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PASCAL DISTRIBUTION SERIES RELATED TO STARLIKE FUNCTIONS WITH RESPECT TO OTHER POINTS

C. RAMACHANDRAN[1*], G. MURUGUSUNDARAMOORTHY[2], L. VANITHA[1], §

Abstract. The aim of the present paper is to find the necessary and sufficient conditions for subclasses of starlike functions with respect to symmetric points, starlike functions with respect to conjugate points, starlike functions with respect to symmetric conjugate points associated with Pascal distribution series and inclusion relations for such subclasses in the open unit disk $\mathbb{U}$. Further, we consider an integral operator related to Pascal distribution series..

Keywords: Analytic functions, Starlike functions with respect to symmetric points, Starlike functions with respect to conjugate points, Starlike functions with respect to symmetric conjugate points, Pascal distribution series.

AMS Subject Classification: 30C45.

1. Introduction and Preliminary results

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0 = f'(0) - 1$.

Let $\mathcal{S}_s^*$ be the subclass of $\mathcal{A}$ consisting of functions given by (1) satisfying

$$\Re\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0, \qquad z \in \mathbb{U}.$$

These functions are called starlike with respect to symmetric points and were introduced by Sakaguchi [22]. The class has also been considered in Robertson [19], Stankiewicz [26],

[1] Department of Mathematics, University College of Engineering Villupuram, Anna University, Villupuram, 605 602, Tamilnadu, India.
e-mail: crjsp2019@gmail.com; ORCID: http://orcid.org/0000-0003-0795-0650.
[*] Corresponding author.
e-mail: swarna.vanitha@gmail.com; ORCID: https://orcid.org/0000-0002-4044-8382.
[2] School of Advanced Sciences, Vellore Institute of Technology, Vellore, 632014, India.
e-mail: gmsmoorthy@yahoo.com; ORCID: http://orcid.org/0000-0001-8285-6619.
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Wu [28] and Owa et al. [15]. El-Ashwah and Thomas in [4], introduced two other classes namely the class $\mathcal{S}_c$ consisting of functions starlike with respect to conjugate points and $\mathcal{S}_{sc}$ consisting of functions starlike with respect to symmetric conjugate points.

Now, we denote $\mathcal{T}$ is a subclass of $\mathcal{A}$ consisting of functions of the form,

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad z \in \mathbb{U}. \tag{2}$$

where a_n is a non negative real number.

For $f \in \mathcal{T}$ Halim et al. [9] studied the class $\mathcal{TS}_s^*(\alpha,\beta)$, $0 \le \alpha \le 1$, $\frac{1}{2} < \beta \le 1$, consisting of functions $f \in \mathcal{T}$ and starlike with respect to symmetric points. An analogous results are also obtained for the class $\mathcal{TS}_c^*(\alpha,\beta)$, $0 \le \alpha \le 1$, $\frac{1}{2} < \beta \le 1$, consisting of functions $f \in \mathcal{T}$ and starlike with respect to conjugate points and the class $\mathcal{TS}_{sc}^*(\alpha,\beta)$, $0 \le \alpha \le 1$, $\frac{1}{2} < \beta \le 1$, consisting of functions $f \in \mathcal{T}$ and starlike with respect to symmetric conjugate points.

In [10], Halim et al. introducted the following subclasses consisting of functions $f \in \mathcal{T}$ and starlike functions with respect to symmetric points, starlike functions with respect to conjugate points, starlike functions with respect to symmetric conjugate points.

Definition 1.1. *A function $f \in \mathcal{TS}_s^*(\alpha,\beta,\sigma,k)$ is said to be starlike functions with respect to symmetric points if it satisfies*

$$\left|\frac{zf'(z)}{f(z)-f(-z)} - k\right| < \beta \left|\frac{\alpha z f'(z)}{f(z)-f(-z)} - (2\sigma - k)\right|$$

for some $0 \le \alpha \le 1$, $0 < \beta \le 1$, $0 \le \sigma \le \frac{1}{2} < k \le 1$ and $z \in \mathbb{U}$.

Definition 1.2. *A function $f \in \mathcal{TS}_c^*(\alpha,\beta,\sigma,k)$ is said to be starlike functions with respect to conjugate points if it satisfies*

$$\left|\frac{zf'(z)}{f(z)+\overline{f(\overline{z})}} - k\right| < \beta \left|\frac{\alpha z f'(z)}{f(z)+\overline{f(\overline{z})}} - (2\sigma - k)\right|$$

for some $0 \le \alpha \le 1$, $0 < \beta \le 1$, $0 \le \sigma \le \frac{1}{2} < k \le 1$ and $z \in \mathbb{U}$.

