent vertex $p_2'$ is called $H$- super subdivision of $G$ and it is denoted by $HSS(G)$, where the $H$- graph is a tree on 6 vertices in which exactly two vertices of degree 3.*

**Definition 10.** [27] *A mean labeling $f$ is an injective function from $V$ to the set $\{0, 1, 2, \cdots, q\}$ such that each edge $uv$ is assigned a label $f^*(uv) = \dfrac{f(u) + f(v)}{2}$ if $f(u) + f(v)$ is even and $f^*(uv) = \dfrac{f(u) + f(v) + 1}{2}$ if $f(u) + f(v)$ is odd, then the resulting edges are distinct.*

**Definition 11.** [24] *A graph $G(p, q)$ is said to be an odd mean graph if there exists an injective function $f$ from the vertex set of $G$ to $\{0, 1, 2, \cdots, 2q - 1\}$ such that the induced map from the edge set of $G$ to $\{1, 3, 5, \cdots, 2q - 1\}$ defined by $f^*(uv) = \dfrac{f(u) + f(v)}{2}$ if $f(u) + f(v)$ is even and $f^*(uv) = \dfrac{f(u) + f(v) + 1}{2}$ if $f(u) + f(v)$ is odd, is a bijection.*

**Definition 12.** [26] *A graph $G(p, q)$ is said to have an odd sequential labeling if there exists a function $f : V(G) \to \{0, 1, 2, \cdots, q\}$ and each edge $uv$ is assigned the label $f(u) + f(v)$ such that the resulting edge labels are $\{1, 3, 5, \cdots, 2q - 1\}$.*

**Definition 13.** [8] *A graph $G(p, q)$ is said to have an even sequential harmonious labeling if there exists a function $f : V(G) \to \{0, 1, 2, \cdots, 2q\}$ such that the induced map $f^* : E(G) \to \{2, 4, \cdots, 2q\}$ defined by $f^*(uv) = f(u)+f(v)$ if $f(u)+f(v)$ is even and $f^*(uv) = f(u) + f(v) + 1$ if $f(u) + f(v)$ is odd, then the resulting edge labels are distinct.*

## 2. Main Results

In this section, first we prove that $Spl(C_{bn})$ and $Spl(B(m)_{(n)})$, slanting ladder $SL_n$, $mG_n$, H-super subdivision of path $P_n$ and cycle $C_n$, $n \equiv 0(mod\ 4)$ admit odd harmonious labeling.

**Theorem 2.1.** *The graph $Spl(Cb_n)$ is odd harmonious.*

*Proof.* Let $v_1, v_2, \cdots, v_n$ and $v_1^{'}, v_2^{'}, \cdots, v_n^{'}$ be the vertices of comb in which $v_1^{'}, v_2^{'}, \cdots, v_n^{'}$ are the pendant vertices. Let $u_1, u_2, \cdots, u_n$ and $u_1^{'}, u_2^{'}, \cdots, u_n^{'}$ be the new added vertices corresponding to $v_1, v_2, \cdots, v_n$ and $v_1^{'}, v_2^{'}, \cdots, v_n^{'}$ and let $G = Spl(Cb_n)$. This graph has $4n$ vertices and $6n - 3$ edges.
Define $f : V(G) \to \{0, 1, 2, \cdots, 2(6n - 3) - 1\}$ as follows:

$$f(v_i) = \begin{cases} 2i - 1 & \text{if } i \text{ is odd} \\ 2i - 2 & \text{if } i \text{ is even.} \end{cases}, 1 \le i \le n;$$

$$f(u_i) = \begin{cases} 4n + 4i - 5 & \text{if } i \text{ is odd} \\ 4n + 4i - 6 & \text{if } i \text{ is even.} \end{cases}, 1 \le i \le n;$$

$$f(v_i^{'}) = \begin{cases} 2(i - 1) & \text{if } i \text{ is odd} \\ 2i - 1 & \text{if } i \text{ is even.} \end{cases}, 1 \le i \le n;$$

$$f(u_i^{'}) = \begin{cases} 12n - 4i - 4 & \text{if } i \text{ is odd} \\ 12n - 4i - 3 & \text{if } i \text{ is even.} \end{cases}, 1 \le i \le n.$$

