TWMS J. App. and Eng. Math. V.12, N.4, 2022, pp. 1329-1335

NEW WAVE FORM SOLUTIONS OF TIME-FRACTIONAL GARDNER EQUATION VIA FRACTIONAL RICCATI EXPANSION METHOD

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ABSTRACT. In this current paper, the fractional Riccati expansion method is proposed for obtaining the new exact solutions of the time-fractional Gardner equation. The fractional derivative is considered in the sense of Jumarie's modified Riemann-Liouville fractional derivative (JMRFD). A travelling wave transformation is firstly utilized to convert the nonlinear fractional partial differential equation (NFPDE) into a fractional ordinary differential equation (FODE). Our main intention in this present paper is to indicate that the suggested method is appropriate to obtain the new exact solutions of fractional partial differential equations. It can be said that the main advantage of the mentioned scheme is very simple and easy to apply. As a result, all the obtained results are presented in the paper.

Keywords: Time-fractional Gardner equation, Fractional Riccati expansion method, Jumarie's modified Riemann-Liouville derivative, Mittag-Leffler function.

AMS Subject Classification: 26A33, 34K37, 34A08.

1. Introduction

Nonlinear fractional partial differential equations (NFPDEs) have impelled common attention in many disciplines [1, 2] in recent years. It is observed that fractional models can be better describe nonlinear physical problems and propagation characteristics in real systems. In order to depict the fractional models of the physical or engineering phenomena, scientists used different definitions of fractional derivative and integral such as Caputo, Riemann-Liouville, Grunwald-Letnikov and Modified Riemann Liouville fractional derivatives, etc [3, 4, 5]. All of them are constructed for remodeling of variety applications such as ocean modeling, atmospheric dynamics, physics turbulence, nonlinear propagation ion-acoustic waves, the description of the interior waves in shallow water, and stochastic dynamical systems [6, 7, 8, 9].

The classical Gardner equation which is combined KdV-mKdV, or eKdV, can describe most of the phenomena such as the interior waves in shallow water [9], ion acoustic waves in plasma with negative ion [10], and the long wave propagation in an homogeneous two-layer shallow liquid [11]. Then, the classical Gardner equation is converted into the

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§ Manuscript received: September 13, 2020; accepted: May 10, 2021.
TWMS Journal of Applied and Engineering Mathematics, Vol.12, No.4

time-fractional Gardner equation by the authors [12]. They are used Agarwal's method to obtain the time-fractional Gardner equation. The time-fractional Gardner equation can be used to designate many physical phenomena such as the nonlinear propagation of ion-acoustic waves in an magnetized plasma [13, 14, 15]. There is a big significant interest in the construction of the new solutions to this equation. Shimin Guo et. al. [12] is used variation iteration method to solve the equation numerically. Another approximate solution of the equation is obtained by using a modified version of the generalized Taylor power series method [9]. A variety kinds of new analytical solutions of the mentioned equation is constructed by the authors [16, 17, 18]. Many important results are obtained in the mentioned studies.

In this paper, we consider the time-fractional Gardner equation as

$$D_t^{\alpha}u + 6(u - \epsilon^2 u^2)u_x + u_{xxx} = 0, \quad 0 < \alpha \leq 1 \tag{1}$$

where ϵ is a nonzero constant, and the D_t^{α} is the JMRFD which is described by Jumarie as

$$D_t^{\alpha}v(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\xi)^{-\alpha}(v(\xi)-v(0))d\xi, & 0<\alpha<1, \\ (v^{(m)}(t))^{\alpha-m}, & m \leq \alpha < m+1, \quad m \geq 1 \end{cases} \tag{2}$$

This definition has some prominent properties. They are given [19] following as

$$D_t^{\alpha}t^p = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)}t^{\alpha-p}, \quad p > 0 \tag{3}$$

$$D_t^{\alpha}(f(t)g(t)) = f(t)D_t^{\alpha}g(t) + g(t)D_t^{\alpha}f(t). \tag{4}$$

To our knowledge, the time-fractional Gardner equation is not solved by using the fractional Riccati expansion method. Therefore, we use the proposed method to find the new solutions of the mentioned equation. Compared with other methods, the main advantage of this method transforms FPDE into FODE with the same order by using traveling wave transformation. Then we can give the exact solutions of fractional ordinary equations using the solutions of the fractional Riccati equation. In addition to this, the mentioned scheme is very simple and easy to apply.