Definition 1.3. *A function $f \in \mathcal{TS}_{sc}^*(\alpha,\beta,\sigma,k)$ is said to be starlike functions with respect to symmetric conjugate points if it satisfies*

$$\left|\frac{zf'(z)}{f(z)-\overline{f(-\overline{z})}} - k\right| < \beta \left|\frac{\alpha z f'(z)}{f(z)-\overline{f(-\overline{z})}} - (2\sigma - k)\right|$$

for some $0 \le \alpha \le 1$, $0 < \beta \le 1$, $0 \le \sigma \le \frac{1}{2} < k \le 1$ and $z \in \mathbb{U}$.

We state the following necessary and sufficient conditions due to Halim et al. [10].

Lemma 1.1. [10] *Let $f \in \mathcal{T}$. A function $f \in \mathcal{TS}_s^*(\alpha,\beta,\sigma,k)$ if and only if*

$$\sum_{n=2}^{\infty}\left(\frac{(1+\beta\alpha)n}{\beta(2(k-2\sigma)+\alpha)-(2k-1)} + \frac{\beta(k-2\sigma)(1-(-1)^n)-k(1-(-1)^n)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\right) a_n \le 1 \tag{3}$$

where $0 \le \alpha \le 1$, $\frac{1}{2} < \beta \le 1$, $0 \le \sigma \le \frac{1}{2} < k \le 1$.

Lemma 1.2. [10] *Let $f \in \mathcal{T}$. A function $f \in \mathcal{TS}_c^*(\alpha,\beta,\sigma,k)$ if and only if*

$$\sum_{n=2}^{\infty}\left(\frac{(1+\beta\alpha)n}{\beta(2(k-2\sigma)+\alpha)-(2k-1)} + \frac{2(\beta(k-2\sigma)-k)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\right) a_n \le 1 \tag{4}$$

where $0 \le \alpha \le 1$, $\frac{1}{2} < \beta \le 1$, $0 \le \sigma \le \frac{1}{2} < k \le 1$.

Lemma 1.3. [10] *Let $f \in \mathcal{T}$. A function $f \in \mathcal{TS}^*_{sc}(\alpha,\beta,\sigma,k)$ if and only if*

$$\sum_{n=2}^{\infty}\left(\frac{(1+\beta\alpha)n}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}+\frac{\beta(k-2\sigma)(1-(-1)^n)-k(1-(-1)^n)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\right)a_n \le 1 \quad (5)$$

where $0 \le \alpha \le 1$, $\frac{1}{2} < \beta \le 1$, $0 \le \sigma \le \frac{1}{2} < k \le 1$.

A variable X is said to be Pascal distribution if it takes the values $0, 1, 2, 3, \cdots$ with probabilities $(1-q)^m$, $\frac{qm(1-q)^m}{1!}$, $\frac{q^2m(m+1)(1-q)^m}{2!}$, $\frac{q^3m(m+1)(m+2)(1-q)^m}{3!}$, $\cdots$ respectively, where q and m are called the parameter, and thus

$$P(X=r)=\binom{r+m-1}{m-1}q^r(1-q)^m, \qquad m \ge 1 \quad 0 \le q \le 1 \quad r=0,1,2,3,\cdots$$

Very recently, El-Deeb et al [6] (see also [1, 14]) introduced a power series whose coefficients are probabilities of Pascal distribution, that is

$$\Psi^m_q(z)=z+\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m z^n, \qquad z \in \mathbb{U},$$

where $m \ge 1; 0 \le q \le 1$ and we note that, by ratio test the radius of convergence of above series is infinity. In [1] , they defined the following series

$$\Phi^m_q(z)=2z-\Psi^m_q(z)=z-\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m z^n, \qquad z \in \mathbb{U}. \quad (6)$$

and considered the linear operator $\mathcal{I}^m_q(z) : \mathcal{A} \to \mathcal{A}$ defined by the convolution or Hadamard product

$$\mathcal{I}^m_q f(z)=\Psi^m_q(z) * f(z)=z+\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m a_n z^n, \qquad z \in \mathbb{U},$$

where $m \ge 1; 0 \le q \le 1$.

Motivated by several earlier results on connections between various subclasses of analytic and univalent functions by using hypergeometric functions (see for example, [2, 8, 11, 20, 23, 24, 25]) and by the recent investigations (see for example, [3, 5, 7, 12, 13, 16, 17, 18, 21]), in the present paper we determine the necessary and sufficient conditions for $\Phi^m_q(z)$ to be in the function classes $\mathcal{TS}^*_s(\alpha,\beta,\sigma,k)$, $\mathcal{TS}^*_c(\alpha,\beta,\sigma,k)$ and $\mathcal{TS}^*_{sc}(\alpha,\beta,\sigma,k)$.We also determine the conditions for the integral operator $\mathcal{G}^m_q(z)=\int\limits_0^z \frac{\Phi^m_q(t)}{t}dt$ belonging to the these classes.

