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ON A SUBCLASS OF MEROMORPHIC FUNCTIONS DEFINED BY HILBERT SPACE OPERATOR

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Abstract. In this paper we introduce and study a new subclass of meromorphic functions associated with a certain differential operator on Hilbert space. For this class, we obtain several properties like the coefficient inequality, growth and distortion theorem, radius of close-to-convexity, starlikeness and meromorphically convexity and integral transforms. Further, it is shown that this class is closed under convex linear combinations.

Keywords: Meromorphic functions, Coefficient estimates, Hadamard product, Hilbert space operators.

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1. Introduction

Let Σ be denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m \tag{1}$$

which are regular in domain $E = \{z : 0 < |z| < 1\}$ with a simple pole at the origin with residue 1 there.

Let Σ_s, $\Sigma^*(\alpha)$ and $\Sigma_k(\alpha), 0 \leq \alpha < 1$ be denote the subclass of Σ that are univalent, moromorphically starlike of order α and meromorphically convex of order α respectively. Analytically $f(z)$ of the form (1) is in $\Sigma^*(\alpha)$ if and only if

$$Re\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha,\ z \in E. \tag{2}$$

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Similarly, $f \in \Sigma_k(\alpha)$ if and only if $f(z)$ is of the form (1) and satisfies

$$Re\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > \alpha,\ z \in E. \tag{3}$$

It being understood that if $\alpha = 1$ then $f(z) = \frac{1}{z}$ is the only function which is $\Sigma^*(1)$ and $\Sigma_k(1)$.

The classes $\Sigma^*(\alpha)$ and $\Sigma_k(\alpha)$ have been extensively studied by Pommerenke [7], Clunie [2], Royster [8]and others.

Since, to a certain extent the work in the meromorphic univalent case has paralleled that of regular univalent case, it is natural to search for a subclass of Σ_s that has properties analogous to those of $T^*(\alpha)$. Juneja and Reddy [6] introduced the class Σ_p of functions of the form

$$f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m,\ a_m \geq 0, \tag{4}$$

$$\Sigma_p^*(\alpha) = \Sigma_p \cap \Sigma^*(\alpha).$$

For functions $f(z)$ in the class Σ_p, we define a linear operator D^n by the following form

$$\begin{aligned}
D^0 f(z) &= f(z)\\
D^1 f(z) &= \frac{1}{z} + 3a_1 z + 4a_2 z^2 + \cdots = \frac{(z^2 f(z))'}{z}\\
D^2 f(z) &= D(D^1 f(z))\\
&\ \ \vdots\\
D^n f(z) &= D(D^{n-1} f(z)) = \frac{1}{z} + \sum_{m=1}^{\infty} (m+2)^n a_m z^m\\
&= \frac{(z^2 D^{n-1} f(z))'}{z},\ \text{for } n = 1, 2, \cdots .
\end{aligned} \tag{5}$$

Let H be a Hilbert space on the complex field and $L(H)$ denote the algebra of all bounded linear operators on H. For a complex- valued function f analytic in a domain E of the complex plane containing the spectrum $\sigma(T)$ of the bounded linear operator T, let $f(T)$ denote the operator on H defined by the Riesz-Dunford integral [3]

$$f(T) = \frac{1}{2\pi i} \int_C (zI - T)^{-1} f(z) dz,$$

where I is the identity operator on H and C is positively oriented simple closed rectifiable closed contour containing the spectrum $\sigma(T)$ in the interior domain [4]. The operator $f(T)$ can also be defined by the following series

$$f(T) = \sum_{m=0}^{\infty} \frac{f^m(0)}{m!} T^m$$

which converges in the norm topology.

The class of all functions $f \in \Sigma$ with $a_m \geq 0$ is denoted by Σ_p. The object of the preset paper is to investigate the following subclass of Σ_p associated with the differential operator $D^n f(z)$.