The structure of the paper is organized as follows. Section 2 introduces the fractional Riccati expansion method. In Section 3, we construct the implementation of the suggested method for the time-fractional Gardner equation. Finally, conclusions of this study are given in Section 4.

2. Fundamental of the fractional Riccati expansion method

In this section, the fractional Riccati expansion method is presented to solve the NFPDE. For a given NFPDE, say in two variables, x and t,

$$P(u, u_t, u_x, D_t^{\alpha}u, D_x^{\alpha}u, \cdots) = 0, \quad 0 < \alpha \leq 1, \tag{5}$$

where $D_t^{\alpha}u$ and $D_x^{\alpha}u$ are JMRFD [5] of the unknown function $u = u(x,t)$, and P is a polynomial of u and its various partial derivatives containing the highest order derivatives and nonlinear terms. There are main steps to implement the proposed method. These are as follows.

Stage 1: A travelling wave transformation is used for reducing fractional partial differential equation into a FODE. In this state, the travelling wave transformation is described as

$$u(x,t) = U(\xi), \quad \xi = x + ct \tag{6}$$

where c is an arbitrary constant to be determined later. Eqn. (5) can be rewritten by using Eqn. (6) as the following FODE for $u = U(\xi)$:

$$\tilde{P}(U, c^{\alpha} D_{\xi}^{\alpha} U, D_{\xi}^{\alpha} U, c^{2\alpha} D_{\xi}^{2\alpha} U, \cdots) = 0 \tag{7}$$

Stage 2: Assuming the exact solution of Eqn. (7) can be represented by a finite power series of $F(\xi)$ as the following form:

$$U(\xi) = a_0 + \sum_{j=1}^{n} a_j F^j(\xi), \quad a_n \neq 0 \tag{8}$$

where a_j $(j = 0, 1, 2, \cdots, n)$ are arbitrary constants to be determined later, n is a positive integer determined by balancing the linear term of the with nonlinear term in (7). Also, $F(\xi)$ satisfies the fractional Riccati equation which is defined by

$$D_{\xi}^{\alpha} F(\xi) = A + BF^2(\xi), \quad 0 < \alpha \leq 1 \tag{9}$$

where A and B are constants. The equation (9) has the following solutions by using the Mittag-Leffler function in one-parameter $E_{\alpha}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1+k\alpha)}$, $\alpha > 0$:

Case 1: If $A = B = 1$, then the solution is $F = \tan(\xi, \alpha)$.

Case 2: If $A = B = -1$, then the solution is $F = \cot(\xi, \alpha)$.

Case 3: If $A = 1$, $B = -1$, then the solutions are $F_1 = \tanh(\xi, \alpha)$, $F_2 = \coth(\xi, \alpha)$.

Case 4: If $A = \frac{1}{2}$, $B = -\frac{1}{2}$, then the solutions are $F_1 = \frac{\tanh(\xi,\alpha)}{1 \pm \operatorname{sech}(\xi,\alpha)}$, $F_2 = \coth(\xi, \alpha) \pm \operatorname{csch}(\xi, \alpha)$.

Case 5: If $A = B = \frac{1}{2}$, then the solutions are $F_1 = \frac{\tan(\xi,\alpha)}{1 \pm \sec(\xi,\alpha)}$, $F_2 = -\cot(\xi, \alpha) + \csc(\xi, \alpha)$ and $F_3 = \tan(\xi, \alpha) \pm \sec \xi, \alpha$.

Case 6: If $A = B = -\frac{1}{2}$, then the solutions are $F_1 = \frac{\cot(\xi,\alpha)}{1 \pm \csc(\xi,\alpha)}$, $F_2 = \sec(\xi, \alpha) - \tan(\xi, \alpha)$ and $F_3 = \cot(\xi, \alpha) \pm \csc \xi, \alpha$.

Case 7: If $A = 1$, $B = -4$, then the solution is $F = \frac{\tanh(\xi,\alpha)}{1+\tanh^2(\xi,\alpha)}$.

Case 8: If $A = 1$, $B = 4$, then the solution is $F = \frac{\tan(\xi,\alpha)}{1-\tan^2(\xi,\alpha)}$.