2. Necessary and Sufficient Conditions

For convenience throughout in the sequel, we use the following identities that hold at least for $m \ge 2$ and $0 \le q < 1$:

$$\begin{aligned}\sum_{n=0}^{\infty}\binom{n+m-1}{m-1}q^n &= \frac{1}{(1-q)^m}\\ \sum_{n=0}^{\infty}\binom{n+m}{m}q^n &= \frac{1}{(1-q)^{m+1}}\\ \sum_{n=0}^{\infty}\binom{n+m+1}{m+1}q^n &= \frac{1}{(1-q)^{m+2}} \qquad (7)\end{aligned}$$

By simple calculation we derive the following relations:

$$\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}=\sum_{n=0}^{\infty}\binom{n+m-1}{m-1}q^{n}-1=\frac{1}{(1-q)^{m}}-1 \tag{8}$$

and

$$\sum_{n=2}^{\infty}(n-1)\binom{n+m-2}{m-1}q^{n-1}=qm\sum_{n=0}^{\infty}\binom{n+m}{m}q^{n}=\frac{qm}{(1-q)^{m+1}} \tag{9}$$

Unless otherwise mentioned, we shall assume in this paper that $0\le\alpha\le1$, $\frac{1}{2}<\beta\le1$, $0\le\sigma\le\frac{1}{2}<k\le1$, while $m\ge1$ and $0\le q<1$

Firstly, we obtain the necessary and sufficient conditions for Φ_q^m to be in class $\mathcal{TS}_s^*(\alpha,\beta,\sigma,k)$.

Theorem 2.1. *Let $\Phi_q^m\in\mathcal{TS}_s^*(\alpha,\beta,\sigma,k)$ if and only if*

$$\begin{aligned}&\left(\frac{1}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\right)\Bigg[(1+\beta\alpha)\left(\frac{mq}{(1-q)^{m+1}}\right)\\&+(\beta(k-2\sigma)-k)\left(\frac{1}{(1+q)^{m}}-\frac{1}{(1-q)^{m}}\right)\Bigg]\le1.\end{aligned} \tag{10}$$

Proof. Since

$$\Phi_q^m(z)=z-\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^{m}z^{n} \tag{11}$$

in view of Lemma 1.1, it suffices to show that

$$\begin{aligned}\sum_{n=2}^{\infty}&\left(\frac{(1+\beta\alpha)n}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}+\frac{\beta(k-2\sigma)(1-(-1)^{n})-k(1-(-1)^{n})}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\right)\\&\times\tbinom{n+m-2}{m-1}q^{n-1}(1-q)^{m}\le1.\end{aligned} \tag{12}$$

Writing $n=(n-1)+1$ in (12) we have

$$\begin{aligned}\mathfrak{P}_1(\alpha,\beta,\sigma,k)&=\sum_{n=2}^{\infty}\left(\frac{(1+\beta\alpha)n}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}+\frac{\beta(k-2\sigma)(1-(-1)^{n})-k(1-(-1)^{n})}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\right)\\&\times\binom{n+m-2}{m-1}q^{n-1}(1-q)^{m}\\&=\sum_{n=2}^{\infty}\left(\frac{(n-1)(1+\beta\alpha)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}+\frac{(1+\beta\alpha)+(1-(-1)^{n})\left[\beta(k-2\sigma)-k\right]}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\right)\\&\times\binom{n+m-2}{m-1}q^{n-1}(1-q)^{m}\\&=\left(\frac{(1+\beta\alpha)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\right)\sum_{n=2}^{\infty}(n-1)\binom{n+m-2}{m-1}q^{n-1}(1-q)^{m}\\&+\left(\frac{(1+\beta\alpha)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\right)\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^{m}\\&+\left(\frac{\beta(k-2\sigma)-k}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\right)\sum_{n=2}^{\infty}(1-(-1)^{n})\binom{n+m-2}{m-1}q^{n-1}(1-q)^{m}.\end{aligned}$$

By using (7), (8) and (9), we get

$$\begin{aligned}
\mathfrak{P}_1(\alpha,\beta,\sigma,k) &= \left(\frac{(1+\beta\alpha)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\right)\left(\frac{mq}{1-q}\right)\\
&+ \left(\frac{(1+\beta\alpha)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\right)(1-(1-q)^m)\\
&\quad+\left(\frac{\beta(k-2\sigma)-k}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\right)\left(1-2(1-q)^m+\left(\frac{1-q}{1+q}\right)^m\right)\\
&= \left(\frac{(1+\beta\alpha)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\right)\left(\frac{mq}{1-q}\right)+(1-(1-q)^m)\\
&\quad+\left(\frac{\beta(k-2\sigma)-k}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\right)\left(\left(\frac{1-q}{1+q}\right)^m-1\right)
\end{aligned}$$