Definition 1.1. *For $0 \leq \beta < 1$ and $0 \leq \alpha < 1$, a function $f \in \Sigma_p$ given by (1) is in the class $\sigma_p(\alpha, \beta, T)$ if*

$$\begin{aligned} & \|T(D^n f(T))' - \{(\beta-1)D^n f(T) + \beta T(D^n f(T))'\}\| \\ < \; & \|T(D^n f(T))' + (1-2\alpha)\{(\beta-1)D^n f(T) + \beta T(D^n f(T))'\}\|. \end{aligned}$$

The main object of the paper is to study some usual properties of the geometric function theory such as coefficient bounds, growth and distortion properties, radius of convexity, convex linear combination and convolution properties, integral operators and $\delta-$neighbourhoods for the class $\sigma_p(\alpha, \beta, T)$.

2. Coefficient Bounds

We first give a characterization of the class $\sigma_p(\alpha, \beta, T)$ by finding necessary and sufficient condition for a functions in the class. This characterization implies coefficient estimates.

Theorem 2.1. *A function $f \in \Sigma_p$ given by (4) is in the class $\sigma_p(\alpha, \beta, T)$ for all contraction T with $T \neq \theta$ if and only if*

$$\sum_{m=1}^{\infty} [m + \alpha - \alpha\beta(m+1)](m+2)^n a_m \; \leq \; (1-\alpha). \tag{6}$$

The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{[m+\alpha-\alpha\beta(m+1)](m+2)^n} z^m, \; m \geq 1. \tag{7}$$

Proof. Suppose that (6) is true for $0 \leq \beta < 1$ and $0 \leq \alpha < 1$. Then

$$\begin{aligned} & \left\|T(D^n f(T))' - \{(\beta-1)D^n f(T) + \beta T(D^n f(T))'\}\right\| \\ & - \left\|T(D^n f(T))' + (1-2\alpha)\{(\beta-1)D^n f(T) + \beta T(D^n f(T))'\}\right\| \\ = \; & \left\| \sum_{m=1}^{\infty} (m+1)(1-\beta)(m+2)^n a_m T^m \right\| \\ & - \left\| 2(1-\alpha)T^{-1} - \sum_{m=1}^{\infty} [m + (1-2\alpha)(\beta - 1 + \beta m)](m+2)^n a_m T^m \right\| \\ \leq \; & \sum_{m=1}^{\infty} (m+1)(1-\beta)(m+2)^n a_m \|T\|^m - 2(1-\alpha)\|T^{-1}\| \\ & + \sum_{m=1}^{\infty} [m + (1-2\alpha)(\beta - 1 + \beta m)](m+2)^n a_m \|T^m\| \\ = \; & 2\sum_{m=1}^{\infty} [m + \alpha - \alpha\beta(m+1)](m+2)^n a_m \|T^m\| - 2(1-\alpha)\|T^{-1}\| \\ \leq \; & 2(1-\alpha) - 2(1-\alpha) = 0, \text{ by using (6)} \end{aligned}$$

and so $f \in \Sigma p$ is in the class $\sigma_p(\alpha, \beta, T)$.

Conversely suppose that $f \in \sigma_p(\alpha, \beta, T)$ satisfies the coefficients inequality (6). Since $f \in \sigma_p(\alpha, \beta, T)$ then

$$\begin{aligned} & \left\| T(D^n f(T))' - \{(\beta-1)D^n f(T) + \beta T(D^n f(T))'\} \right\| \\ < & \left\| T(D^n f(T))' + (1-2\alpha)\{(\beta-1)D^n f(T) + \beta T(D^n f(T))'\} \right\|. \end{aligned}$$

From this inequality, it is obtained that

$$\begin{aligned} & \left\| \sum_{m=1}^{\infty} (m+1)(1-\beta)(m+2)^n a_m T^{m-1} \right\| \\ < & \left\| 2(1-\alpha) - \sum_{m=1}^{\infty} [m + (1-2\alpha)(\beta - 1 + \beta m)](m+2)^n a_m T^{m+1} \right\|. \end{aligned}$$

By choosing $T = rI(0 < r < 1)$ in above inequality, we get

$$\frac{\sum\limits_{m=1}^{\infty} (m+1)(1-\beta)(m+2)^n a_m r^{m+1}}{2(1-\alpha) - \sum\limits_{m=1}^{\infty} [m + (1-2\alpha)(\beta - 1 + \beta m)](m+2)^n a_m r^{m+1}} < 1.$$

Letting $r \to 1$ in the above inequality, we obtain the assertion (6).
This completes the proof of the theorem. □

From Theorem 2.1, we have the following result.