The induced edge labels are
$f^*(v_i v_{i+1}) = 4i - 1$, $1 \le i \le n - 1$;
$f^*(v_i v_i{'}) = 4i - 3$, $1 \le i \le n$;
$f^*(u_i v_i{'}) = 4n + 6i - 7$, $1 \le i \le n$;
$f^*(u_i v_{i+1}) = 4n + 6i - 5$, $1 \le i \le n - 1$;
$f^*(v_i u_{i+1}) = 4n + 6i - 3$, $1 \le i \le n - 1$;
$f^*(v_i u_i{'}) = 12n - 2i - 5$, $1 \le i \le n$.
In view of the above defined labeling pattern, the split of Comb $Cb_n$ is an odd harmonious graph. □

An odd harmonious labeling of split of comb $Cb_4$ is shown in Figure 1.

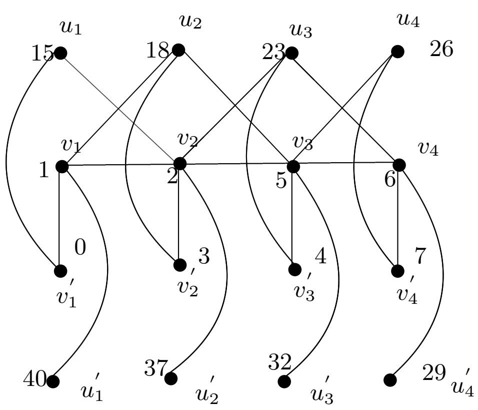


FIGURE 1. An odd harmonious labeling of $Spl(Cb_4)$

**Theorem 2.2.** *The graph $Spl(B(m)_{(n)})$, $m$ is even, is odd harmonious.*

*Proof.* Let the vertices be $u_i, v_i, u_{ij}, v_{ij}$, $1 \le i \le n$, $1 \le j \le m$ and $u_i^{'}, v_i^{'}, u_{ij}^{'}, v_{ij}^{'}$, $1 \le i \le n$, $1 \le j \le m$ be the new vertices added to the corresponding vertices $u_i, v_i, u_{ij}, v_{ij}$ to obtain $Spl(B(m)_{(n)})$ . This graph has $4(m+1)n$ vertices and $6(m+1)n-3$ edges.
Define $f : V(G) \to \{0, 1, 2, \cdots, 2[6(m+1)n-3]-1\}$ as follows:

$f(v_i) = \begin{cases} 3(i-1) & \text{if } i \text{ is odd} \\ 1+3(i-2) & \text{if } i \text{ is even.} \end{cases}, 1 \le i \le 2n;$

$f(u_i) = \begin{cases} 2+3(i-1) & \text{if } i \text{ is odd} \\ 5+3(i-2) & \text{if } i \text{ is even.} \end{cases}, 1 \le i \le 2n;$