But $\mathfrak{P}_1(\alpha,\beta,\sigma,k)$ is bounded above by 1 if and only if (10) holds. □

Theorem 2.2. *A function $\Phi_q^m(z)$ is in $\mathcal{TS}_c^*(\alpha,\beta,\sigma,k)$ if and only if*

$$\left(\frac{(1+\beta\alpha)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\right)\left(\frac{mq}{(1-q)^{m+1}}\right)\le 1. \tag{13}$$

Proof. In view of Lemma 1.2, it suffices to show that

$$\mathfrak{P}_2(\alpha,\beta,\sigma,k)=\sum_{n=2}^{\infty}\left(\frac{(1+\beta\alpha)n}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}+\frac{2(\beta(k-2\sigma)-k)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\right)a_n\le 1 \tag{14}$$

Writing $n=(n-1)+1$ in (14) we have

$$\begin{aligned}
\mathfrak{P}_2(\alpha,\beta,\sigma,k) &= \sum_{n=2}^{\infty}\left(\frac{(n-1)(1+\beta\alpha)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}+\frac{(1+\beta\alpha)+2\left[\beta(k-2\sigma)-k\right]}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\right)\\
&\times \binom{n+m-2}{m-1}q^{n-1}(1-q)^m\\
&= \left(\frac{(1+\beta\alpha)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\right)\sum_{n=2}^{\infty}(n-1)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\\
&\quad+\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m.\\
&= \left(\frac{(1+\beta\alpha)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\right)\left(\frac{mq}{1-q}\right)+(1-(1-q)^m)
\end{aligned}$$

Therefore, we see $\mathfrak{P}_2(\alpha,\beta,\sigma,k)$ is bounded above by 1 if and only if (13) is satisfied. □

Applying Lemma 1.3 and using the same technique as in the proof of Theorem 2.1 we have the following result:

Theorem 2.3. *A function $\Phi_q^m(z)$ is in $\mathcal{TS}_{sc}^*(\alpha,\beta,\sigma,k)$ if and only if*

$$\begin{aligned}
&\left(\frac{1}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\right)\left[(1+\beta\alpha)\left(\frac{mq}{(1-q)^{m+1}}\right)\right.\\
&\qquad\left.+\left(\beta(k-2\sigma)-k\right)\left(\frac{1}{(1+q)^m}-\frac{1}{(1-q)^m}\right)\right]\le 1.
\end{aligned} \tag{15}$$

3. Inclusion Properties

A function $f \in \mathcal{A}$ is said to be in the class $\mathrm{R}^{\tau}(\eta, \upsilon)$, $(\tau \in \mathbb{C}\backslash\{0\},\ 0 < \eta \leq 1; \upsilon < 1)$, if it satisfies the inequality

$$\left| \frac{(1-\eta)\frac{f(z)}{z} + \eta f'(z) - 1}{2\tau(1-\upsilon) + (1-\eta)\frac{f(z)}{z} + \eta f'(z) - 1} \right| < 1, \quad (z \in \mathbb{D}).$$

The class $\mathrm{R}^{\tau}(\eta, \upsilon)$ was introduced earlier by Swaminathan [27](for special cases see the references cited there in) and obtained the following estimate.

Lemma 3.1. [27] *If $f \in \mathrm{R}^{\tau}(\eta, \upsilon)$ is of form (1), then*

$$|a_n| \leq \frac{2\,|\tau|\,(1-\upsilon)}{1+\eta(n-1)}, \quad n \in \mathbb{N} \setminus \{1\}. \tag{16}$$

The bounds given in (16) are sharp.

Making use of Lemma 3.1, we will study the action of the Pascal distribution series on the class $\mathcal{TS}_s^*(\alpha, \beta, \sigma, k)$, $\mathcal{TS}_c^*(\alpha, \beta, \sigma, k)$ and $\mathcal{TS}_{sc}^*(\alpha, \beta, \sigma, k)$.

