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# A NOTE ON HERMITE MATRIX BASED NARUMI POLYNOMIALS

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Abstract. In this article, the 2-index 2-variable Hermite matrix based Narumi polynomials are introduced by means of generating function. Some important properties including operational representation and quasi-monomiality of these polynomials are established. For suitable values of indices and variables, the 2-index 2-variable Hermite matrix polynomials yield several special matrix polynomials. Consequently, the results for the corresponding new special polynomials related to Narumi polynomials are presented.

Keywords: 2-index 2-variable Hermite matrix polynomials; Narumi polynomials; Monomiality principle; Operational techniques.

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## 1. Introduction and preliminaries

The special polynomials with matrix parameters provide the solutions of special matrix differential equations. These matrix differential equations are the systems of differential equations, each of which is satisfied by the corresponding scalar special polynomial. In the same way the other results for special matrix polynomials like generating functions, series definitions, recurrence relations *etc.* can be viewed as the systems of equations, which are satisfied by the corresponding scalar special polynomials.

If $A$ is a matrix in $\mathbb{C}^{r\times r}$ $(r \in \mathbb{N})$, its spectrum $\sigma(A)$ denotes the set of all the eigenvalues of $A$ and the 2-norm of $A$, denoted by $\|A\|$, is defined by

$$\|A\| = \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}, \tag{1}$$

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where $\|y\|_2 = (y^T y)$ is the Euclidean norm of any $y$ in $\mathbb{C}^{r\times r}$ $(r \in \mathbb{N})$. The real numbers $\alpha(A)$ and $\beta(A)$ are defined by

$$\alpha(A) = \max\{Re(z) : z \in \sigma(A)\}, \quad \beta(A) = \min\{Re(z) : z \in \sigma(A)\}. \tag{2}$$

A matrix $A$ in $\mathbb{C}^{r\times r}$ $(r \in \mathbb{N})$ is said to be positive stable if

$$Re(\mu) \nleq 0, \quad \mu \in \sigma(A), \quad \sigma(A) := \text{spectrum of A}. \tag{3}$$

If $f(z)$ and $g(z)$ are holomorphic functions in an open set $\Omega$ of the complex plane and if $\sigma(A) \subset \Omega$, then from the Riesz-Dunford functional calculus [5, p. 558]:

$$f(A)g(A) \ = \ g(A)f(A), \tag{4}$$

where $f(A)$ and $g(A)$ denote the images of functions $f(z)$ and $g(z)$ respectively, acting on the matrix $A$.

If $D_0$ is the complex plane cut along the negative real axis and $\log(z)$ denotes the principal logarithm of $z$, then $z^{1/2}$ represents $\exp(\frac{1}{2}\log(z))$. If matrix $A \in \mathbb{C}^{r\times r}$ $(r \in \mathbb{N})$ with $\sigma(A) \subset D_0$, then $A^{1/2} = \sqrt{A}$ denotes the image by $z^{1/2}$ of the matrix functional calculus acting on the matrix A.

Recall that the 2-index 2-variable Hermite Matrix polynomials (2I2VHMP) $H_{n,m}(x, y; A)$ are defined by the following generating function [9]:

$$\exp(xt\sqrt{mA} - yt^m I) = \sum_{n=0}^{\infty} H_{n,m}(x, y; A)\frac{t^n}{n!}, \tag{5}$$

with $m$ must be a positive integer and $I$ is the unit matrix in $\mathbb{C}^{r\times r}$ $(r \in \mathbb{N})$.

The $H_{n,m}(x, y; A)$ have the following explicit representation:

$$H_{n,m}(x, y; A) = n! \sum_{k=0}^{[\frac{n}{m}]} \frac{(-1)^k y^k (x\sqrt{mA})^{n-mk}}{k!(n-mk)!} \tag{6}$$

and the differential equation satisfied by $H_{n,m}(x, y; A)$ is [9]

$$\left(my(\sqrt{mA})^{-m}\frac{\partial^m}{\partial x^m} - x\frac{\partial}{\partial x} + n\right) H_{n,m}(x, y; A) = 0. \tag{7}$$

For suitable values of the indices and variables, number of known special polynomials as special cases of the 2I2VHMP $H_{n,m}(x, y; A)$ are mentioned in the following table.