If $i$ is odd, $f(v_{ij}) = 12n-5+(4m-3)(i-1)+4(j-1)$; $j = 1, 3, \cdots, m$;
If $i$ is even, $f(v_{ij}) = 12n+4m-8+4(j-1)+(4m-3)(i-2)+2$, $j = 1, 3, \cdots, m-1$;
$f(v_{ij}) = 12n+4m-8+4(j-1)+(4m-3)(i-2)$, $j = 2, 4, \cdots, m$;
If $i$ is odd, $f(u_{ij}) = 12n+4m+6(m-1)+(4m-3)(2n-2)+1+2(j-1)+(2m+3)(2n-1-i)$, $j = 1, , \cdots, m$;
If $i$ is even, $f(u_{ij}) = 12n+4m-2+4(m-1)+(4m-3)(2n-2)+2(j-1)+(2m+3)(2n-i)$, $j = 1, , \cdots, m$.
The induced edge labels are
$f^*(v_iv_{i+1}) = 6i-5$, $1 \le i \le 2n-1$;
$f^*(v_iu_{i+1}) = 6i-1$, if $i$ is odd, $1 \le i \le 2n-1$;
$f^*(v_iu_{i+1}) = 6i-3$, if $i$ is even, $1 \le i \le 2n-1$;
$f^*(u_iv_{i+1}) = 6i-3$, if $i$ is odd, $1 \le i \le 2n-1$;
$f^*(u_iv_{i+1}) = 6i-1$, if $i$ is even, $1 \le i \le 2n-1$;
If $i$ is even, $f^*(v_iv_{ij}) = 1+3(i-2)+12n+4m-8+4(j-1)+(4m-3)(i-2)+2$, $j = 1, 3, \cdots, m-1$;
$f^*(v_iv_{ij}) = 1+3(i-2)+12n+4m-8+4(j-1)+(4m-3)(i-2)$, $j = 2, 4, \cdots, m$;
If $i$ is odd, $f^*(v_iv_{ij}) = 3(i-1)+12n-5+(4m-3)(i-1)+4(j-1)$, $j = 1, 2, \cdots, m$;
If $i$ is odd , $f^*(u_iv_{ij}) = 2+3(i-1)+12n-5+(4m-3)(i-1)+4(j-1)$;$j = 1, 2, \cdots, m$;
If $i$ is even, $f^*(u_iv_{ij}) = 5+3(i-2)+12n+4m-8+4(j-1)+(4m-3)(i-2)+2$, $j = 1, 3, \cdots, m-1$;
$f^*(u_iv_{ij}) = 5+3(i-2)+12n+4m-8+4(j-1)+(4m-3)(i-2)$, $j = 2, 4, \cdots, m$;
If $i$ is odd, $f^*(v_iu_{ij}) = 3(i-1)+12n+4m+6(m-1)+(4m-3)(2n-2)+1+2(j-1)+$

$(2m+3)(2n-1-i)$, $j=1,2,\cdots,m$;
If $i$ is even, $f^*(v_iu_{ij}) = 1+3(i-2)+12n+4m-2+4(m-1)+(4m-3)(2n-2)+2(j-1)+(2m+3)(2n-i)$, $j=1,2,\cdots,m$.
In view of the above defined labeling pattern, the split of $B(m)_{(n)}$ is an odd harmonious graph. $\square$

An odd harmonious labeling of $B(4)_{(2)}$ is shown in Figure 2.

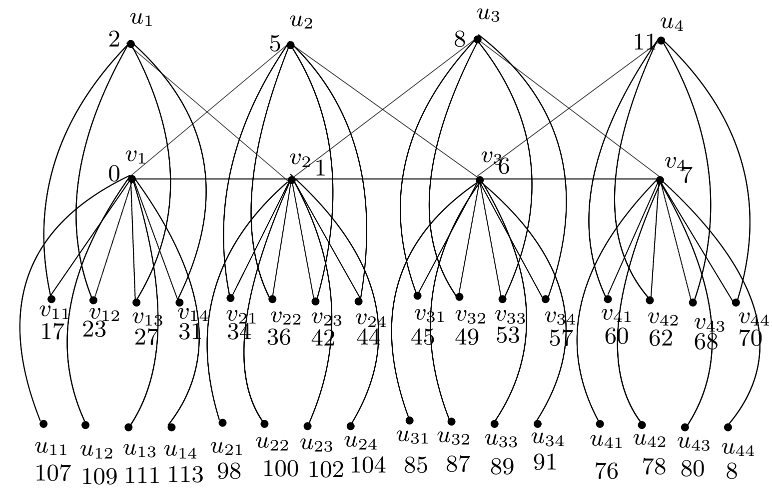

FIGURE 2. An odd harmonious labeling of $Spl(B(4)_{(2)})$

**Theorem 2.3.** *The slanting ladder $SL_n$ is odd harmonious.*

*Proof.* Let the vertices of $G = SL_n$ be $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_n$. This graph has $2n$ vertices and $3(n-1)$ edges.
Define $f : V(G) \to \{0,1,2,\cdots,6(n-1)-1\}$ as follows:
$f(u_i) = i-1$, $1 \le i \le n$; $f(v_i) = 2n+i-3$, $1 \le i \le n$.
The induced edge labels are
$f^*(u_iu_{i+1}) = 2i-1$, $1 \le i \le n-1$; $f^*(v_iv_{i+1}) = 4n+2i-5$, $1 \le i \le n-1$;
$f^*(u_iv_{i+1}) = 2n+2i-3$, $1 \le i \le n-1$. In view of the above defined labeling pattern, the slanting ladder $SL_n$ is an odd harmonious graph. $\square$

An odd harmonious labeling of slanting ladder $SL_4$ is shown in Figure 3.