Theorem 3.1. *If $f \in \mathrm{R}^{\tau}(\eta, \upsilon)$, then $\Phi_q^m \in \mathcal{TS}_s^*(\alpha, \beta, \sigma, k)$ if and only if*

$$\begin{aligned} &\frac{2\,|\tau|\,(1-\upsilon)}{\eta[\beta(2(k-2\sigma)+\alpha)-(2k-1)]} \times \\ &\qquad \Big[\ (1+\beta\alpha)\,(1-(1-q)^m) \\ &+ \quad (\beta(k-2\sigma)-k)\left((1-q)-(1+q)\left(\frac{1-q}{1+q}\right)^m - 2q(m-1)(1-q)^m\right)\Big] \leq 1. \end{aligned} \tag{17}$$

Proof. In view of Lemma 1.1, it suffices to show that

$$\begin{aligned} \mathfrak{P}_3(\alpha, \beta, \sigma, k) \quad &= \quad \sum_{n=2}^{\infty} \left(\frac{(1+\beta\alpha)n}{\beta(2(k-2\sigma)+\alpha)-(2k-1)} + \frac{\beta(k-2\sigma)(1-(-1)^n) - k(1-(-1)^n)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)} \right) \\ &\times \quad \binom{n+m-2}{m-1} q^{n-1}(1-q)^m |a_n| \leq 1 \end{aligned}$$

Since $f \in \mathrm{R}^\tau(\eta, \upsilon)$, then by Lemma 3.1, we have $|a_n| \leq \frac{2|\tau|(1-\upsilon)}{1+\eta(n-1)}, n \in \mathbb{N} \setminus \{1\}$ and also we note that $1 + \eta(n-1) \geq n\eta$, Thus, we have

$$\begin{aligned}
&\mathfrak{P}_3(\alpha, \beta, \sigma, k) \\
&\leq \frac{2|\tau|(1-\upsilon)}{\eta} \times \\
&\quad \left[\sum_{n=2}^{\infty} \left(\frac{(1+\beta\alpha)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)} + \frac{(1-(-1)^n)(\beta(k-2\sigma)-k)}{n\left(\beta(2(k-2\sigma)+\alpha)-(2k-1)\right)} \right) \right. \\
&\quad \left. \times \binom{n+m-2}{m-1} q^{n-1}(1-q)^m \right] \\
&= \frac{2|\tau|(1-\upsilon)(1-q)^m}{\eta[\beta(2(k-2\sigma)+\alpha)-(2k-1)]} \left[(1+\beta\alpha) \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1} \right. \\
&\quad \left. + (\beta(k-2\sigma)-k) \sum_{n=2}^{\infty} \frac{(1-(-1)^n)}{n} \binom{n+m-2}{m-1} q^{n-1} \right] \\
&= \frac{2|\tau|(1-\upsilon)}{\eta[\beta(2(k-2\sigma)+\alpha)-(2k-1)]} \left[(1+\beta\alpha)\left[1-(1-q)^m\right] + (\beta(k-2\sigma)-k) \right. \\
&\quad \left. \times \left((1-q) - (1+q)\left(\frac{1-q}{1+q}\right)^m - 2q(m-1)(1-q)^m \right) \right]
\end{aligned}$$

But $\mathfrak{P}_3(\alpha, \beta, \sigma, k)$ is bounded by 1, if (17) holds. This completes the proof of Theorem 3.1. $\square$

Applying Lemma 1.2 and using the same method as in the proof of Theorem 3.1, we have the following result.

Theorem 3.2. *If $f \in \mathrm{R}^\tau(\eta, \upsilon)$, then $\Phi_q^m \in \mathcal{TS}_c^*(\alpha, \beta, \sigma, k)$ if and only if*

$$\begin{aligned}
&\frac{2|\tau|(1-\upsilon)}{\eta[\beta(2(k-2\sigma)+\alpha)-(2k-1)]}\left[(1+\beta\alpha)\left[1-(1-q)^m\right]\right. \\
&\left.+ 2\left(\beta(k-2\sigma)-k\right)\left[(1-q)-(1-q)^m - q(m-1)(1-q)^m\right]\right] \leq 1. \qquad (18)
\end{aligned}$$

Applying Lemma 1.3 and using the same technique as in the proof of Theorem 3.1, we have the following result.

Theorem 3.3. *If $f \in \mathrm{R}^\tau(\eta, \upsilon)$, then $\Phi_q^m \in \mathcal{TS}_{sc}^*(\alpha, \beta, \sigma, k)$ if and only if*

$$\begin{aligned}
&\frac{2|\tau|(1-\upsilon)}{\eta[\beta(2(k-2\sigma)+\alpha)-(2k-1)]}\left[(1+\beta\alpha)\left[1-(1-q)^m\right]\right. \\
&\left.+ \left(\beta(k-2\sigma)-k\right)\left((1-q)-(1+q)\left(\frac{1-q}{1+q}\right)^m - 2q(m-1)(1-q)^m\right)\right] \leq 1. \qquad (19)
\end{aligned}$$

4. An integral operator

Theorem 4.1. *If $m > 1$, then the integral operator*

$$\mathcal{G}_q^m(z) = \int_0^z \frac{\Phi_q^m(t)}{t} dt \qquad (20)$$

is in $\mathcal{TS}_s^*(\alpha,\beta,\sigma,k)$ *if and only if*

$$\frac{1}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\left[(1+\beta\alpha)\left[1-(1-q)^m\right]\right.$$
$$\left.+\left(\beta(k-2\sigma)-k\right)\left((1-q)-(1+q)\left(\frac{1-q}{1+q}\right)^m-2q(m-1)(1-q)^m\right)\right]\leq 1. \quad (21)$$