Table 1. Some special cases of the 2I2VHMP $H_{n,m}(x,y;A)$

| S. No. | Values of the indices and variables | Relation between $H_{n,m}(x,y;A)$ and its special cases | Name of the known polynomials | Generating function |
|---|---|---|---|---|
| I. | $m=2$ | $H_{n,2}(x,y;A)=H_n(x,y,A)$ | 2-variable Hermite matrix polynomials $H_n(x,y;A)$ [2] | $\exp(xt\sqrt{2A}-yt^2I)$ $=\sum_{n=0}^{\infty}H_n(x,y;A)\frac{t^n}{n!}$ |
| II. | $m=2;\ x\to\frac{x}{2}$, $y\to-y$ | $H_{n,2}\left(\frac{x}{2},-y;A\right)=\mathcal{H}_n(x,y;A)$ | 2-variable Hermite matrix polynomials of the second form $\mathcal{H}_n(x,y;A)$ [8] | $\exp(xt\sqrt{\frac{A}{2}}+yt^2I)$ $=\sum_{n=0}^{\infty}\mathcal{H}_n(x,y;A)\frac{t^n}{n!}$ |
| III. | $y=1$ | $H_{n,m}(x,1;A)=H_{n,m}(x,A)$ | Generalized Hermite matrix polynomials $H_{n,m}(x;A)$ [11] | $\exp(xt\sqrt{mA}-t^mI)$ $=\sum_{n=0}^{\infty}H_{n,m}(x;A)\frac{t^n}{n!}$ |
| IV. | $m=2;\ y=1$ | $H_{n,2}(x,1;A)=H_n(x,A)$ | Hermite matrix polynomials $H_n(x;A)$ [6] | $\exp(xt\sqrt{2A}-t^2I)$ $\sum_{n=0}^{\infty}H_n(x;A)\frac{t^n}{n!}$ |

The idea of monomiality arised within the context of poweroid, suggested by J. F. Steffenson [12]. The monomiality principle is reformulated and developed by Dattoli [3], according to which, the polynomial set $\{p_n(x)\}_{n\in\mathbb{N}}$ is "quasi-monomial", provided there exist two operators $\hat{M}$ and $\hat{P}$ playing, respectively, the role of multiplicative and derivative operators, for the family of polynomials. These operators satisfy the following identities:

$$\hat{M}\{p_n(x)\}=p_{n+1}(x) \tag{8}$$

and

$$\hat{P}\{p_n(x)\}=n\ p_{n-1}(x), \tag{9}$$

for all $n\in\mathbb{N}$. The operators $\hat{M}$ and $\hat{P}$ also satisfy the commutation relation

$$[\hat{P},\hat{M}]=\hat{P}\hat{M}-\hat{M}\hat{P}=\hat{1} \tag{10}$$

and thus display the Weyl group structure. If the considered polynomial set $\{p_n(x)\}_{n\in\mathbb{N}}$ is quasi-monomial, its properties can easily be derived from those of the $\hat{M}$ and $\hat{P}$ operators. In fact:

(i) Combining recurrences (8) and (9), we have

$$\hat{M}\,\hat{P}\{p_n(x)\}=n\ p_n(x), \tag{11}$$

which can be interpreted as the differential equation satisfied by $p_n(x)$, if $\hat{M}$ and $\hat{P}$ have a differential realization.

(ii) Assuming here and in the sequel $p_0(x)=1$, then $p_n(x)$ can be explicitly constructed as:

$$p_n(x)=\hat{M}^n\{1\}, \tag{12}$$

which yields the series definition for $p_n(x)$.

(iii) Identity (12) implies that the exponential generating function of $p_n(x)$ can be given in the form:

$$\exp(t\hat{M})\{1\}=\sum_{n=0}^{\infty}p_n(x)\frac{t^n}{n!},\quad |t|<\infty. \tag{13}$$

The 2I2VHMP $H_{n,m}(x,y;A)$ are shown to be quasi-monomial under the action of the following multiplicative and derivative operators [9]:

$$\hat{M}_H:=x\sqrt{mA}-my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}} \tag{14}$$

and

$$\hat{P}_H := \frac{1}{\sqrt{mA}}\frac{\partial}{\partial x}, \tag{15}$$

respectively.

The Narumi polynomials $N_n^{(a)}(x)$ [1] are defined by the following generating function:

$$\left(\frac{t}{\ln(1+t)}\right)^a (1+t)^x = \sum_{n=0}^{\infty} N_n^{(a)}(x)\frac{t^n}{n!} \tag{16}$$

or

$$\left(\frac{t}{\ln(1+t)}\right)^a \exp\left(x\ln(1+t)\right) = \sum_{n=0}^{\infty} N_n^{(a)}(x)\frac{t^n}{n!}. \tag{17}$$

The article is organized as: In Section 2, the 2-index 2-variable Hermite matrix based Narumi polynomials are introduced and framed within the context of monomiality principle. In Section 3, some examples are considered.

## 2. 2-index 2-variable Hermite matrix based Narumi polynomials

To introduce the 2-index 2-variable Hermite matrix based Narumi polynomials (2I2VHMNP) denoted by ${}_HN_{n,m}^{(a)}(x,y;A)$, we prove the following result:

**Theorem 2.1.** *For the 2I2VHMNP ${}_HN_{n,m}^{(a)}(x,y;A)$, the following generating function holds true:*

$$\left(\frac{t}{\ln(1+t)}\right)^a \exp\left(x\sqrt{mA}\ln(1+t)) - y(\ln(1+t))^m I\right) = \sum_{n=0}^{\infty} {}_HN_{n,m}^{(a)}(x,y;A)\frac{t^n}{n!}. \tag{18}$$