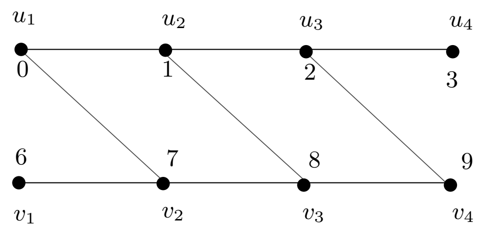


FIGURE 3. An odd harmonious labeling of $SL_4$

**Theorem 2.4.** *The graph $mG_n$ is odd harmonious.*

*Proof.* Let $G = mG_n$. Let $\{s_i : 1 \le i \le m\} \cup \{t_i : 1 \le i \le m\} \cup \{u_i : 1 \le i \le m\} \cup \{s_{ij}, t_{ij} : 1 \le i \le m, 1 \le j \le n\}$ be the vertices of $G$ . This graph has $2m(n+1)+1$ vertices and $2m(n+1)$ edges.
Define $f : V(G) \to \{0, 1, 2, \cdots, 4m(n+1)-1\}$ as follows:
$f(s_i) = 1 + 4(i-1)$, $1 \le i \le m$; $f(t_i) = 3 + 4(i-1)$, $1 \le i \le m$;
$f(u_i) = 2n(i-1)$, $1 \le i \le m$;
$f(s_{ij}) = 4 + 2mn + 2(j-1) + 6(m-i) + 2(m-i)(n-1)$, $1 \le i \le m,\ 1 \le j \le n$;
$f(t_{ij}) = 2i + 2(j-1) + 2(i-1)(n-1)$, $1 \le i \le m,\ 1 \le j \le n$;
The induced edge labels are
$f^*(s_iu_i) = 1 + 4(i-1) + 2n(i-1)$, $1 \le i \le m$;
$f^*(t_iu_i) = 3 + 4(i-1) + 2n(i-1)$, $1 \le i \le m$;
$f^*(s_is_{ij}) = 1+4(i-1)+4+2mn+2(j-1)+6(m-i)+2(m-i)(n-1)$, $1 \le i \le m,\ 1 \le j \le n$;
$f^*(t_it_{ij}) = 3 + 4(i-1) + 2i + 2(j-1) + 2(i-1)(n-1)$, $1 \le i \le m,\ 1 \le j \le n$;
In view of the above defined labeling pattern, the graph $mG_n$ is odd harmonious. □

An odd harmonious labeling of $3G_2$ is shown in Figure 4.

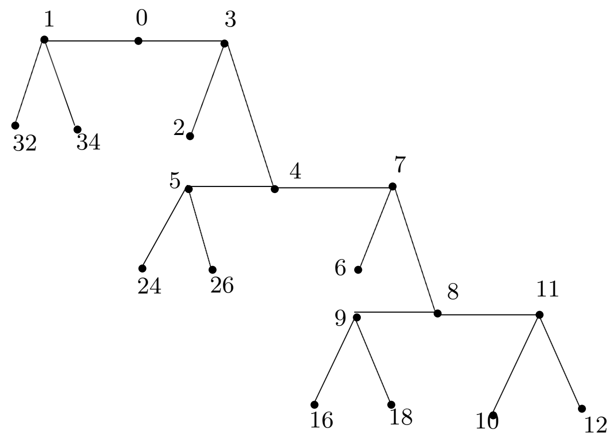


FIGURE 4. An odd harmonious labeling of $3G_2$

**Theorem 2.5.** *The graph $HSS(P_n)$ is odd harmonious.*

*Proof.* Let $G = HSS(P_n)$. Let the vertices be $u_i, u^{(1)}_{i(i+1)}, u^{(1)}_{(i+1)i}, u^{(2)}_{i(i+1)}, u^{(2)}_{(i+1)i}, u_{n+1}$; $1 \le i \le n$ and the corresponding edges are $u_iu^{(1)}_{i(i+1)}, u^{(1)}_{i(i+1)}u^{(2)}_{i(i+1)}, u^{(1)}_{i(i+1)}u^{(1)}_{(i+1)i}, u^{(1)}_{(i+1)i}u^{(2)}_{(i+1)i}, u^{(1)}_{(i+1)i}u_{i+1}$; $1 \le i \le n$ where $n \ge 1$. This graph has $5n+1$ vertices and $5n$ edges.