Corollary 2.1. *If a function $f(z) \in \Sigma_p$ given by (4) is in the class $\sigma_p(\alpha, \beta, T)$ then*

$$a_m \le \frac{(1-\alpha)}{\sum\limits_{m=1}^{\infty} [m + \alpha - \alpha\beta(m+1)](m+2)^n}, \quad (m \ge 1). \tag{8}$$

The result is sharp for the function f of the form (7).

3. Distortion Bounds

In this section, we obtain growth and the distortion bounds for the class $\sigma_p(\alpha, \beta, T)$.

Theorem 3.1. *If $f \in \sigma_p(\alpha, \beta, T)$ then $0 < |z| = r < 1$,*

$$\begin{aligned} \|f(T)\| &\ge \frac{1}{\|T\|} - \frac{(1-\alpha)}{3^n[1+\alpha-2\alpha\beta]}\|T\|, \\ \|f(T)\| &\le \frac{1}{\|T\|} + \frac{(1-\alpha)}{3^n[1+\alpha-2\alpha\beta]}\|T\|. \end{aligned} \tag{9}$$

The result is sharp for

$$f(z) = \frac{1}{z} + \frac{(1-\alpha)}{3^n[1+\alpha-2\alpha\beta]}\, z. \tag{10}$$

Proof. Suppose $f(z)$ is in $\sigma_p(\alpha, \beta, T)$. By Theorem 2.1, we have
$3^n[1+\alpha-2\alpha\beta] \sum\limits_{m=1}^{\infty} a_m \le \sum\limits_{m=1}^{\infty} (m+2)^n[m+\alpha-\alpha\beta(m+1)]a_m \le (1-\alpha)$.
Therefore $\sum\limits_{m=1}^{\infty} a_m \le \frac{1-\alpha}{3^n[1+\alpha-2\alpha\beta]}$.

Also, if $f(T) = T^{-1} + \sum\limits_{m=1}^{\infty} a_m T^m$, then

$$\frac{1}{\|T\|} - \sum_{m=1}^{\infty} a_m \|T\|^m \leq \|f(T)\| \leq \frac{1}{\|T\|} + \sum_{m=1}^{\infty} a_m \|T\|^m. \tag{11}$$

Since $\|T\| < 1$, the above inequality becomes

$$\frac{1}{\|T\|} - \|T\| \sum_{m=1}^{\infty} a_m \leq \|f(T)\| \leq \frac{1}{\|T\|} + \|T\| \sum_{m=1}^{\infty} a_m. \tag{12}$$

Using (11), we get the result. □

Theorem 3.2. *If $f(z) \in \sigma_p(\alpha, \beta, T)$ then*

$$\begin{aligned} \|f'(T)\| &\geq \frac{1}{\|T\|^2} - \frac{(1-\alpha)}{3^n[1+\alpha-2\alpha\beta]}, \\ \|f'(T)\| &\leq \frac{1}{\|T\|^2} + \frac{(1-\alpha)}{3^n[1+\alpha-2\alpha\beta]} \ . \end{aligned} \tag{13}$$

The result is sharp for

$$f(z) = \frac{1}{z} + \frac{(1-\alpha)}{3^n[1+\alpha-2\alpha\beta]} \ z. \tag{14}$$

4. Extreme points

In this section, we obtain extreme bounds for the class $\sigma_p(\alpha, \beta, T)$.