*Proof.* Replacing $x$ in the l.h.s. and r.h.s of equation (17) by the multiplicative operator $\hat{M}_H$ of the 2I2VHMP we have,

$$\left(\frac{t}{\ln(1+t)}\right)^a \exp(\hat{M}_H\ln(1+t)) = \sum_{n=0}^{\infty} N_n^{(a)}(\hat{M}_H)\frac{t^n}{n!}. \tag{19}$$

On using equation (14) and denoting resultant 2I2VHMNP in the r.h.s. by ${}_HN_{n,m}^{(a)}(x,y;A)$ that is

$$N_n^{(a)}(\hat{M}_H) = N_n^{(a)}\left(x\sqrt{mA} - my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}}\right) = {}_HN_{n,m}^{(a)}(x,y;A), \tag{20}$$

we obtain

$$\left(\frac{t}{\ln(1+t)}\right)^a \exp\left(\left(x\sqrt{mA} - my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}}\right)\ln(1+t)\right) = \sum_{n=0}^{\infty} {}_HN_{n,m}^{(a)}(x,y;A))\frac{t^n}{n!} \tag{21}$$

or

$$\left(\frac{t}{\ln(1+t)}\right)^a \exp\left(\left(x\sqrt{mA} - my\frac{\partial^{m-1}}{\partial(x\sqrt{mA})^{m-1}}\right)\ln(1+t)\right) = \sum_{n=0}^{\infty} {}_HN_{n,m}^{(a)}(x,y;A)\frac{t^n}{n!}. \tag{22}$$

Now, decoupling the exponential in l.h.s of equation (22) by using the Crofton-type identity [4, p. 12]:

$$f\left(x+m\lambda\frac{d^{m-1}}{dx^{m-1}}\right)\{1\}=\exp\left(\lambda\frac{d^m}{dx^m}\right)\{f(x)\}, \tag{23}$$

we obtain

$$\left(\frac{t}{\ln(1+t)}\right)^a \exp\left(-y\frac{\partial^m}{\partial(x\sqrt{mA})^m}\right)\exp\left(x\sqrt{mA}\ln(1+t)\right)=\sum_{n=0}^{\infty}{}_HN_{n,m}^{(a)}(x,y;A)\frac{t^n}{n!}. \tag{24}$$

On expanding the first exponential in above equation, we obtain assertion (18).

□

From equation (18), the following relation is obtained:

$$\frac{\partial}{\partial y}{}_HN_{n,m}^{(a)}(x,y;A)=-(\sqrt{mA})^{-m}\frac{\partial^m}{\partial x^m}{}_HN_{n,m}^{(a)}(x,y;A). \tag{25}$$

**Corollary 2.1.** *The following operational representation connecting the 2I2VHMNP* ${}_HN_{n,m}^{(a)}(x,y;A)$ *with the Narumi polynomials* $N_n^{(a)}(x)$ *holds true:*

$${}_HN_{n,m}^{(a)}(x,y;A)=\exp\left(-y\frac{\partial^m}{\partial(x\sqrt{mA})^m}\right)N_n^{(a)}(x\sqrt{mA}). \tag{26}$$

*Proof.* From equation (20), we can write

$${}_HN_{n,m}^{(a)}(x,y;A)=N_n^{(a)}\left(x\sqrt{mA}-my\frac{\partial^{m-1}}{\partial(x\sqrt{mA})^{m-1}}\right). \tag{27}$$

Using equation (23), assertion (26) follows.

□

In order to frame the 2I2VHMNP ${}_HN_{n,m}^{(a)}(x,y;A)$ within the context of monomiality principle formalism, we prove the following result:

**Theorem 2.2.** *The 2I2VHMNP* ${}_HN_{n,m}^{(a)}(x,y;A)$ *are quasi monomial under the action of the following multiplicative and derivative operators:*

$$\hat{M}_{HN}:=\left(x\sqrt{mA}-my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}}+\frac{a\left(e^{\frac{D_x}{\sqrt{mA}}}\left(\frac{D_x}{\sqrt{mA}}-1\right)+1\right)}{\left(e^{\frac{D_x}{\sqrt{mA}}}-1\right)\frac{D_x}{\sqrt{mA}}}\right)\frac{1}{e^{\frac{D_x}{\sqrt{mA}}}} \tag{28}$$

*and*

$$\hat{P}_{HN}:=e^{\frac{D_x}{\sqrt{mA}}}-1, \tag{29}$$

*respectively.*

*Proof.* Consider the following identity:

$$D_x\left\{\left(\frac{t}{\ln(1+t)}\right)^a\exp\left(x\sqrt{mA}\ \ln(1+t)-y(\ln(1+t))^mI\right)\right\}$$
$$=\sqrt{mA}\ \ln(1+t)\left\{\left(\frac{t}{\ln(1+t)}\right)^a\exp\left(x\sqrt{mA}\ \ln(1+t)-y(\ln(1+t))^mI\right)\right\} \tag{30}$$

or

$$\left(e^{\frac{D_x}{\sqrt{mA}}}-1\right)\left\{\left(\frac{t}{\ln(1+t)}\right)^a \exp\left(x\sqrt{mA}\ \ln(1+t)-y(\ln(1+t))^m I\right)\right\}$$
$$=t\left\{\left(\frac{t}{\ln(1+t)}\right)^a \exp\left(x\sqrt{mA}\ \ln(1+t)-y(\ln(1+t))^m I\right)\right\}. \quad (31)$$