Proof. Since

$$\mathcal{G}_q^m(z)=z-\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\frac{z^n}{n}$$

then by Lemma 1.1, we need only to show that

$$\sum_{n=2}^{\infty}\left(\frac{n(1+\beta\alpha)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}+\frac{(1-(-1)^n)(\beta(k-2\sigma)-k)}{(\beta(2(k-2\sigma)+\alpha)-(2k-1))}\right)\times$$
$$\frac{1}{n}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m$$

or, equivalently

$$\frac{(1-q)^m}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}$$
$$\times\left[(1+\beta\alpha)\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}+(\beta(k-2\sigma)-k)\sum_{n=2}^{\infty}\frac{(1-(-1)^n)}{n}\binom{n+m-2}{m-1}q^{n-1}\right]. \quad (22)$$

The remaining part of the proof of Theorem 4.1 is similar to that of Theorem 3.1, and so we omit the details. □

Theorem 4.2. *If* $m>1$*, then the integral operator* $\mathcal{G}_q^m(z)$ *given by* (20) *is in* $\mathcal{TS}_c^*(\alpha,\beta,\sigma,k)$ *if and only if*

$$\frac{1}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\left[\,(1+\beta\alpha)\left[1-(1-q)^m\right]\right.$$
$$\left.+2\left(\beta(k-2\sigma)-k\right)\left[(1-q)-(1-q)^m-q(m-1)(1-q)^m\right]\right]\leq 1. \quad (23)$$

The proof of Theorem 4.2 is lines similar to the proof of Theorem 4.1, so we omitted the proof of Theorem 4.2.

Theorem 4.3. *If* $m>1$*, then the integral operator* $\mathcal{G}_q^m(z)$ *given by* (20) *is in* $\mathcal{TS}_{sc}^*(\alpha,\beta,\sigma,k)$ *if and only if*

$$\frac{1}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\left[\,(1+\beta\alpha)\left[1-(1-q)^m\right]\right.$$
$$\left.+\left(\beta(k-2\sigma)-k\right)\left((1-q)-(1+q)\left(\frac{1-q}{1+q}\right)^m-2q(m-1)(1-q)^m\right)\right]\leq 1. \quad (24)$$

The technique used for proving Theorem 4.3 is similar to that used in Theorem 4.1, so we omitted the proof of Theorem 4.3.

5. Corollaries and consequences

By specializing the parameter $\sigma=0$ and $k=1$ in Theorems 2.1-2.3, Theorems 3.1-3.3 and Theorems 4.1-4.3 we obtain the following corollaries.

Corollary 5.1. *A function $\Phi_q^m \in \mathcal{TS}_s^*(\alpha,\beta)$ if and only if*

$$\left(\frac{1}{\beta(2+\alpha)-1}\right)\left[(1+\beta\alpha)\left(\frac{mq}{(1-q)^{m+1}}\right)+(\beta-1)\left(\frac{1}{(1+q)^m}-\frac{1}{(1-q)^m}\right)\right]\leq 1. \quad (25)$$

Corollary 5.2. *A function $\Phi_q^m \in \mathcal{TS}_c^*(\alpha,\beta)$, if and only if*

$$\left(\frac{(1+\beta\alpha)}{\beta(2+\alpha)-1}\right)\left(\frac{mq}{(1-q)^{m+1}}\right)\leq 1. \quad (26)$$

Corollary 5.3. *A function $\Phi_q^m \in \mathcal{TS}_{sc}^*(\alpha,\beta)$, if and only if*

$$\left(\frac{1}{\beta(2+\alpha)-1}\right)\left[(1+\beta\alpha)\left(\frac{mq}{(1-q)^{m+1}}\right)+(\beta-1)\left(\frac{1}{(1+q)^m}-\frac{1}{(1-q)^m}\right)\right]\leq 1. \quad (27)$$

Corollary 5.4. *If $f \in \mathrm{R}^\tau(\eta,\upsilon)$, then Φ_q^m is in $\mathcal{TS}_s^*(\alpha,\beta)$ if and only if*

$$\begin{aligned}&\frac{2|\tau|(1-\upsilon)}{\eta[\beta(2+\alpha)-1]}\left[(1+\beta\alpha)\left[1-(1-q)^m\right]\right.\\&\left.+(\beta-1)\left((1-q)-(1+q)\left(\frac{1-q}{1+q}\right)^m-2q(m-1)(1-q)^m\right)\right]\leq 1. \quad (28)\end{aligned}$$