We define $f : V(G) \to \{0, 1, 2, \cdots, 4m(n+1)-1\}$ as follows:
$f(u_i) = 3(i-1)$, $1 \le i \le n+1$;
$f(u^{(1)}_{i(i+1)}) = 1 + 3(i-1)$, $1 \le i \le n$;
$f(u^{(1)}_{(i+1)i}) = 2 + 3(i-1)$, $1 \le i \le n$;
$f(u^{(2)}_{i(i+1)}) = 3n + 5 + 7(n-i)$, $1 \le i \le n$;
$f(u^{(2)}_{(i+1)i}) = 3n + 2 + 7(n-i)$, $1 \le i \le n$;

The induced edge labels are

$f(u_iu^{(1)}_{i(i+1)}) = 6i - 5, \ \ 1 \le i \le n;$
$f(u_{i+1}u^{(1)}_{(i+1)i}) = 6i - 1, \ \ 1 \le i \le n;$
$f(u^{(1)}_{i(i+1)}u^{(1)}_{(i+1)i}) = 6i - 3, \ \ 1 \le i \le n;$
$f(u^{(1)}_{i(i+1)}u^{(2)}_{i(i+1)}) = 10n - 4i + 3, \ \ 1 \le i \le n;$
$f(u^{(1)}_{(i+1)i}u^{(2)}_{(i+1)i}) = 10n - 4i + 1, \ \ 1 \le i \le n;$

In view of the above defined labeling pattern, the graph $HSS(P_n)$ is odd harmonious. □

An odd harmonious labeling of $HSS(P_3)$ is shown in Figure 5.

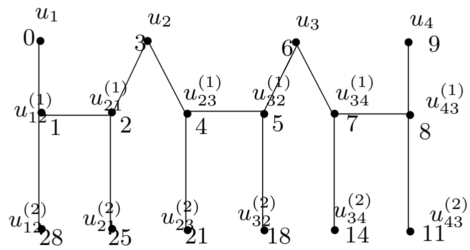


FIGURE 5. An odd harmonious labeling of $HSS(P_3)$

**Theorem 2.6.** *The graph $HSS(C_n)$, $n \equiv 0(mod\ 4)$ is odd harmonious.*

*Proof.* Let $G = HSS(C_n)$. Let the vertices be $u_i, u^{(1)}_{i(i+1)}, u^{(1)}_{(i+1)i}, u^{(2)}_{i(i+1)}, u^{(2)}_{(i+1)i}$; $1 \le i \le n-1$ and $u_n, u^{(1)}_{n1}, u^{(2)}_{n1}, u^{(1)}_{1n}, u^{(2)}_{1n}$. The corresponding edges are $u_iu^{(1)}_{i(i+1)}$, $u^{(1)}_{i(i+1)}u^{(2)}_{i(i+1)}, u^{(1)}_{i(i+1)}u^{(1)}_{(i+1)i}, u^{(1)}_{(i+1)i}u^{(2)}_{(i+1)i}, u^{(1)}_{(i+1)i}u_{(i+1)}; 1 \le i \le n-1$ and $u_nu^{(1)}_{n1}, u^{(1)}_{n1}u^{(2)}_{n1}$, $u^{(1)}_{n1}u^{(1)}_{1n}, u^{(1)}_{1n}u^{(2)}_{1n}, u^{(1)}_{1n}u_1$. This graph has $5n$ vertices and edges.
We define $f : V(G) \to \{0, 1, 2, \cdots, 10n - 1\}$ as follows:
$f(u_i) = 3i - 3$, if $i$ is odd;
$f(u_i) = 3i - 3$, if $i$ is even and $2 \le i \le \frac{n}{2}$;
$f(u_i) = 3i - 1, \ \ \frac{n}{2} \le i \le n;$
$f(u^{(1)}_{i(i+1)}) = 3i - 2, \ \ 1 \le i \le \frac{n}{2};$
$f(u^{(1)}_{i(i+1)}) = 3i - 2$, if $i$ is even and $\frac{n}{2} + 2 \le i \le n - 2$;
$f(u^{(1)}_{i(i+1)}) = 3i$, if$i$ is odd and $\frac{n}{2} + 2 \le i \le n - 1$;
$f(u^{(1)}_{n1}) = 3n - 2;$
$f(u^{(1)}_{(i+1)i}) = 3i - 1, \ \ 1 \le i \le \frac{n}{2} + 1;$
$f(u^{(1)}_{(i+1)i}) = 3i + 1, \ \ \frac{n}{2} + 2 \le i \le n$ and $i$ is even;
$f(u^{(1)}_{(i+1)i}) = 3i - 1, \ \ \frac{n}{2} + 3 \le i \le n - 1$ and $i$ is odd;
$f(u^{(1)}_{1n)}) = 3n + 1;$
$f(u^{(2)}_{i(i+1)}) = 10n - 7i + 5, \ \ 1 \le i \le \frac{n}{2};$
$f(u^{(2)}_{i(i+1)}) = 10n - 7i + 5, \ \ \frac{n}{2} \le i \le n - 2$ and $i$ is even;