Differentiating equation (19) partially with respect to $t$ and using equation (20) in the r.h.s. of the resultant equation, we find

$$\left(\left(\hat{M}_H+\frac{a((1+t)\ln(1+t)-t)}{t\ln(1+t)}\right)\frac{1}{(1+t)}\right)\left\{\left(\frac{t}{\ln(1+t)}\right)^a \exp(\hat{M}_H\ln(1+t))\right\}$$
$$=\sum_{n=0}^{\infty} {}_HN^{(a)}_{n+1,m}(x,y;A)\frac{t^n}{n!},$$

which on using expression of $\hat{M}_H$ and equation (23) gives

$$\left(\left(x\sqrt{mA}-my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}}+\frac{a((1+t)\ln(1+t)-t)}{t\ln(1+t)}\right)\frac{1}{(1+t)}\right)$$
$$\left\{\left(\frac{t}{\ln(1+t)}\right)^a \exp\left(x\sqrt{mA}\ \ln(1+t)-y(\ln(1+t))^m I\right)\right\}=\sum_{n=0}^{\infty} {}_HN^{(a)}_{n+1,m}(x,y;A)\frac{t^n}{n!}.$$

In view of relation (31), the above equation becomes

$$\left(\left(x\sqrt{mA}-my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}}+\frac{a\left(e^{\frac{D_x}{\sqrt{mA}}}\left(\frac{D_x}{\sqrt{mA}}-1\right)+1\right)}{\left(e^{\frac{D_x}{\sqrt{mA}}}-1\right)\frac{D_x}{\sqrt{mA}}}\right)\frac{1}{e^{\frac{D_x}{\sqrt{mA}}}}\right)\sum_{n=0}^{\infty} {}_HN^{(a)}_{n,m}(x,y;A)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty} {}_HN^{(a)}_{n+1,m}(x,y;A)\frac{t^n}{n!}. \quad (32)$$

Rearranging the summation and then equating the coefficients of like powers of $t$ on both sides of the above equation, assertion (28) is obtained.

Making use of generating function (18) in both sides of identity (31), we have

$$\left(e^{\frac{D_x}{\sqrt{mA}}}-1\right)\sum_{n=0}^{\infty} {}_HN^{(a)}_{n,m}(x,y;A)\frac{t^n}{n!}=\sum_{n=0}^{\infty} n\ {}_HN^{(a)}_{n-1,m}(x,y;A)\frac{t^n}{n!}. \quad (33)$$

Rearranging the summation in the l.h.s. and then equating the coefficients of like powers of $t$ on both sides of the above equation, assertion (29) is obtained. □

**Remark 2.1.** *Using equations* (28) *and* (29) *in monomiality equation* (11)*, we deduce the following consequence of Theorem 2.2:*

**Corollary 2.2.** *The 2I2VHMNP* ${}_HN^{(a)}_{n,m}(x,y;A)$ *satisfy the following differential equation:*

$$\left(\left(x\sqrt{mA}-my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}}+\frac{a\left(e^{\frac{D_x}{\sqrt{mA}}}\left(\frac{D_x}{\sqrt{mA}}-1\right)+1\right)}{\left(e^{\frac{D_x}{\sqrt{mA}}}-1\right)\frac{D_x}{\sqrt{mA}}}\right)\frac{\left(e^{\frac{D_x}{\sqrt{mA}}}-1\right)}{e^{\frac{D_x}{\sqrt{mA}}}}-n\right)$$
$$\times {}_HN^{(a)}_{n,m}(x,y;A)=0. \quad (34)$$

We have mentioned special cases of the 2I2VHMP $H_{n,m}(x,y;A)$ in Table 1. Now, for the same choice of the variables and indices, the 2I2VHMNP ${}_HN^{(a)}_{n,m}(x,y;A)$ reduce to the corresponding special case. We mention these new special polynomials related to the Narumi polynomials in the following table.

Table 2. Special cases of the 2I2VHMNP ${}_HN^{(a)}_{n,m}(x,y;A)$

| **S. No.** | **Values of the indices and variables** | **Relation between $H_{n,m}(x,y;A)$ and its special cases** | **Name of the polynomials** |
|---|---|---|---|
| I. | $m=2$ | ${}_HN^{(a)}_{n,2}(x,y;A) = {}_HN^{(a)}_{n}(x,y;A)$ | 2-variable Hermite matrix-Narumi polynomials ${}_HN^{(a)}_{n}(x,y;A)$ |
| II. | $m=2$; $x \to \frac{x}{2}$, $y \to -y$ | ${}_HN^{(a)}_{n,2}(\frac{x}{2},-y;A) = {}_{\mathcal{H}}N^{(a)}_{n}(x,y;A)$ | 2-variable Hermite matrix-Narumi polynomials of the second form ${}_{\mathcal{H}}N^{(a)}_{n}(x,y;A)$ |
| III. | $y=1$ | ${}_HN^{(a)}_{n,m}(x,1;A) = {}_HN^{(a)}_{n,m}(x;A)$ | Generalized Hermite matrix-Narumi polynomials ${}_HN^{(a)}_{n,m}(x;A)$ |
| IV. | $m=2$; $y=1$ | ${}_HN^{(a)}_{n,2}(x,1;A) = {}_HN^{(a)}_{n}(x;A)$ | Hermite matrix-Narumi polynomials ${}_HN^{(a)}_{n}(x;A)$ |