Case 9: If $A = -1$, $B = -4$, then the solution is $F = \frac{\cot(\xi,\alpha)}{1-\cot^2(\xi,\alpha)}$.

In this study, the generalized hyperbolic and trigonometric functions are defined as:

$$\cosh(\xi, \alpha) = \frac{E_{\alpha}(\xi^{\alpha}) + E_{\alpha}(-\xi^{\alpha})}{2} \quad , \sinh(\xi, \alpha) = \frac{E_{\alpha}(\xi^{\alpha}) - E_{\alpha}(-\xi^{\alpha})}{2}$$

$$\cos(\xi, \alpha) = \frac{E_{\alpha}(i\xi^{\alpha}) + E_{\alpha}(-i\xi^{\alpha})}{2} \quad , \sin(\xi, \alpha) = \frac{E_{\alpha}(i\xi^{\alpha}) - E_{\alpha}(-i\xi^{\alpha})}{2}$$

$$\tanh(\xi, \alpha) = \frac{\sinh(\xi, \alpha)}{\cosh(\xi, \alpha)}, \coth(\xi, \alpha) = \frac{\cosh(\xi, \alpha)}{\sinh(\xi, \alpha)}$$

$$\operatorname{sech}(\xi, \alpha) = \frac{1}{\cosh(\xi, \alpha)}, csch(\xi, \alpha) = \frac{1}{\sinh(\xi, \alpha)}$$

$$\tan(\xi, \alpha) = \frac{\sin(\xi, \alpha)}{\cos(\xi, \alpha)}, \cot(\xi, \alpha) = \frac{\cos(\xi, \alpha)}{\sin(\xi, \alpha)}$$

$$\sec(\xi, \alpha) = \frac{1}{\cos(\xi, \alpha)}, \csc(\xi, \alpha) = \frac{1}{\sin(\xi, \alpha)}$$

Stage 3: After applying the fractional Riccati expansion method (8) into the FODE (7) the left hand side (7) can be converted into a polynomial in $F(\xi)$. Setting each coefficient

of the polynomial to zero yields system of algebraic equations for $a_0, a_1, a_2, \cdots, a_n$ and c.
Stage 4: By solving the system obtained in Stage 3, the constants $a_0, a_1, a_2, \cdots, a_n$ and c can be expressed by the parameters A and B. Depending on the chosen values A and B the function $F(\xi)$ possesses the travelling wave solutions are given above, then the fractional Riccati expansion method (8) has the travelling wave solution of the NFDE (5).

3. Governing of the fractional Riccati expansion scheme to the time-fractional Gardner equation

In this section, we will implement the suggested techniques described in the above to obtain the exact wave solution of the time-fractional Gardner equation. By substituting the travelling wave transformation (6) into Eqn. (1), the following FODE can be found

$$c^{\alpha} D_{\xi}^{\alpha} U + 6(U - \epsilon^2 U^2)U' + U''' = 0 \tag{10}$$

Balancing number is

$$\begin{aligned} N+3 &= 2N + (N+1) \\ N &= 1 \end{aligned}$$

We suppose that $U(\xi)$ can be expressed by a finite power series of $F(\xi)$ as follows:

$$U(\xi) = a_0 + a_1 F(\xi), \quad a_1 \neq 0 \tag{11}$$

where a_0, a_1 are arbitrary constants to be determined later. Now substituting Eqn. (11) into Eqn. (10), we get a polynomial in $F(\xi)$. Then collecting the coefficients of $(F(\xi))^j, (j = 0, 1, 2, 3, 4)$ and setting them to be zero and to solve the obtained system of equations with the help of Maple, we have the following solution set:

$$a_0 = \pm \frac{B}{2\epsilon^3}, a_1 = \pm \frac{B}{\epsilon}, \epsilon = \pm B, c = \exp\left(\frac{\ln(-\frac{4AB^3+3}{2B^2})}{\alpha}\right)$$

We get the following solutions of (1) by choosing the special values of A and B and the corresponding function $F(\xi)$,