Corollary 5.5. *If $f \in \mathrm{R}^\tau(\eta,\upsilon)$, then Φ_q^m is in $\mathcal{TS}_c^*(\alpha,\beta)$ if and only if*

$$\begin{aligned}&\frac{2|\tau|(1-\upsilon)}{\eta[\beta(2+\alpha)-1]}\left[(1+\beta\alpha)\left[1-(1-q)^m\right]\right.\\&\left.+2(\beta-1)\left[(1-q)-(1-q)^m-q(m-1)(1-q)^m\right]\right]\leq 1. \quad (29)\end{aligned}$$

Corollary 5.6. *If $f \in \mathrm{R}^\tau(\eta,\upsilon)$, then Φ_q^m is in $\mathcal{TS}_{sc}^*(\alpha,\beta)$ if and only if*

$$\begin{aligned}&\frac{2|\tau|(1-\upsilon)}{\eta[\beta(2+\alpha)-1]}\left[(1+\beta\alpha)\left[1-(1-q)^m\right]\right.\\&\left.+(\beta-1)\left[(1-q)-(1+q)\left(\frac{1-q}{1+q}\right)^m-2q(m-1)(1-q)^m\right]\right]\leq 1. \quad (30)\end{aligned}$$

Corollary 5.7. *If $m>1$, then the integral operator $\mathcal{G}_q^m(z)$ given by* (20) *is in $\mathcal{TS}_s^*(\alpha,\beta)$ if and only if*

$$\begin{aligned}&\frac{1}{\beta(2+\alpha)-1}\left[(1+\beta\alpha)\left[1-(1-q)^m\right]\right.\\&\left.+(\beta-1)\left[(1-q)-(1+q)\left(\frac{1-q}{1+q}\right)^m-2q(m-1)(1-q)^m\right]\right]\leq 1. \quad (31)\end{aligned}$$

Corollary 5.8. *If $m>1$, then the integral operator $\mathcal{G}_q^m(z)$ given by* (20) *is in $\mathcal{TS}_c^*(\alpha,\beta)$ if and only if*

$$\begin{aligned}&\frac{1}{\beta(2+\alpha)-1}\left[(1+\beta\alpha)\left[1-(1-q)^m\right]\right.\\&\left.+2(\beta-1)\left[(1-q)-(1-q)^m-q(m-1)(1-q)^m\right]\right]\leq 1. \quad (32)\end{aligned}$$

Corollary 5.9. *If $m>1$, then the integral operator $\mathcal{G}_q^m(z)$ given by* (20) *is in $\mathcal{TS}_{sc}^*(\alpha,\beta)$ if and only if*

$$\begin{aligned}&\frac{1}{\beta(2+\alpha)-1}\left[(1+\beta\alpha)\left[1-(1-q)^m\right]\right.\\&\left.+(\beta-1)\left[(1-q)-(1+q)\left(\frac{1-q}{1+q}\right)^m-2q(m-1)(1-q)^m\right]\right]\leq 1. \quad (33)\end{aligned}$$

6. CONCLUSION

In this article, we construct a new class of analytic function whose power series representation is associated with Pascal distribution. Such a new research paves a progressive path to the young researchers for extending their investigation to approach a new dimension in the field of geometric theory through probability distribution.

References

[1] Bulboaca, T. and Murugusundaramoorthy, G., (2020), Univalent functions with positive coefficients involving Pascal distribution series, Commun. Korean Math. Soc.,35, (3), pp. 867-877.

[2] Cho, N. E., Woo, S. Y., Owa, S., (2002), Uniform convexity properties for hypergeometric functions, Fract. Cal. Appl. Anal., 5, (3), pp. 303-313.

[3] El-Ashwah, R. M., Kota, W. Y., (2017), Some condition on a Poisson distribution series to be in subclasses of univalent functions, Acta Universitatis Apulensis, (51), pp. 89 - 103.

[4] El-Ashwah, R. M., Thomas, D. K., (1987), Some subclasses of close-to-convex functions, J. Ramanujan Math. Soc., 2, pp. 86-100.

[5] El-Deeb, S. M., Bulboacă, T., Differential sandwich-type results for symmetric functions associated with Pascal distribution series, J. Contemporary Math. Anal., In press.

[6] El-Deeb, S. M., Bulboacă, T., Dziok, J., (2019), Pascal distribution series connected with certain subclasses of univalent functions, Kyungpook Math. J., 59, pp. 301-314.

[7] Frasin, B. A., (2019), On certain subclasses of analytic functions associated with Poisson distribution series, Acta Univ. Sapientiae, Mathematica, 11, (1), pp. 78 - 86.