**Remark 2.2.** *To find recurrence three term formula and investigating orthogonality problem are not possible at this stage. The orthogonality of mixed special polynomials have not been studied so far. Efforts are being done to study orthogonality of mixed special polynomials in future works. To study combinatorial properties of the mixed special polynomials will also be taken as future aspect.*

In the next section, corresponding to the new families of special matrix polynomials related to the Narumi polynomials given in Table 2, we obtain the results for these mixed type special matrix polynomials.

## 3. Examples

In order to obtain the results for the corresponding new special matrix polynomials related to the Narumi polynomials, we consider the following examples:

**1.** Taking $m=2$ in Theorems 2.1, 2.2 and Corollary 2.2, we get the following results for 2-variable Hermite matrix-Narumi polynomials ${}_HN^{(a)}_{n}(x,y;A)$:

Table 3. Results for the ${}_HN^{(a)}_{n}(x,y;A)$

| **S.No.** | **Results** | **Mathematical Expressions** |
|---|---|---|
| **1.** | **Generating function** | $\left(\frac{t}{\ln(1+t)}\right)^a \exp\left(x\sqrt{2A}\ln(1+t)) - y(\ln(1+t))^2 I\right) = \sum_{n=0}^{\infty} {}_HN^{(a)}_{n}(x,y;A)\frac{t^n}{n!}$ |
| **2.** | **Multiplicative and derivative operators** | $\hat{M} := \left(\left(x\sqrt{2A} - 2y(\sqrt{2A})^{-1}\frac{\partial}{\partial x} + \frac{a\left(e^{\frac{D_x}{\sqrt{2A}}}\left(\frac{D_x}{\sqrt{2A}}-1\right)+1\right)}{\left(e^{\frac{D_x}{\sqrt{2A}}}-1\right)\frac{D_x}{\sqrt{mA}}}\right)\frac{1}{e^{\frac{D_x}{\sqrt{2A}}}}\right)$ <br> $\hat{P} := e^{\frac{D_x}{\sqrt{2A}}} - 1$ |
| **3.** | **Differential equation** | $\left(\left(x\sqrt{2A} - 2y(\sqrt{2A})^{-1}\frac{\partial}{\partial x} + \frac{a\left(e^{\frac{D_x}{\sqrt{2A}}}\left(\frac{D_x}{\sqrt{2A}}-1\right)+1\right)}{\left(e^{\frac{D_x}{\sqrt{2A}}}-1\right)\frac{D_x}{\sqrt{mA}}}\right)\frac{e^{\frac{D_x}{\sqrt{2A}}}-1}{e^{\frac{D_x}{\sqrt{2A}}}} - n\right){}_HN^{(a)}_{n}(x,y;A) = 0$ |

**2.** Taking $m = 2$; $x \to \frac{x}{2}$, $y \to -y$ in Theorems 2.1, 2.2 and Corollary 2.2, we get the following results for 2-variable Hermite matrix-Narumi polynomials of the second form ${}_{\mathcal{H}}N_n^{(a)}(x, y; A)$:

Table 4. Results for the ${}_{\mathcal{H}}N_n^{(a)}(x, y; A)$

| **S.No.** | **Results** | **Mathematical Expressions** |
|---|---|---|
| **1.** | **Generating function** | $\left(\frac{t}{\ln(1+t)}\right)^a \exp\left(x\sqrt{\frac{A}{2}}\ln(1+t)) + y(\ln(1+t))^2 I\right) = \sum_{n=0}^{\infty} {}_{\mathcal{H}}N_n^{(a)}(x, y; A)$ |
| **2.** | **Multiplicative and derivative operators** | $\hat{M} := \left(\left(x\sqrt{\frac{A}{2}} + 2y\sqrt{\frac{2}{A}}\frac{\partial}{\partial x} + \frac{a\left(e^{\sqrt{\frac{2}{A}}D_x}\left(\sqrt{\frac{2}{A}}D_x - 1\right)+1\right)}{\left(e^{\sqrt{\frac{2}{A}}D_x} - 1\right)\frac{D_x}{\sqrt{mA}}}\right)\frac{1}{e^{\sqrt{\frac{2}{A}}D_x}}\right)$<br>$\hat{P} := e^{\sqrt{\frac{2}{A}}D_x} - 1$ |
| **3.** | **Differential equation** | $\left(\left(x\sqrt{\frac{A}{2}} + 2y\sqrt{\frac{2}{A}}\frac{\partial}{\partial x} + \frac{a\left(e^{\sqrt{\frac{2}{A}}D_x}\left(\sqrt{\frac{2}{A}}D_x - 1\right)+1\right)}{\left(e^{\sqrt{\frac{2}{A}}D_x} - 1\right)\frac{D_x}{\sqrt{mA}}}\right)\frac{e^{\sqrt{\frac{2}{A}}D_x} - 1}{e^{\sqrt{\frac{2}{A}}D_x}} - n\right) {}_{\mathcal{H}}N_n^{(a)}(x, y; A) = 0$ |