Theorem 4.1. *Let $f_0(z) = \frac{1}{z}$ and*

$$f_m(z) = \frac{1}{z} + \frac{(1-\alpha)}{[m+\alpha-\alpha\beta(m+1)](m+2)^m} \ z^m, \ m = 1, 2, \cdots . \tag{15}$$

Then $f \in \sigma_p(\alpha, \beta, T)$ if and only if can be expressed in the form

$$f(z) = \sum_{m=0}^{\infty} \tau_m f_m(z), \ \tau_m \geq 0 \ \textit{and} \ \sum_{m=0}^{\infty} \tau_m = 1.$$

Proof. Assume that $f(z) = \sum\limits_{m=1}^{\infty} \tau_m f_m(z), \ (\tau_m \geq 0, \sum\limits_{m=0}^{\infty} \tau_m = 1, \ m = 0, 1, 2, \cdots)$.
Then we have

$$\begin{aligned} f(z) &= \sum_{m=0}^{\infty} \tau_m f_m(z) \\ &= \tau_0 f_0(z) + \sum_{m=1}^{\infty} \tau_m f_m(z) \\ &= \frac{1}{z} + \sum_{m=1}^{\infty} \frac{1-\alpha}{[m+\alpha-\alpha\beta(m+1)](m+2)^n} z^m. \end{aligned}$$

Therefore $\sum\limits_{m=1}^{\infty} [m+\alpha-\alpha(m+1)](m+2)^n \tau_m \frac{1-\alpha}{[m+\alpha-\alpha\beta(m+1)](m+2)^n}$
$= (1-\alpha) \sum\limits_{m=1}^{\infty} \tau_m = (1-\alpha)(1-\tau_0) \leq (1-\alpha)$.
Hence by Theorem 2.1, $f \in \sigma_p(\alpha, \beta, T)$.

Conversely, suppose that $f \in \sigma_p(\alpha, \beta, T)$. Since, by Corollary 2.1,

$$a_m \leq \frac{1-\alpha}{[m+\alpha-\alpha\beta(m+1)](m+2)^n}, \ m \geq 1,$$

setting $\tau_m = \frac{[m+\alpha-\alpha\beta(m+1)](m+2)^n}{1-\alpha} a_m, \ m \geq 1$ and $\tau_0 = 1 - \sum_{m=1}^{\infty} \tau_m$.
We obtain $f(z) = \tau_0 f_0(z) + \sum_{m=1}^{\infty} \tau_m f_m(z)$.
This completes the proof of the theorem. □

Theorem 4.2. *The class $f \in \sigma_p(\alpha, \beta, T)$ is closed under convex combination.*

Proof. Let the functions $f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m$ and $g(z) = \frac{1}{z} + \sum_{m=1}^{\infty} b_m z^m$ be in the class $\sigma_p(\alpha, \beta, T)$. Then by Theorem 2.1, we have

$$\sum_{m=1}^{\infty} [m+\alpha-\alpha\beta(m+1)](m+2)^n a_m \leq (1-\alpha),$$
$$\sum_{m=1}^{\infty} [m+\alpha-\alpha\beta(m+1)](m+2)^n b_m \leq (1-\alpha).$$

For $0 \leq \tau \leq 1$, define the function $h(z)$ as $h(z) = \tau f(z) + (1-\tau) g(z)$.
Then we get $h(z) = \frac{1}{z} + \sum_{m=1}^{\infty} [\tau a_m + (1-\tau) b_m] z^m$. Now we obtain

$$\begin{aligned}
&\sum_{m=1}^{\infty} [m+\alpha-\alpha\beta(m+1)](m+2)^n [\tau a_m + (1-\tau) b_m] \\
&= \tau \sum_{m=1}^{\infty} [m+\alpha-\alpha\beta(m+1)](m+2)^n a_m + (1-\tau) \sum_{m=1}^{\infty} [m+\alpha-\alpha\beta(m+1)](m+2)^n b_m \\
&\leq \tau(1-\alpha) + (1-\tau)(1-\alpha) \\
&= (1-\alpha).
\end{aligned}$$

So, $h \in \sigma_p(\alpha, \beta, T)$. □

5. Radii of close-to-convexity, starlikeness and convexity

Theorem 5.1. *Let $f \in \sigma_p(\alpha, \beta, T)$. Then f is meromorphically close-to-convex of order γ $(0 \leq \gamma < 1)$ in the disc $|z| < r_1$, where*

$$r_1 = \inf_{m \in N} \left[\frac{(1-\gamma)[m+\alpha-\alpha\beta(m+1)](m+2)^n}{m(1-\alpha)} \right]^{\frac{1}{m+1}}. \tag{16}$$

The result is sharp for the extremal function given by (7).