$f(u_{n1}^{(2)}) = 3n + 5;$
$f(u_{i(i+1)}^{(2)}) = 10n - 7i + 3,\ \dfrac{n}{2} + 1 \leq i \leq n - 1$ and $i$ is odd;
$f(u_{(i+1)i}^{(2)}) = 10n - 7i + 2,\ \ 1 \leq i \leq \dfrac{n}{2} + 1;$
$f(u_{(i+1)i}^{(2)}) = 10n - 7i + 2,\ \ \dfrac{n}{2} + 3 \leq i \leq n - 1$ and $i$ is odd;
$f(u_{(i+1)i}^{(2)}) = 10n - 7i,\ \ \dfrac{n}{2} + 2 \leq i \leq n - 2$ and $i$ is odd;
The induced edge labels are
$f^*(u_i u_{i(i+1)}^{(1)}) = 6i - 5,\ \ 1 \leq i \leq \dfrac{n}{2};$
$f^*(u_i u_{i(i+1)}^{(1)}) = 6i - 5,\ \ \dfrac{n}{2} + 1 \leq i \leq n;$
$f^*(u_n u_{n1}^{(1)}) = 6n - 3;$
$f^*(u_1 u_{1n}^{(1)}) = 3n + 1;$
$f^*(u_{i+1} u_{(i+1)i}^{(1)}) = 6i - 1,\ \ 1 \leq i \leq \dfrac{n}{2} + 1;$
$f^*(u_{i+1} u_{(i+1)i}^{(1)}) = 6i + 1,\ \ \dfrac{n}{2} + 3 \leq i \leq n - 1;$
$f^*(u_{i(i+1)}^{(1)} u_{i(i+1)}^{(2)}) = 10n - 4i + 3,\ \ 1 \leq i \leq n - 1;$
$f^*(u_{(i+1)i}^{(1)} u_{(i+1)i}^{(2)}) = 10n - 4i + 1,\ \ 1 \leq i \leq n - 1;$
$f^*(u_{n(n-1)}^{(1)} u_{n(n-1)}^{(2)}) = 6n + 5;$
$f^*(u_{n1}^{(1)} u_{n1}^{(2)}) = 6n + 3;$

In view of the above defined labeling pattern, the graph $HSS(C_n)$, $n \equiv 0(mod\ 4)$ is odd harmonious. □

An odd harmonious labeling of $HSS(C_8)$ is shown in Figure 6.