**3.** Taking $y = 1$ in Theorems 2.1, 2.2 and Corollary 2.2, we get the following results for Generalized Hermite matrix-Narumi polynomials ${}_HN_{n,m}^{(a)}(x; A)$:

Table 5. Results for the ${}_HN_{n,m}^{(a)}(x; A)$

| **S.No.** | **Results** | **Mathematical Expressions** |
|---|---|---|
| **1.** | **Generating function** | $\left(\frac{t}{\ln(1+t)}\right)^a \exp\left(x\sqrt{mA}\ln(1+t)) - (\ln(1+t))^m I\right) = \sum_{n=0}^{\infty} {}_HN_{n,m}^{(a)}(x; A)\frac{t^n}{n!}$ |
| **2.** | **Multiplicative and derivative operators** | $\hat{M} := \left(\left(x\sqrt{mA} - m(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}} + \frac{a\left(e^{\frac{D_x}{\sqrt{mA}}}\left(\frac{D_x}{\sqrt{mA}} - 1\right)+1\right)}{\left(e^{\frac{D_x}{\sqrt{mA}}} - 1\right)\frac{D_x}{\sqrt{mA}}}\right)\frac{1}{e^{\frac{D_x}{\sqrt{mA}}}}\right)$<br>$\hat{P} := e^{\frac{D_x}{\sqrt{mA}}} - 1$ |
| **3.** | **Differential equation** | $\left(\left(x\sqrt{mA} - m(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}} + \frac{a\left(e^{\frac{D_x}{\sqrt{mA}}}\left(\frac{D_x}{\sqrt{mA}} - 1\right)+1\right)}{\left(e^{\frac{D_x}{\sqrt{mA}}} - 1\right)\frac{D_x}{\sqrt{mA}}}\right)\frac{e^{\frac{D_x}{\sqrt{mA}}} - 1}{e^{\frac{D_x}{\sqrt{mA}}}} - n\right) {}_HN_{n,m}^{(a)}(x; A) = 0$ |

**4.** Taking $m = 2$; $y = 1$ in Theorems 2.1, 2.2 and Corollary 2.2, we get the following results for Hermite matrix-Narumi polynomials ${}_HN_n^{(a)}(x; A)$:

Table 6. Results for the ${}_HN_n^{(a)}(x; A)$

| **S.No.** | **Results** | **Mathematical Expressions** |
|---|---|---|
| **1.** | **Generating function** | $\left(\frac{t}{\ln(1+t)}\right)^a \exp\left(x\sqrt{2A}\ln(1+t)) - (\ln(1+t))^2 I\right) = \sum_{n=0}^{\infty} {}_HN_n^{(a)}(x; A)\frac{t^n}{n!}$ |
| **2.** | **Multiplicative and derivative operators** | $\hat{M} := \left(\left(x\sqrt{2A} - 2(\sqrt{mA})^{-1}\frac{\partial}{\partial x} + \frac{a\left(e^{\frac{D_x}{\sqrt{2A}}}\left(\frac{D_x}{\sqrt{2A}} - 1\right)+1\right)}{\left(e^{\frac{D_x}{\sqrt{2A}}} - 1\right)\frac{D_x}{\sqrt{mA}}}\right)\frac{1}{e^{\frac{D_x}{\sqrt{2A}}}}\right)$<br>$\hat{P} := e^{\frac{D_x}{\sqrt{2A}}} - 1$ |
| **3.** | **Differential equation** | $\left(\left(x\sqrt{2A} - 2(\sqrt{mA})^{-1}\frac{\partial}{\partial x} + \frac{a\left(e^{\frac{D_x}{\sqrt{2A}}}\left(\frac{D_x}{\sqrt{2A}} - 1\right)+1\right)}{\left(e^{\frac{D_x}{\sqrt{2A}}} - 1\right)\frac{D_x}{\sqrt{mA}}}\right)\frac{e^{\frac{D_x}{\sqrt{2A}}} - 1}{e^{\frac{D_x}{\sqrt{2A}}}} - n\right) {}_HN_n^{(a)}(x; A) = 0$ |

The Narumi polynomials $N_n^{(a)}(x)$ reduce to the Bernoulli polynomials of the second kind $b_n(x)$ for $a = 1$. The Bernoulli polynomials of the second kind $b_n(x)$ are defined by the following generating function [10]:

$$\left(\frac{t}{\ln(1+t)}\right)(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!} \tag{35}$$

or

$$\left(\frac{t}{\ln(1+t)}\right)\exp\left(x\ln(1+t)\right) = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}. \tag{36}$$