[8] Frasin, B. A., Al-Hawary, T., Yousef, F., (2019), Necessary and sufficient conditions for hypergeometric functions to be in a subclass of analytic functions, Afrika Matematika, 30, (1-2), pp. 223-230.

[9] Halim, S. A., Janteng, A., Darus, M., (2005), Coefficient properties for classes with negative coefficients and starlike with respect to other points, 13th Mathematical Sciences National Symposium Proceeding.

[10] Halim, S. A., Janteng, A., Darus, M., (2006), Classes with Negative Coefficients and Starlike with respect to other Points II, Tamkang Journal of Mathematics, 37, (4), pp. 345-354.

[11] Merkes, E. and Scott, B. T., (1961), Starlike hypergeometric functions, Proc. Amer. Math. Soc., 12, pp. 885-888.

[12] Murugusundaramoorthy, G., (2017), Subclasses of starlike and convex functions involving Poisson distribution series , Afr. Mat. 28, pp. 357-366.

[13] Murugusundaramoorthy, G., Vijaya, K., Porwal, S., (2016), Some inclusion results of certain subclass of analytic functions associated with Poisson distribution series, Hacettepe J. Math. Stat. 45, (4), pp. 1101-1107.

[14] Murugusundaramoorthy, G., Frasin, B. A., Al-Hawary, T., (2001), Uniformly convex spiral functions and uniformly spirallike function associated with Pascal distribution series, arXiv:2001.07517.

[15] Owa, S., Wu, Z., Ren, F., (1988), A note on certain subclass of Sakaguchi functions, Bull. de la Royale de Liege, 57, (3), pp. 143-150.

[16] Porwal, S., (2014), An application of a Poisson distribution series on certain analytic functions, J. Complex Anal., Art. ID 984135, pp. 1-3.

[17] Porwal, S., (2013), Mapping properties of generalized Bessel functions on some subclasses of univalent functions, Anal. Univ. Oradea Fasc. Matematica, 20, (2), pp. 51-60.

[18] Porwal, S., Kumar, M., (2016), A unified study on starlike and convex functions associated with Poisson distribution series, Afr. Mat., 27, (5), pp. 1021-1027.

[19] Robertson, M. S., (1961), Applications of the subordination principle to univalent functions, Pacific J. of Math., 11, pp. 315-324.

[20] Ramachandran, C., Vanitha, L., Murugusundaramoorthy, G., (2014), Starlike And Uniformly Convex Functions Involving Generalized Hypergeometric Function, International Journal of Pure and Applied Mathematics, 92, (5), pp. 691-701.

[21] Ramachandran, C., Vanitha, L., Kanas, S., (2018), Certain results on q-starlike and q-convex error functions Mathematica Slovaca, 68, (2), pp. 361-368.

[22] Sakaguchi, K., (1959), On certain univalent mapping, J. Math. Soc. Japan. 11, pp. 72-75.

[23] Silverman, H., (1993), Starlike and convexity properties for hypergeometric functions, J. Math. Anal. Appl. 172, pp. 574-581.

[24] Srivastava, H. M., El-Deeb, S. M., (2020), The Faber polynomial expan-sion method and the Taylor-Maclaurin coe¢ cient estimates of bi-close-to-convex functions connected with the q-convolution, AIMS Math., 5, (6), pp. 7087-7106.

[25] Srivastava, H. M., Murugusundaramoorthy, G., Sivasubramanian, S., (2007), Hypergeometric functions in the parabolic starlike and uniformly convex domains, Integr. Transf. Spec. Func. 18, pp. 511-520.

[26] Stankiewicz, J., (1965), Some remarks on functions starlike w.r.t symmetric points, Ann. Univ. Marie Curie Sklodowska, 19, (7), pp. 53-59.

[27] Swaminathan, A., (2004), Certain suffienct conditions on Gaussian hypergeometric functions,Journal of Inequalities in Pure and Applied Mathematics., 5, (4), Art.83 pp. 1-10.

[28] Wu, Z., (1987), On classes of Sakaguchi functions and Hadamard products, Sci. Sinica Ser. A, 30, pp. 128-135.

C. Ramachandran received his Ph.D. from Anna University, Chennai, Tamil Nadu, India in 2007. Since 2008, he has been working as an associate professor in the Department of Mathematics at University College of Engineering, Villupuram. His research interests include Complex Analysis, Geometric Function Theory and Special Functions.

G. Murugusundaramoorthy for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.11, N.2.

L. Vanitha is working in the Department of Mathematics at University College of Engineering Villupuram, Anna University, India. She received her Bachelor and Master Degree in Mathematics from Madras University and Master of Philosophy at Madurai Kamaraj University. She received Ph. D. Degree in Mathematics from Anna University in 2017. Her area of interest includes Geometric Function Theory and Special Function.