Proof. It sufficient to show that

$$\|f'(T)T^2 + 1\| < (1-\gamma). \tag{17}$$

By Theorem 2.1, we have

$$\sum_{m=1}^{\infty} \frac{[m+\alpha-\alpha\beta(m+1)](m+2)^n}{1-\alpha} a_m \leq 1.$$

So the inequality

$$\|f'(T)T^2+1\| = \|\sum_{m=1}^{\infty} m a_m T^{m+1}\| \le \sum_{m=1}^{\infty} m a_m \|T\|^{m+1} < (1-\gamma)$$

holds true if

$$\frac{m\|T\|^{m+1}}{1-\gamma} \le \frac{[m+\alpha-\alpha\beta(m+1)](m+2)^n}{1-\alpha}.$$

Then (15) holds true if

$$\|T\|^{m+1} \le \frac{(1-\gamma)[m+\alpha-\alpha\beta(m+1)](m+2)^n}{m(1-\alpha)},\ m\ge 1$$

which yields the close-to-convexity of the function and completes the proof. □

Theorem 5.2. *Let $f\in\sigma_p(\alpha,\beta,T)$. Then f is meromorphically starlike of order γ $(0\le\gamma<1)$ in the disc $|z|<r_2$, where*

$$r_2 = \inf_{m\in N}\left[\frac{(1-\gamma)[m+\alpha-\alpha\beta(m+1)](m+2)^n}{(m+2-\gamma)(1-\alpha)}\right]^{\frac{1}{m+1}}. \tag{18}$$

The result is sharp for the extremal function given by (7).

Proof. Let $f(T)=T^{-1}+\sum\limits_{m=1}^{\infty} a_m T^m$. Since $f\in\sigma_p(\alpha,\beta,T)$ is meromorphically starlike of order γ,

$$\left\|\frac{Tf'(T)}{f(T)}+1\right\| \le (1-\delta). \tag{19}$$

Substituting for f, the above inequality becomes,

$$\sum_{m=1}^{\infty}\left(\frac{m+2-\gamma}{1-\gamma}\right)\|T\|^{m+1}a_m \le 1. \tag{20}$$

By Theorem 2.1,

$$\sum_{m=1}^{\infty}\frac{[m+\alpha-\alpha\beta(m+1)](m+2)^n}{1-\alpha}a_m \le 1. \tag{21}$$

Then, (20) will be true if $\left(\frac{m+2-\gamma}{1-\gamma}\right)\|T\|^{m+1} \le \frac{[m+\alpha-\alpha\beta(m+1)](m+2)^n}{1-\alpha}$.

That is $\|T\| \le \left[\frac{(1-\gamma)[m+\alpha-\alpha\beta(m+1)](m+2)^n}{(1-\alpha)(m+2-\gamma)}\right]^{\frac{1}{m+1}}$. □

Theorem 5.3. *Let $f\in\sigma_p(\alpha,\beta,T)$. Then f is meromorphically convex of order γ $(0\le\gamma<1)$ in the disc $|z|<r_3$, where*

$$r_3 = \inf_{m\in N}\left[\frac{(1-\gamma)[m+\alpha-\alpha\beta(m+1)](m+2)^n}{m(m+2-\gamma)(1-\alpha)}\right]^{\frac{1}{m+1}}.$$

The result is sharp for the extremal function given by (7).

Proof. By using the technique employed in the proof of the Theorem 5.1, we can show that

$$\left\|\frac{Tf''(T)}{f'(T)}+2\right\| < 1-\gamma$$

for $|z|<r_3$ and prove that the assertion of the theorem is true. □

6. Hadamard product

Theorem 6.1. *For function $f, g \in \Sigma_p$ defined by (1) and (2) respectively, let $f, g \in \sigma_p(\alpha, \beta, T)$. Then the Hadamard product $f * g \in \sigma_p(\rho, \beta, T)$, where*

$$\rho \leq 1 - \frac{(1-\alpha)^2(m+1)(1-\beta)}{(1-\alpha)^2[1-\beta(m+1)] + [m+\alpha-\alpha\beta(m+1)]^2(m+2)^n}.$$