Thus, taking $a = 1$ in the results of the 2I2VHMNP ${}_HN_{n,m}^{(a)}(x, y; A)$, we obtain the following results for the 2-index 2-variable Hermite matrix-Bernoulli polynomials of the second kind ${}_Hb_{n,m}(x, y; A)$:

Table 7. Results for the ${}_Hb_{n,m}(x, y; A)$

| S.No. | Results | Mathematical Expressions |
|---|---|---|
| 1. | Generating function | $\left(\frac{t}{\ln(1+t)}\right)\exp\left(x\sqrt{mA}\ln(1+t)) - y(\ln(1+t))^m I\right) = \sum_{n=0}^{\infty} {}_Hb_{n,m}(x, y; A)\frac{t^n}{n!}$ |
| 2. | Multiplicative and derivative operators | $\hat{M} := \left(\left(x\sqrt{mA} - my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}} + \frac{\left(e^{\frac{D_x}{\sqrt{mA}}}\left(\frac{D_x}{\sqrt{mA}}-1\right)+1\right)}{\left(e^{\frac{D_x}{\sqrt{mA}}}-1\right)\frac{D_x}{\sqrt{mA}}}\right)\frac{1}{e^{\frac{D_x}{\sqrt{mA}}}}\right)$ <br> $\hat{P} := e^{\frac{D_x}{\sqrt{mA}}} - 1$ |
| 3. | Differential equation | $\left(\left(x\sqrt{mA} - my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}} + \frac{\left(e^{\frac{D_x}{\sqrt{mA}}}\left(\frac{D_x}{\sqrt{mA}}-1\right)+1\right)}{\left(e^{\frac{D_x}{\sqrt{mA}}}-1\right)\frac{D_x}{\sqrt{mA}}}\right)\frac{e^{\frac{D_x}{\sqrt{mA}}}-1}{e^{\frac{D_x}{\sqrt{mA}}}} - n\right){}_Hb_{n,m}(x, y; A) = 0$ |

The Narumi polynomials $N_n^{(a)}(x)$ reduce to the lower factorial polynomials $(x)_n$ for $a = 0$. The lower factorial polynomials $(x)_n$ are defined by the following generating function [10]:

$$(1+t)^x = \sum_{n=0}^{\infty}(x)_n\frac{t^n}{n!} \tag{37}$$

or

$$\exp(x\ln(1+t)) = \sum_{n=0}^{\infty}(x)_n\frac{t^n}{n!}. \tag{38}$$

Thus, taking $a = 0$ in the results of the 2I2VHMNP ${}_HN_{n,m}^{(a)}(x, y; A)$, we obtain the following results for the 2-index 2-variable Hermite matrix-lower factorial polynomials ${}_H(x, y; A)_{n,m}$:

Table 8. Results for the ${}_H(x,y;A)_{n,m}$

| S.No. | Results | Mathematical Expressions |
|---|---|---|
| 1. | **Generating function** | $\exp\left(x\sqrt{mA}\ln(1+t)) - y(\ln(1+t))^m I\right) = \sum_{n=0}^{\infty} {}_H(x,y;A)_{n,m}\frac{t^n}{n!}$ |
| 2. | **Multiplicative and derivative operators** | $\hat{M} := \left(x\sqrt{mA} - my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}}\right)\frac{1}{e^{\frac{D_x}{\sqrt{mA}}}}$ <br> $\hat{P} := e^{\frac{D_x}{\sqrt{mA}}} - 1$ |
| 3. | **Differential equation** | $\left(\left(x\sqrt{mA} - my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}}\right)\frac{\left(e^{\frac{D_x}{\sqrt{mA}}}-1\right)}{e^{\frac{D_x}{\sqrt{mA}}}} - n\right){}_H(x,y;A)_{n,m} = 0$ |

## 4. Concluding remarks

Recently, the Hermite-Appell matrix polynomials (HAMP) ${}_HR_n^{(m,s)}(x,y,z;A)$ are introduced by the following generating function [7]:

$$A(t)\exp(xt\sqrt{mA} - yt^m + zt^s) = \sum_{n=0}^{\infty} {}_HR_n^{(m,s)}(x,y,z;A)\frac{t^n}{n!}. \tag{39}$$

The HAMP ${}_HR_n^{(m,s)}(x,y,z;A)$ are shown to be quasi-monomial w.r.t to the following multiplicative and derivative operators [7]:

$$\hat{M}_{HA} := \left(x\sqrt{mA} - my(\sqrt{mA})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}} + sz(\sqrt{mA})^{-(s-1)}\frac{\partial^{s-1}}{\partial x^{s-1}} + \frac{A'\left(\frac{D_x}{\sqrt{mA}}\right)}{A\left(\frac{D_x}{\sqrt{mA}}\right)}\right) \tag{40}$$

and

$$\hat{P}_{HA} := \frac{1}{\sqrt{mA}}D_x, \tag{41}$$

respectively.

Here, we introduce the Hermite-Appell matrix based Narumi polynomials.