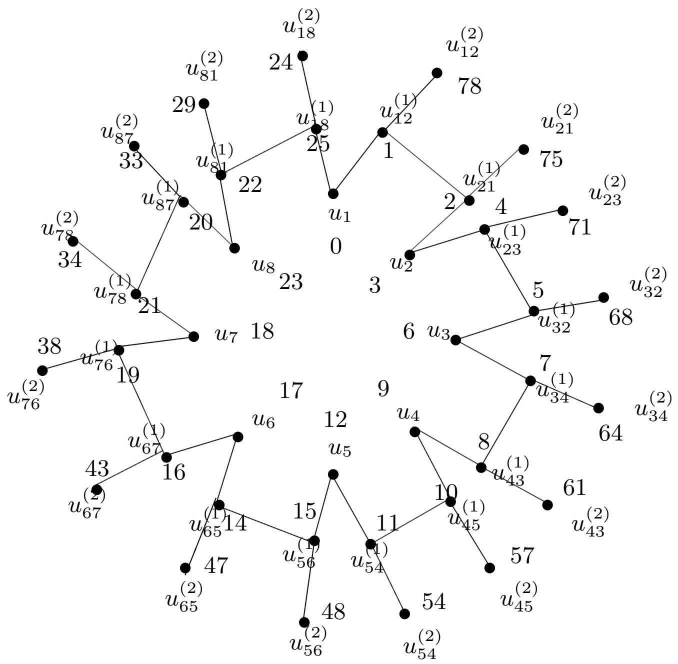

FIGURE 6. A odd harmonious labeling of $HSS(C_8)$

Further, we conclude the paper with the following observations. Since it is easier to prove the observations, we omit the proofs.

**Observation 2.1.** *Every strongly odd harmonious graph admits an odd sequential labeling.*

**Observation 2.2.** *Every strongly odd harmonious graph admits mean labeling.*

**Observation 2.3.** *Every strongly odd harmonious graph admits an odd mean labeling.*

**Observation 2.4.** *Every odd sequential graph admits an odd harmonious labeling.*

**Observation 2.5.** *Every odd harmonious graph $G$ admits an even sequential harmonious labeling.*

## 3. Conclusion

In this paper, we obtain some new results showing that the graphs $Spl(C_{bn})$, $Spl(B(m)_{(n)})$, slanting ladder $SL_n$, $mG_n$, H-super subdivision of path $P_n$ and cycle $C_n$, $n \equiv 0(mod\ 4)$ are odd harmonious. Also, we observe that all strongly odd harmonious graphs admit mean labeling, odd mean labeling, odd sequential labeling and all odd sequential graphs are odd harmonious and all odd harmonious graphs are even sequential harmonious.

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## References

[1] Abdel-Aal M. E., (2013), Odd Harmonious Labeling of Cyclic Snakes, International Journal on Applications of Graph Theory in Wireless Adhoc Networks and Sensor Networks, 5, (3), pp. 1-13.

[2] Abdel-Aal, (2014), New Families of Odd Harmonious Graphs, International Journal of Soft Computing, Mathematics and Control, 3, (1), pp. 1-13.

[3] Abdel-Aal, (2016), Further Results on Odd Harmonious Graphs, International Journal on Applications of Graph Theory in Wireless Adhoc Networks and Sensor Networks, Vol.8, No.3/4, pp. 1-14.

[4] Esakkiammal E., Thirusangu, K. and Seethalakshmi, S., (2017), Lucky edge labeling of $H$-super subdivision of graphs, Annals of Pure and Applied Mathematics, Vol.14, No.3, pp.601-610.

[5] Firmansah, F., Yuwono, M. R., (2017), Odd Harmonious Labeling of Pleated of the Dutch Windmill Graphs, $\chi$ Cauchy Jurnal Matematika Murni dan Aplikasi, Vol.4, No.4, pp.161-166.

[6] Firmansah, F., (2017), The Odd Harmonious Labeling on Variation of the Double Quadrilateral Windmill Graphs, Jurnal ILMU DASAR, Vol.18, No.2, pp. 109-118.

[7] Gallian J. A., (2018), A Dynamic Survey of Graph Labeling, The Electronic Journal of Combinatorics, #DS6.

[8] Gayathri, B., Hemalatha, V., (2008), Even Sequential Harmonious Labeling of Some Graphs, National Conference held at Govt. College for Women, Pudukottai.

[9] Graham, R. L. and Sloane, N. J. A., (1980), On Additive bases and Harmonious Graphs, SIAM J. Algebr. Disc. Meth., 4, pp. 382-404.