Proof. We need to find the largest ρ such that

$$\sum_{m=1}^{\infty} \frac{[m+\rho-\rho\beta(m+1)](m+2)^n}{1-\rho} a_m b_m \leq 1.$$

Since $f, g \in \sigma_p(\alpha, \beta, T)$, by Theorem 2.1, we have

$$\sum_{m=1}^{\infty} \frac{[m+\alpha-\alpha\beta(m+1)](m+2)^n}{1-\alpha} a_m \leq 1 \tag{22}$$

$$\text{and} \sum_{m=1}^{\infty} \frac{[m+\alpha-\alpha\beta(m+1)](m+2)^n}{1-\alpha} b_m \leq 1. \tag{23}$$

From (22) and (23), we find , by means of the Cauchy-Schwartz inequality, that

$$\sum_{m=1}^{\infty} \frac{[m+\alpha-\alpha\beta(m+1)](m+2)^n}{1-\alpha} \sqrt{a_m b_m} \leq 1. \tag{24}$$

We want only to show that

$$\frac{[m+\rho-\rho\beta(m+1)](m+2)^n}{1-\rho} a_m b_m \leq \frac{[m+\alpha-\alpha\beta(m+1)](m+2)^n}{1-\alpha} \sqrt{a_m b_m}$$

$$\Rightarrow \sqrt{a_m b_m} \leq \frac{(1-\rho)[m+\alpha-\alpha\beta(m+1)]}{(1-\alpha)[m+\rho-\rho\beta(m+1)]}. \tag{25}$$

On the other hand, from (24), we have

$$\sqrt{a_m b_m} \leq \frac{(1-\alpha)}{[m+\alpha-\alpha\beta(m+1)](m+2)^n}. \tag{26}$$

Therefore in view of (25) and (26), it is enough to find the largest ρ that

$$\frac{(1-\alpha)}{[m+\alpha-\alpha\beta(m+1)](m+2)^n} \leq \frac{(1-\rho)[m+\alpha-\alpha\beta(m+1)]}{(1-\alpha)[m+\rho-\rho\beta(m+1)]}$$

which yields

$$\rho \leq \frac{[m+\alpha-\alpha\beta(m+1)]^2(m+2)^n - n(1-\alpha)^2}{[m+\alpha-\alpha\beta(m+1)]^2(m+2)^n + (1-\alpha)^2[1-\beta(m+1)]}$$

$$\Rightarrow \rho \leq 1 - \frac{(1-\alpha)^2(m+1)(1-\beta)}{(1-\alpha)^2[1-\beta(m+1)] + [m+\alpha-\alpha\beta(m+1)]^2(m+2)^n}.$$

□

Theorem 6.2. *For function $f, g \in \Sigma_p$ defined by (1) and (2) respectively, let $f, g \in \sigma_p(\alpha, \beta, T)$. Then the function $k(z) = \frac{1}{z} + \sum_{m=1}^{\infty}(a_m^2 + b_m^2)z^m$ is in the class $\sigma_p(\rho, \beta, T)$, where*

$$\rho \leq 1 - \frac{2(1-\alpha)^2(m+2)^n[1-\beta(m+1)+m]}{\{[m+\alpha-\alpha\beta(m+1)](m+2)^n\}^2 + 2(1-\alpha)^2(m+2)^n[1-\beta(m+1)]}.$$

Proof. Since $f, g \in \sigma_p(\alpha, \beta, T)$, we have

$$\sum_{m=1}^{\infty}\left[\frac{[m+\alpha-\alpha\beta(m+1)](m+2)^n}{1-\alpha}a_m\right]^2 \leq 1 \tag{27}$$

$$\text{and}\ \sum_{m=1}^{\infty}\left[\frac{[m+\alpha-\alpha\beta(m+1)](m+2)^n}{1-\alpha}b_m\right]^2 \leq 1. \tag{28}$$

Combining the last two inequalities, we get

$$\sum_{m=1}^{\infty}\left[\frac{[m+\alpha-\alpha\beta(m+1)](m+2)^n}{1-\alpha}\right]^2 (a_m^2+b_m^2) \leq 1. \tag{29}$$