In order to derive the generating function for the Hermite-Appell matrix based Narumi polynomials, we take the Hermite-Appell matrix polynomials ${}_HR_n^{(m,s)}(x,y,z;A)$ as base in generating function (17) of the Narumi polynomials. Thus, replacing $x$ by the multiplicative operator $\hat{M}_{HA}$ of the Hermite-Appell matrix polynomials ${}_HR_n^{(m,s)}(x,y,z;A)$ in the l.h.s. of equation (17) and denoting the resultant Hermite-Appell matrix based Narumi polynomials in the r.h.s. by ${}_{HR}N_n^{(m,s,a)}(x,y,z;A)$, we have

$$\left(\frac{t}{\ln(1+t)}\right)^a \exp\left(\hat{M}_{HA}\ln(1+t)\right) = \sum_{n=0}^{\infty} {}_{HR}N_n^{(m,s,a)}(x,y,z;A)\frac{t^n}{n!}, \tag{42}$$

which by virtue of equation (13) with $t$ replaced by $\ln(1+t)$ and then using equation (39) in the resultant equation, the following generating function for the Hermite-Appell matrix based Narumi polynomials ${}_HR_n^{(m,s)}(x,y,z;A)$ is obtained:

$$\left(\frac{t}{\ln(1+t)}\right)^a A(\ln(1+t))\exp(x(\ln(1+t))\sqrt{mA} - y(\ln(1+t))^m + z(\ln(1+t))^s)$$

$$= \sum_{n=0}^{\infty} {}_{HR}N_n^{(m,s,a)}(x,y,z;A)\frac{t^n}{n!}. \tag{43}$$

Since for $A(t) = \left(\frac{t}{e^t-1}\right)$, $A(t) = \left(\frac{2}{e^t+1}\right)$ and $A(t) = \left(\frac{2t}{e^t+1}\right)$, the 2D-Appell polynomials $R_n^{(s)}(x,y)$ become the 2D-Bernoulli polynomials $B_n^{(s)}(x,y)$, 2D-Euler polynomials $E_n^{(s)}(x,y)$ and 2D-Genocchi polynomials $G_n^{(s)}(x,y)$, respectively. Therefore, taking $A(\ln(1+t)) = \left(\frac{\ln(1+t)}{e^{\ln(1+t)}-1}\right)$, $A(\ln(1+t)) = \left(\frac{2}{e^{\ln(1+t)}+1}\right)$ and $A(\ln(1+t)) = \left(\frac{2\ln(1+t)}{e^{\ln(1+t)}+1}\right)$ in equation (43), we get the following generating functions for the Hermite-2D-Bernoulli matrix based Narumi polynomials ${}_HB_n^{(m,s)}(x,y,z;A)$, Hermite-2D-Euler matrix based Narumi polynomials ${}_HE_n^{(m,s)}(x,y,z;A)$ and Hermite-2D-Genocchi matrix based Narumi polynomials ${}_HG_n^{(m,s)}(x,y,z;A)$, respectively:

$$\left(\frac{t}{\ln(1+t)}\right)^a \left(\frac{\ln(1+t)}{e^{\ln(1+t)}-1}\right) \exp(x(\ln(1+t))\sqrt{mA} - y(\ln(1+t))^m + z(\ln(1+t))^s)$$

$$= \sum_{n=0}^{\infty} {}_{HB}N_n^{(m,s,a)}(x,y,z;A)\frac{t^n}{n!}, \tag{44}$$

$$\left(\frac{t}{\ln(1+t)}\right)^a \left(\frac{2}{e^{\ln(1+t)}+1}\right) \exp(x(\ln(1+t))\sqrt{mA} - y(\ln(1+t))^m + z(\ln(1+t))^s)$$

$$= \sum_{n=0}^{\infty} {}_{HE}N_n^{(m,s,a)}(x,y,z;A)\frac{t^n}{n!} \tag{45}$$

and

$$\left(\frac{t}{\ln(1+t)}\right)^a \left(\frac{2\ln(1+t)}{e^{\ln(1+t)}+1}\right) \exp(x(\ln(1+t))\sqrt{mA} - y(\ln(1+t))^m + z(\ln(1+t))^s)$$

$$= \sum_{n=0}^{\infty} {}_{HG}N_n^{(m,s,a)}(x,y,z;A)\frac{t^n}{n!}. \tag{46}$$

Advancement in the theory of generalized and multi-variable forms of special functions serves as an analytical foundation for the majority of problems in mathematical physics that have been solved exactly and find broad practical applications. For example, the generalized Hermite polynomials are used to deal with quantum mechanical and optical beam transport problems. Further, an important generalization of special functions is special matrix functions. The study of special matrix polynomials is important due to their applications in certain areas of statistics, physics and engineering. In this paper, the hybrid special matrix polynomials are introduced by making use of operational identities for decoupling of exponential operators. Also, the concept associated with monomiality principle are used to establish their properties.

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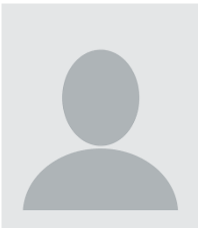

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**M. I. Qureshi** for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.11, N.4.