[10] Gusti Saputri, A., Sugeng, K. A., and Froncek D., (2013), The Odd Harmonious Labeling of Dumbell and Generalized Prism Graphs, AKCE Int. J. Graphs Comb., 10, (2), pp. 221-228.

[11] Harary F., (1972), Graph theory, Addison Wesley, Massachusetts.

[12] Jeyanthi, P., Philo, S. and Kiki, A. S., (2015), Odd Harmonious Labeling of Some New Families of Graphs, SUT Journal of Mathematics, Vol.51, No.2, pp. 53-65.

[13] Jeyanthi P., Philo, S., (2016), Odd Harmonious Labeling of Some Cycle Related Graphs, Proyecciones Journal of Mathematics, Vol.35, No.1, pp. 85-98.

[14] Jeyanthi, P., Philo, S., (2017), Odd Harmonious Labeling of Plus Graphs, Bulletin of the International Mathematical Virtual Institute, Vol.7, pp. 515-526.

[15] Jeyanthi, P., Philo, S., (2019), Siddiqui, M. K., Odd Harmonious Labeling of Super Subdivision Graphs, Proyecciones Journal of Mathematics, Vol.38, No.1, pp. 1-11.
[16] Jeyanthi, P., Philo, S., (2019), Odd Harmonious Labeling of Subdivided Shell Graphs, International Journal of Computer Sciences and Engineering, Vol.7, (5), pp.77-80.
[17] Jeyanthi, P., Philo, S., (2019), Odd Harmonious Labeling of Certain Graphs, Journal of Applied Science and Computations, Vol.6, (4), pp. 1224-1232.
[18] Jeyanthi, P., Philo S., (2019), Some Results on Odd Harmonious Labeling of Graphs, Bulletin of the International Mathematical Virtual Institute, Vol. 9, pp. 567-576.
[19] Jeyanthi, P., Philo, S., (2019), Odd Harmonious Labeling of Some New Graphs, Southeast Asian Bulletin of Mathematics, Vol. 43, pp. 509-523.
[20] Jeyanthi, P., Philo, S., Youssef, M. Z., (2019), Odd Harmonious Labeling of Grid Graphs, Proyecciones Journal of Mathematics, Vol. 38, No.3, pp. 411-429.
[21] Jeyanthi, P., Philo, S., (2020), Odd Harmonious Labeling of Some Classes of Graphs, CUBO, A Mathematical Journal, Vol.22, No.3, pp. 299-314.
[22] Jeyanthi P, Philo S., Odd Harmonious Labeling of Step Ladder Graphs, Utilitas Mathematica, preprint.
[23] Liang Z, Bai Z., (2009), On the Odd Harmonious Graphs with Applications, J. Appl. Math. Comput., 29, pp. 105-116.
[24] Manikam K, Marudai M., (2006), Odd Mean Labeling of Graphs, Bulletin of Pure and Applied Sciences, 25, E(1), pp. 149-153.
[25] Selvaraju, P., Balaganesan, P. and Renuka, J., (2013), Odd Harmonious Labeling of Some Path Related Graphs, International Journal of Mathematical Sciences and Engg. Appls., 7, (3), pp. 163-170.
[26] Singh, G.S., Varkey, T. K. M., On Odd Sequential and bi Sequential Graphs, preprint.
[27] Somasundaram, S., Ponraj, R., (2003), Mean Labeling of Graphs, National. Acad. Sci., Vol.26, pp. 210-213.
[28] Subbiah, M., (2011), A Study on Some Variations of Graph Labeling and its Applications in Various Fields, Unpublished Thesis, Submitted to Bharathidasan University.
[29] Vaidya, S. K., Shah, N. H., (2011), Some New Odd Harmonious Graphs, International Journal of Mathematics and Soft Computing, Vol.1, No.1, pp. 9-16.
[30] Vaidya, S. K., Shah, N. H., (2012), Odd Harmonious Labeling of Some Graphs, International J. Math. Combin, Vol.3, pp. 105-112.
[31] Vasuki, R., Arockiaraj, S., (2013), On Mean Graphs, International J. Math. Combin., Vol. 3, pp. 22-34.

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