But we need to find the largest ρ such that

$$\sum_{m=1}^{\infty}\left[\frac{[m+\rho-\rho\beta(m+1)](m+2)^n(a_m^2+b_m^2)}{1-\alpha}\right] \leq 1. \tag{30}$$

The inequality (30) would hold if

$$\frac{[m+\rho-\rho\beta(m+1)](m+2)^n}{1-\rho} \leq \frac{1}{2}\left[\frac{[m+\alpha-\alpha\beta(m+1)](m+2)^n}{1-\alpha}\right]^2.$$

Then we have

$$\begin{aligned}\rho &\leq \frac{([m+\alpha-\alpha\beta(m+1)](m+2)^n)^2-2m(1-\alpha)^2(m+2)^n}{([m+\alpha-\alpha\beta(m+1)](m+2)^n)^2+2(1-\alpha)^2(m+2)^n[1-\beta(m+1)]}\\ &= 1-\frac{2(1-\alpha)^2(m+2)^n[1-\beta(m+1)+m]}{([m+\alpha-\alpha\beta(m+1)](m+2)^n)^2+2(1-\alpha)^2(m+2)^n[1-\beta(m+1)]}.\end{aligned}$$

□

7. Integral operators

In this section, we consider integral transforms of functions in the class $\sigma_p(\alpha, \beta, T)$ of the type considered by Goel and Sohi [5].

Theorem 7.1. *Let the function $f \in \Sigma_p$ given by (1) be in the class $\sigma_p(\alpha, \beta, T)$. Then the integral operator*

$$F(z) = c\int_0^1 u^c f(uz)du,\ 0 < u \leq 1,\ 0 < c < \infty \tag{31}$$

is in the class $\sigma_p(\rho, \beta, T)$, where

$$\rho = 1-\frac{(1-\alpha)(1+2\beta)+c}{(1+\alpha-2\alpha\beta)(c+2)+(1-\alpha)(1-2\beta)c}.$$

The result is sharp for the function

$$f(z) = \frac{1}{z}+\frac{(1-\alpha)}{3^n[1+\alpha-2\alpha\beta]}z.$$

Proof. Let $f \in \Sigma_p$ given by (1) be in the class $\sigma_p(\alpha, \beta, T)$. Then

$$F(z) = c\int_0^1 u^c f(uz)du = \frac{1}{z}+\frac{c}{c+m+1}a_m z^m. \tag{32}$$

We have to show that

$$\sum_{m=1}^{\infty}\left[\frac{c[m+\rho-\rho\beta(m+1)](m+2)^n}{(1-\rho)(c+m+1)}\right]a_m \leq 1. \tag{33}$$

Since $f \in \sigma_p(\alpha,\beta,T)$, we have

$$\sum_{m=1}^{\infty}\frac{[m+\alpha-\alpha\beta(m+1)](m+2)^n}{1-\alpha}a_m \leq 1.$$

The inequality (33) satisfied if

$$\frac{c[m+\rho-\rho\beta(m+1)]}{(1-\rho)(c+m+1)} \leq \frac{[m+\alpha-\alpha\beta(m+1)]}{1-\alpha}.$$

Then we get

$$\begin{aligned}\rho &\leq \frac{[m+\alpha-\alpha\beta(m+1)](m+c+1)-(1-\alpha)cm}{[m+\alpha-\alpha\beta(m+1)](m+c+1)+c(1-\alpha)[1-\beta(m+1)]}\\ &= 1-\frac{(1-\alpha)(1+\beta(m+1)+cm}{[m+\alpha-\alpha\beta(m+1)](m+c+1)+c(1-\alpha)[1-\beta(m+1)]}.\end{aligned}$$

$$\text{Since } \phi(m) = 1-\frac{(1-\alpha)(1+\beta(m+1)+cm}{[m+\alpha-\alpha\beta(m+1)](m+c+1)+c(1-\alpha)[1-\beta(m+1)]}$$

is an increasing function of $m,\ m \geq 1$. We obtained the desired result. □

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