Case I: The solutions are obtained by choosing the values of $A = B = 1$, we can obtain trigonometric solutions of Eqn. (1),

$$u_1(x,t) = \frac{1}{2} + \tan(x + ct, \alpha) \tag{12}$$

$$u_2(x,t) = \frac{1}{2} + \cot(x + ct, \alpha) \tag{13}$$

Case II: For $A = 1$ and $B = -1$, the hyperbolic solutions are

$$u_3(x,t) = \frac{1}{2} + \tanh(x + ct, \alpha) \tag{14}$$

$$u_4(x,t) = \frac{1}{2} + \coth(x + ct, \alpha) \tag{15}$$

Case III: For $A = \frac{1}{2}$, $B = -\frac{1}{2}$, we have

$$u_5(x,t) = 2 + \frac{\tanh(x + ct, \alpha)}{1 \pm \operatorname{sech}(x + ct, \alpha)} \tag{16}$$

$$u_6(x,t) = 2 + \frac{\coth(x + ct, \alpha)}{1 \pm \operatorname{sech}(x + ct, \alpha)} \tag{17}$$

Other remaining results can be generated in a similar fashion. By selecting the value of $\alpha = 1$, the classical Gardner equation is obtained as a special form of the equation (1).

4. Graphical Representation of the Solutions

In this section, we will presented only a few figures of the wave solutions of the time-fractional Gardner equation by setting specific values of the unknown parameters. Figs. 1-2 are the graphical visualization of the fractional soliton solutions for Eqn. (1) with $\alpha = 0.5$ and $\epsilon = 1$.

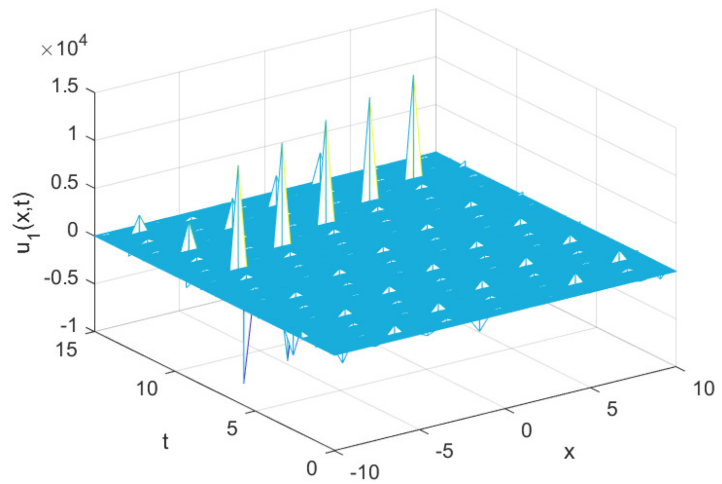

Figure 1. Graphical illustration of the wave solution for $u_1(x,t)$ and $\alpha = 0.5$

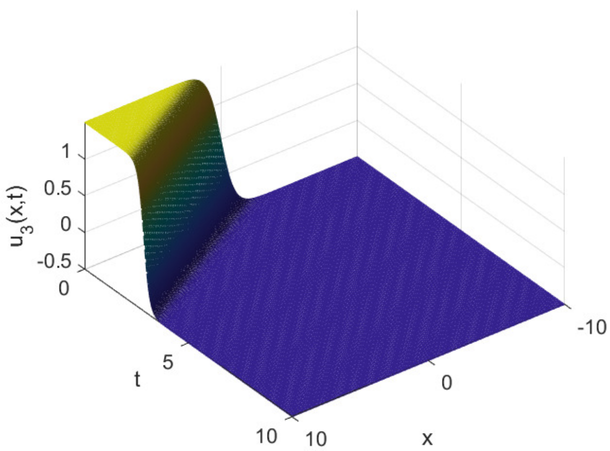

Figure 2. Graphical illustration of the wave solution for $u_3(x,t)$ and $\alpha = 0.5$

5. Conclusion

In this study, we have obtained new exact solutions of the time-fractional Gardner equation by using the fractional Riccati expansion method. Jumarie's modified Riemann-Liouville fractional derivative is preferred for time fractional derivative. Besides, a travelling wave transform that is simple and effective, is implemented to convert FPDE into an FODE. We would also like to say that the method can be used for many other nonlinear fractional differential equations. Another point that is worthy of being emphasize the fractional Riccati expansion technique satisfies that it reliable and effective to seek many other nonlinear fractional differential equations in mathematical physics and engineering and the computation procedure of the suggested method is simple and easy.

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